HOMOLOGY OPERATIONS AND POWER SERIES

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1. Introduction. Bullett and Macdonald [**1**] have used power series to simplify the statement and proof of the Adem relations for Steenrod cohomology operations. In this paper I give a similar treatment of May's generalized Adem relations [**4**, §4] and of the Nishida relations ([**6**], [**2**, I.1.1(9)], [**5**, 3.1(7)]). Both sets of relations apply to Dyer-Lashof operations in E_∞ spaces such as infinite loop spaces ([**3**], [**2**, I.1]) and in H_∞ ring spectra ([**5**, §3]).

The results on the Adem relations are very similar to those of [**1**], and the proof mostly comes from [**4**]. The surprising result of this paper is that the Adem and Nishida relations can be expressed by virtually the same formulae ((2*a*) and (5*a*), (2*c*) and (5*b*) below). My proof of the Nishida relations is designed to explain this, and is a bit simpler than the original proof.

Throughout the paper we work modulo p, where p is a prime number.

2. Statement of the Adem relations. Recall that in [**4**] May constructs operations in the homology of certain modulo p chain complexes. There are operations P_i for all p and βP_i for odd p, where i is any integer. The degree of P_i is i if $p=2$ or $2i(p-1)$ if p is odd, and the degree of βP_i is $2i(p-1)-1$. The Dyer-Lashof operations denoted Q^i and βQ^i are special cases of the P_i and βP_i, and the Steenrod cohomology operations denoted P^i (or Sq^i if $p=2$) and βP^i are special cases of the P_{-i} and βP_{-i} (a qth cohomology group being regarded as a $(-q)$th homology group).

We consider complexes for which all the operations are defined and the Adem relations hold: Adem objects of $\mathscr{C}(p,\infty)$ in the terminology of [**4**]. Let u and v be indeterminates commuting with each other and with anything else that occurs, and put

$$t=v(1-u^{-1}v)^{p-1},\qquad s=u(1-v^{-1}u)^{p-1}=u^pv^{-p}t=u^pv^{-(p-1)}(1-u^{-1}v)^{p-1}. \tag{1}$$

For any z, write

$$P(z)=\sum_i z^iP_i,\qquad \beta P(z)=\sum_i z^i\beta P_i;$$

(all summations are over the integers, and terms usually regarded as undefined are to be taken as zero). The Adem relations can then be written: for all p,

$$P(u)P(t)=P(v)P(s), \tag{2a}$$

and for odd p,

$$\beta P(u)P(t)=\beta P(v)P(s), \tag{2b}$$

$$P(u)\beta P(t)=(1-u^{-1}v)\beta P(v)P(s)+u^{-1}vP(v)\beta P(s), \tag{2c}$$

$$\beta P(u)\beta P(t)=u^{-1}v\beta P(v)\beta P(s). \tag{2d}$$

The usual Adem relations are derived in Section 5, and (2*a*)–(2*d*) are proved in Section 7.

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REMARK 1. In the applications mentioned above, βP_i is the composite of P_i with a Bockstein β satisfying $\beta^2=0$, so that (2a) implies (2b) and (2c) implies (2d). But this is not true in general (despite the notation), which is why (2b) and (2d) are stated separately.

REMARK 2. To get formulae for cohomology operations, write $P^*(z)=\sum_i z^i P^i$, etc. ($P^*(z)$ is denoted $P(z)$ in [**1**]). Since P^i is an example of P_{-i}, the relations (2a)–(2d) imply

$$P^*(u^{-1})P^*(t^{-1})=P^*(v^{-1})P^*(s^{-1}),$$

etc. These relations are not the same as in [**1**], but are equivalent, as is shown in [**1**, §4] for $p=2$.

3. Statement of the Nishida relations. The Nishida relations apply to the homology of an E_∞ space or H_∞ ring spectrum. They link the Dyer–Lashof operations Q^i, the Bockstein β, and the duals P_*^i of the Steenrod cohomology operations. With u, v, s and t as in (1) and with the obvious meanings for $Q(z)$ and $P_*(z)$ they say: for all p,

$$P_*(u^{-1})Q(v)=Q(t)P_*(s^{-1}), \tag{3a}$$

and for odd p,

$$P_*(u^{-1})\beta Q(v)=(1-u^{-1}v)^{-1}[\beta Q(t)P_*(s^{-1})-u^{-1}vQ(t)P_*(s^{-1})\beta]. \tag{3b}$$

These can be rewritten to look like the Adem relations (2a) and (2c) if one uses the conjugate cohomology operations cP^i defined by

$$\sum_{i+j=k}(cP^i)P^j = \begin{cases}1 & \text{if} \quad k=0,\\ 0 & \text{if} \quad k\neq 0.\end{cases}$$

This definition can be written $cP^*(z)P^*(z)=1$ with the obvious notation, and dualization yields

$$cP_*(z)=P_*(z)^{-1}. \tag{4}$$

So (3a) is equivalent to: for all p,

$$cP_*(u^{-1})Q(t)=Q(v)cP_*(s^{-1}). \tag{5a}$$

Substitution of (3a) in (3b) yields

$$P_*(u^{-1})\beta Q(v)=(1-u^{-1}v)^{-1}[\beta Q(t)P_*(s^{-1})-u^{-1}vP_*(u^{-1})Q(v)\beta],$$

which by (4) is similarly equivalent to: for odd p,

$$cP_*(u^{-1})\beta Q(t)=(1-u^{-1}v)\beta Q(v)cP_*(s^{-1})+u^{-1}vQ(v)\beta cP_*(s^{-1}). \tag{5b}$$

The usual Nishida relations will be derived from (3a)–(3b) in Section 6 and (5a)–(5b) will be proved from (2a) and (2c) in Section 8.

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4. The Cartan formulae. For completeness we note that the Cartan formulae

$$P_k(x\otimes y)=\sum_{i+j=k} P_ix\otimes P_jy$$

[**4**, 2.7] are obviously equivalent to

$$P(z)(x\otimes y)=P(z)x\otimes P(z)y,$$

and similarly for other Cartan formulae. One, which will be used in Section 8, is

$$cP_*(z)(x\otimes y)=cP_*(z)x\otimes cP_*(z)y. \tag{6}$$

5. Derivation of the usual Adem relations. The usual Adem relations [**4**, 4.7] are: for all p,

$$P_aP_b=\sum_i (-1)^{a+i}\binom{(i-b)(p-1)-1}{ip-a}P_{a+b-i}P_i,$$

and for odd p,

$$\beta P_aP_b=\sum_i (-1)^{a+i}\binom{(i-b)(p-1)-1}{ip-a}\beta P_{a+b-i}P_i,$$

$$P_a\beta P_b=\sum_i (-1)^{a+i}\binom{(i-b)(p-1)}{ip-a}\beta P_{a+b-i}P_i-\sum_i (-1)^{a+i}\binom{(i-b)(p-1)-1}{ip-a-1}P_{a+b-i}\beta P_i,$$

$$\beta P_a\beta P_b=-\sum_i (-1)^{a+i}\binom{(i-b)(p-1)-1}{ip-a-1}\beta P_{a+b-i}\beta P_i.$$

REMARK. The binomial coefficients are defined by

$$\binom{n}{k}=\begin{cases}\dfrac{n(n-1)\ldots(n-k+1)}{k!}, & \text{if } k>0,\\ 1 & \text{if } k=0,\\ 0 & \text{if } k<0,\end{cases}$$

regardless of the sign of n. With these conventions, the relations are true for all a and b.

The relations are derived from (2a)–(2d) respectively by the method of [**1**, §3]: express the right hand sides of (2a)–(2d) as power series in u, v, t by putting $s=u^pv^{-p}t$ (see (1)), then express them as power series in u and t by putting

$$v^r=\sum_k (-1)^{k-r}\binom{-k(p-1)-1}{k-r}u^{r-k}t^k \tag{7}$$

(which is true modulo p and will be proved below), and finally equate the coefficients of u^at^b.

Proof of (7). Put $V=u^{-1}v$, $T=u^{-1}t$; then (7) is equivalent to

$$V^r=\sum_k (-1)^{k-r}\binom{-k(p-1)-1}{k-r}T^k.$$

But (1) gives

$$T = V(1-V)^{p-1},$$

which modulo p gives

$$dT = (1-V)^{p-2}\,dV,$$

so

$$V^r = \sum_k \left(\operatorname*{res}_{T=0} V^r T^{-k-1}\,dT\right)T^k = \sum_k \left(\operatorname*{res}_{V=0} V^{r-k-1}(1-V)^{-k(p-1)-1}\,dV\right)T^k$$

$$= \sum_k (-1)^{k-r}\binom{-k(p-1)-1}{k-r}T^k,$$

as required.

6. Derivation of the usual Nishida relations. The usual Nishida relations ([**6**], [**2**, I.1.1(9)], [**5**, 3.1(7)]) are: for all p,

$$P_*^m Q^n = \sum_i (-1)^{m+i}\binom{(n-m)(p-1)}{m-ip}Q^{n-m+i}P_*^i,$$

and for odd p,

$$P_*^m \beta Q^n = \sum_i (-1)^{m+i}\binom{(n-m)(p-1)-1}{m-ip}\beta Q^{n-m+i}P_*^i$$

$$= +\sum_i (-1)^{m+i}\binom{(n-m)(p-1)-1}{m-ip-1}Q^{n-m+i}P_*^i\beta.$$

They are easily derived from (3a)–(3b) respectively: express the right hand sides of (3a)–(3b) as power series in u, v and $(1-u^{-1}v)$ by (1), then express them as power series in u and v by the binomial theorem, and finally equate the coefficients of $u^{-m}v^n$.

7. Proof of the Adem relations (2a)–(2d). This is based on May's proof [**4**, §4]; we shall show that (2a)–(2d) are linear combinations of equations derived in that proof. First we recall some notation and the definition of the operations. Throughout this section x is a homology class of degree q. If p is odd, then we write

$$m = \tfrac{1}{2}(p-1), \qquad \nu(2r) = (-1)^r, \qquad \nu(2r+1) = (-1)^r m! \tag{8a}$$

where r is any integer. Modulo p we have $(p-1)! = -1$ (Wilson's theorem), so

$$(m!)^2 = (-1)^{m+1}. \tag{8b}$$

For all p and for $i \geqslant 0$, let e_i be the standard generator of $H_i(\mathbf{Z}/p;\mathbf{Z}/p) \cong \mathbf{Z}/p$ as in [**4**, 1.2]; for $i<0$, let e_i be 0. Let θ be the structure map inducing the operations and put

$$D_i x = \theta_*(e_i \otimes x^p). \tag{9}$$

The definition [**4**, 2.2] of the operations then amounts to: for $p=2$,

$$D_{i-q}x = P_i x, \tag{10a}$$

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and for odd p,

$$D_{(2i-q)(p-1)}x = (-1)^i \nu(q)^{-1} P_i x, \qquad D_{(2i-q)(p-1)-1}x = (-1)^i \nu(q)^{-1} \beta P_i x. \tag{10b}$$

Next we quote the necessary equations from [**4**, 4]. They are equations (e), (g) and (i) of the proof of [**4**, 4.7], and say: for $p=2$,

$$\sum_k \binom{s-k}{k} D_{r+2k-s} D_{s-k} x = \sum_l \binom{r-l}{l} D_{s+2l-r} D_{r-l} x; \tag{11a}$$

for odd p and for $\varepsilon = 0$ or 1,

$$\sum_k (-1)^k \nu(2s) \binom{s-k(p-1)}{k} D_{2r+(2kp-2s)(p-1)-\varepsilon} D_{2s-2k(p-1)} x$$
$$= \sum_l (-1)^{l+qm} \nu(2r) \binom{r-l(p-1)}{l} D_{2s+(2lp-2r)(p-1)-\varepsilon} D_{2r-2l(p-1)} x \tag{11b}$$

and

$$\sum_k (-1)^{k+qm} \nu(2s-1) \binom{s-k(p-1)-1}{k} D_{2r+(2kp-2s+1)(p-1)-\varepsilon} D_{2s-2k(p-1)-1} x$$
$$= (1-\varepsilon) \sum_l (-1)^{l+qm} \nu(2r) \binom{r-l(p-1)}{l} D_{2s+(2lp-2r)(p-1)-1} D_{2r-2l(p-1)} x$$
$$- \sum_l (-1)^l \nu(2r-1) \binom{r-l(p-1)-1}{l} D_{2s+(2lp-2r+1)(p-1)-\varepsilon} D_{2r-2l(p-1)-1} x. \tag{11c}$$

Finally we prove ($2a$)–($2d$) by forming linear combinations of ($11a$)–($11c$). First take $p=2$. Put $r=a-q$, $s=b-q$ in ($11a$), multiply both sides by $u^a v^b$, and sum over a and b to get

$$\sum_{a,b,k} \binom{b-k-q}{k} u^a v^b D_{a-b+2k} D_{b-k-q} x = \sum_{a,b,l} \binom{a-l-q}{l} u^a v^b D_{-a+b+2l} D_{a-l-q} x. \tag{12}$$

Change the summation variables on the left hand side to i, j, k, where $i=a+k+q$, $j=b-k$. The left hand side becomes

$$\sum_{i,j,k} \binom{j-q}{k} u^{i-k-q} v^{j+k} D_{i-(q+j)} D_{j-q} x.$$

By the binomial theorem and ($10a$), this is

$$\sum_{i,j} u^{i-q} v^j (1+u^{-1}v)^{j-q} P_i P_j x$$

(note that $\deg D_{j-q}x = \deg P_j x = q+j$). By (1) therefore (since we are working modulo 2), the left hand side of (12) is $(u+v)^{-q} P(u) P(t) x$. Similarly the right hand side is $(u+v)^{-q} P(v) P(s) x$. So (12) implies ($2a$), for $p=2$, as required.

Next take p to be odd. The calculations are similar, but more complicated. First put $r=a(p-1)-qm$, $s=b(p-1)-qm$ in ($11b$) and ($11c$), multiply both sides by $(-1)^{a+b} u^a v^b$,

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and sum over a and b to get equations analogous to (12). Transform the left hand sides as for $p=2$, using the change of variables $i=a+k+qm$, $j=b-k$, and eliminating the $\nu(d)$ by (8a). Transform the right hand sides in a similar manner. After using (8b) to eliminate $(m!)^2$ one obtains

$$(-1)^q(v-u)^{-qm}\beta^\varepsilon P(u)P(t)x=(-1)^q(v-u)^{-qm}\beta^\varepsilon P(v)P(s)x$$

and

$$\begin{aligned}(-1)^q(v-u)^{-qm}(1-u^{-1}v)^{-1}\beta^\varepsilon P(u)\beta P(t)x\\ =(1-\varepsilon)(-1)^q(v-u)^{-qm}\beta P(v)P(s)x\\ -(-1)^q(v-u)^{-qm}(1-v^{-1}u)^{-1}\beta^\varepsilon P(v)\beta P(s)x.\end{aligned}$$

Multiply these equations by $(-1)^q(v-u)^{qm}$ and $(-1)^q(v-u)^{qm}(1-u^{-1}v)$, respectively. The first equation gives (2a) for $\varepsilon=0$ and (2b) for $\varepsilon=1$, and the second gives (2c) for $\varepsilon=0$ and (2d) for $\varepsilon=1$. This completes the proof.

8. Proof of the Nishida relations (5a)–(5b). The proof uses the formally similar Adem relations (2a) and (2c). To be precise, it uses the Adem relations for cohomology operations (Section 2, Remark 2) dualized and conjugated. Since the dual of the Bockstein is the Bockstein (up to sign) and the conjugate $c\beta$ of the Bockstein is $-\beta$, these relations are: for all p,

$$cP_*(u^{-1})cP_*(t^{-1})=cP_*(v^{-1})cP_*(s^{-1}), \tag{13a}$$

and for odd p,

$$cP_*(u^{-1})\beta cP_*(t^{-1})=(1-u^{-1}v)\beta cP_*(v^{-1})cP_*(s^{-1})+u^{-1}vcP_*(v^{-1})\beta cP_*(s^{-1}). \tag{13b}$$

The idea of the proof is to express the Q^i in terms of the cP^i_*, which is essentially done in [**4**, 9.1], and so reduce (5a) and (5b) to (13a) and (13b). To express the Q^i in terms of the cP^i_*, we proceed as follows. Let X be the E_∞ space or H_∞ ring spectrum we are considering. We have a composite map

$$\theta d: B\mathbf{Z}/p\times X\to X,$$

where $B\mathbf{Z}/p$ is the classifying space of $\mathbf{Z}/p$ (so that $H_*(B\mathbf{Z}/p;\mathbf{Z}/p)=H_*(\mathbf{Z}/p;\mathbf{Z}/p)$), θ is the structure map defining the operations $Q^i=P_i$ by (9), (10a), (10b) and d is the "diagonal" map whose effect in homology is described by [**4**, 9.1]. Let $x\in H_s(X;\mathbf{Z}/p)$. From [**4**, 9.1] we obtain the equations: for $p=2$,

$$\theta_*d_*(e_r\otimes x)=\sum_k\theta_*(e_{r+2k-s}\otimes(P^k_*x)^2), \tag{14a}$$

and for odd p,

$$\theta_*d_*(e_{2r(p-1)}\otimes x)=\nu(s)\sum_k(-1)^k\theta_*(e_{(2r+2kp-s)(p-1)}\otimes(P^k_*x)^p). \tag{14b}$$

Write

$$e(z)=\begin{cases}\sum_i z^ie_i & \text{if}\quad p=2,\\ \sum_i z^ie_{2i(p-1)} & \text{if}\quad p \text{ is odd.}\end{cases} \tag{15}$$

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We now claim that, for all p,

$$\theta_* d_*(e(-z)\otimes x) = Q(z)P_*(z^{-1})x. \tag{16}$$

Indeed one can prove this by applying (15) to the left hand side, then (14a)–(14b), then (9), then (10a)–(10b) (note that P_i is Q^i in the present application, and that deg $P^i_* x$ is $s-i$ if $p=2$ or $s-2i(p-1)$ if p is odd), and finally (8a). Since $cP_*(z^{-1}) = P_*(z^{-1})^{-1}$ (by (4)), (16) gives: for all p

$$Q(z)x = \theta_* d_*(e(-z)\otimes cP_*(z^{-1})x). \tag{17}$$

Equation (17) expresses the Q^i in terms of the cP^i_*. To proceed further, we need the values of the Steenrod operations in $H_*(\mathbf{Z}/p; \mathbf{Z}/p)$. These are well-known: from [**4**, proof of 4.6, equations (c) and (d)] and from [**4**, 1.2(2)] we find: for $p=2$,

$$P^k_* e_r = \binom{r-k}{k} e_{r-k},$$

and for odd p,

$$\beta e_{2r(p-1)} = e_{2r(p-1)-1},$$

$$P^k_* e_{2r(p-1)} = \binom{(r-k)(p-1)}{k} e_{2(r-k)(p-1)},$$

$$P^k_* e_{2r(p-1)-1} = \binom{(r-k)(p-1)-1}{k} e_{2(r-k)(p-1)-1}.$$

With $e(z)$ as in (15), these equations combined with the binomial theorem give: for all p,

$$P_*(u^{-1})e(-v) = e(-t),$$

and for odd p,

$$P_*(u^{-1})\beta e(-v) = (1-u^{-1}v)^{-1}\beta e(-t),$$

where $t = v(1-u^{-1}v)^{p-1}$ as in (1). Inverting $P_*(u^{-1})$ by (4) gives: for all p,

$$cP_*(u^{-1})e(-t) = e(-v), \tag{18a}$$

and for odd p,

$$cP_*(u^{-1})\beta e(-t) = (1-u^{-1}v)\beta e(-v). \tag{18b}$$

We can now prove (5a). For all p, (17) gives

$$cP_*(u^{-1})Q(t)x = cP_*(u^{-1})\theta_* d_*(e(-t)\otimes cP_*(t^{-1})x).$$

But $\theta_* d_*$ commutes with $cP_*(u^{-1})$, since it is induced by a map θd, so the right hand side is

$$\theta_* d_* cP_*(u^{-1})(e(-t)\otimes cP_*(t^{-1})x).$$

Apply the Cartan formula (6), then (18a) and the Adem relation (13a), to get

$$\theta_* d_*(e(-v)\otimes cP_*(v^{-1})cP_*(s^{-1})x).$$

By (17) this is $Q(v)cP_*(s^{-1})x$, as required. This completes the proof of (5a).

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Similarly we can prove ($5b$). Here p is odd. By (17)

$$cP_*(u^{-1})\beta Q(t)x = cP_*(u^{-1})\beta\theta_*d_*(e(-t)\otimes cP_*(t^{-1})x).$$

Now β commutes with θ_*d_* just as $cP_*(u^{-1})$ does, and β is a derivation, so the right hand side is

$$\theta_*d_*cP_*(u^{-1})(\beta e(-t)\otimes cP_*(t^{-1})x + e(-t)\otimes \beta cP_*(t^{-1})x).$$

Apply (6), then ($18a$)–($18b$) and ($13a$)–($13b$), to get

$$(1-u^{-1}v)\theta_*d_*(\beta e(-v)\otimes cP_*(v^{-1})cP_*(s^{-1})x + e(-v)\otimes \beta cP_*(v^{-1})cP_*(s^{-1})x) \\ + u^{-1}v\theta_*d_*(e(-v)\otimes cP_*(v^{-1})\beta cP_*(s^{-1})x).$$

Since β is a derivation commuting with θ_*d_*, this is

$$(1-u^{-1}v)\beta\theta_*d_*(e(-v)\otimes cP_*(v^{-1})cP_*(s^{-1})x) + u^{-1}v\theta_*d_*(e(-v)\otimes cP_*(v^{-1})\beta cP_*(s^{-1})x),$$

and by (17) this is

$$(1-u^{-1}v)\beta Q(v)cP_*(s^{-1})x + u^{-1}vQ(v)\beta cP_*(s^{-1})x,$$

as required. This completes the proof.

REFERENCES

1. S. R. Bullett and I. G. Macdonald, On the Adem relations, *Topology* **21** (1982), 329–332.

2. F. R. Cohen, T. J. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics **533** (Springer-Verlag, 1976).

3. E. Dyer and R. K. Lashof, Homology of iterated loop spaces, *Amer. J. Math.* **84** (1962), 35–88.

4. J. P. May, A general algebraic approach to Steenrod operations, in F. P. Peterson, ed., *The Steenrod algebra and its applications*, Lecture Notes in Mathematics **168** (Springer-Verlag, 1970), 153–231.

5. J. P. May, H_∞ ring spectra and their applications, in R. J. Milgram, ed., Algebraic and geometric topology, *Proc. Symp. Pure Math.* **32** (1978), part 2, 229–243.

6. G. Nishida, Cohomology operations in iterated loop spaces, *Proc. Japan Acad.* **44** (1968), 104–109.

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