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THE AUGMENTATION TERMINALS OF CERTAIN LOCALLY FINITE GROUPS

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1. Introduction

1.1. Terminals. Let G be a group and $\mathbf{Z}G$ be the integral group ring of G. We shall write $\mathfrak{g}$ for the augmentation ideal of G; that is to say, the kernel of the homomorphism of $\mathbf{Z}G$ onto $\mathbf{Z}$ which sends each group element to 1. The powers $\mathfrak{g}^\lambda$ of $\mathfrak{g}$ are defined inductively for ordinals λ by $\mathfrak{g}^\lambda = \mathfrak{g}^\mu\mathfrak{g}$, if $\lambda = \mu + 1$, and $\mathfrak{g}^\lambda = \bigcap_{\mu<\lambda}\mathfrak{g}^\mu$, otherwise. The first ordinal λ for which $\mathfrak{g}^\lambda = \mathfrak{g}^{\lambda+1}$ is called the *augmentation terminal* or simply the *terminal* of G. For example, if G is either a cyclic group of prime order or else isomorphic with the additive group of rational numbers then $\mathfrak{g}^n > \mathfrak{g}^\omega = 0$ for all finite n, so that these groups have terminal ω.

The groups with finite terminal are well-known and easily described. If G is one such, then every homomorphic image of G must also have finite terminal. It follows that the derived quotient G/G' of G must be periodic and divisible. It is simple to see that this implies $\mathfrak{g}^2 = \mathfrak{g}^3$ (see, for example, [**3**, § 4.3]). Since the additive group $\mathfrak{g}/\mathfrak{g}^2$ is isomorphic with G/G', the deduction is that *G has terminal* 1 *if G coincides with G' and* 2 *if not.*

We are primarily concerned with finding all the locally finite groups whose terminals are infinite but less than $\omega 2$; and with describing the terminals of such groups as precisely as possible in group theoretic terms.

1.2. The main theorem. There are five characteristic subgroups of G which play a rôle. The first is K which is to be the intersection of all the normal subgroups X of G for which G/X is nilpotent and of finite exponent. *We shall assume throughout that G/K' is periodic.* One consequence of this assumption, which we prove in § 3.1, is that K' is contained in the commutator subgroup $[K, {}_nG]$ for any n. Therefore K' is contained in the ωth term of the lower central series of G. This is the second of the five. We shall denote it by M; and we shall write the lower central series as

$$G = \gamma_1(G) \geqq \gamma_2(G) \geqq \dots,$$

so that

$$M = \gamma_\omega(G) = \bigcap (\gamma_n(G)\colon n \geqq 1).$$

The third, L, is to be $(1 + \mathfrak{g}^\omega) \cap G$. That L comes between K and M will be clear from § 3.

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For any group X and set $\tilde{\omega}$ of primes, we shall write $X_{\tilde{\omega}}$ for the subgroup generated by all the elements of X whose orders are finite $\tilde{\omega}$-numbers. Writing H for G/K', we define N by saying that N/K' is to be the product of the subgroups $[H_p, H_q]$ taken over all pairs p, q of distinct primes. The subgroup O is defined similarly by requiring that O/K' is the product of all the subgroups $[H_p, H_q, H_r]$ for distinct primes p, q and r. Obviously O is contained in N and, since two elements of a nilpotent group commute if they have coprime orders, N is contained in M.

Most of our calculations will take place inside H. In other words we shall usually be able to assume that K' is trivial. The diagram which illustrates this case may be useful.

$$\begin{array}{l} G \\ K = \bigcap_n G^{n!}\gamma_n(G) \\ L = (1 + \mathfrak{g}^{\omega}) \cap G \\ M = \bigcap_n \gamma_n(G) \\ N = \prod\,[G_p, G_q] \\ O = \prod\,[G_p, G_q, G_r] \\ K' = 1 \end{array}$$

For any right $\mathbf{Z}G$-module A we define $A_{(0)} = A$ and, for non-zero ordinals λ, we set $A_{(\lambda)} = A_{(\mu)}\mathfrak{g}$ if $\lambda = \mu + 1$, and $A_{(\lambda)} = \bigcap_{\mu<\lambda} A_{(\mu)}\mathfrak{g}$ if λ is a limit ordinal. The first ordinal λ for which $A_{(\lambda)} = A_{(\lambda+1)}$ is the *G-depth of A*; we shall denote it with $d(A, G)$. For example, $d(\mathbf{Z}G, G)$ is none other than the terminal of G.

Now K/K', as an Abelian normal subgroup of H, is an H-module in the usual way. Important for our purposes is the factor N/O. More precisely it is the H_p-depths of the p-parts $(N/O)_p$ which are significant. Accordingly we shall write

(1) $$d_p = d((N/O)_p, H_p),$$

and set

(2) $$d = \operatorname*{Max}_p d_p.$$

With these notations we may state the main result as

THEOREM 1. *Suppose G/K' is locally finite. The terminal of G is less than $\omega 2$ if and only if*

(3) *K/N is divisible*

and

(4) *d is finite.*

When this is so, and when $G > K$, the terminal is $\omega + d$ if d is positive; if d is zero the terminal is ω if $L = N$ and $\omega + 1$ if not.

Of course when $G = K$ then G/G' is periodic and divisible and the terminal is finite.

Inasmuch as L occurs in the description when d is zero, the theorem cannot be said to describe the terminal group theoretically. However, it may well be that L and M coincide for all groups, although we have been able neither to prove nor disprove this. If the so-called dimension subgroup conjecture: that $(1 + \mathfrak{g}^n) \cap G$ coincides with $\gamma_n(G)$ for $n = 1, 2, 3, \ldots$: were true then there would be no doubt. On the other hand seemingly little can be deduced about dimension subgroups from knowing that $L = M$. Indeed the dimension subgroup question is effectively about finite prime power groups G and for these it is well-known (see, for example, [**2**]) that $\mathfrak{g}^\omega$ is trivial.

It is perhaps worth remarking that even periodic Abelian groups may fail to have terminals less than $\omega 2$. For, by the theorem, if G is any Abelian p-group in which $K = \bigcap (G^{p^n}: n \geqq 1)$ has exponent p, then G has terminal at least $\omega 2$. Probably the simplest example of such a group G is the Abelian p-group generated by elements $x_1, x_2, \ldots,$ subject only to the relations $x_n^{p^n} = 1,\ x_{n+1}^{p^n} = x_1,\ n = 1, 2, \ldots$: here K is generated by x_1.

1.3. Finite groups. Suppose G is finite. Evidently K coincides with M. Moreover, setting $M' = 1$, $M_p = [M_p, G_{p'}]$ for all primes p so that M coincides with N. Therefore (3) is satisfied; since N/O is finite, so also is (4). As a special case of the theorem, therefore, there is the first

COROLLARY. *If G is finite and $G > G'$ then the terminal of G is $\omega + d$.*

This result allows us to construct, for any given ordinal λ with $\omega \leqq \lambda < \omega 2$, a finite group whose terminal is λ. Indeed if λ is ω then any non-trivial finite nilpotent group will do.

Suppose $\lambda = \omega + d$ with $d > 0$. Let p be an odd prime greater than d and suppose that M is a vector space with basis $m_1, \ldots, m_d$ over the field with p elements. There is an automorphism a of M defined by

$$m_i^{\,a} = m_{i+1} - m_i, \quad (1 \leqq i < d)$$
$$m_d^{\,a} = -m_d,$$

so that we may form the natural split extension $G = M \rtimes \langle a \rangle$ of M by $\langle a \rangle$. It is easy to see that a^p is multiplication by -1. Therefore $M(a^p - 1) = M$. It follows that M is the limit of the lower central series of G. The order of G is $2p^{d+1}$, so that O is trivial. The terminal of G is therefore precisely $\omega + d(M, G_p)$.

Now $G_p = \langle M, a^2 \rangle$ so that $d(M, G_p)$ is the first integer n for which $(a^2 - 1)^n = 0$. Since $a - 1$ is an automorphism of M, this is the first n such that $(a + 1)^n = 0$. It follows that $d(M, G_p) = d$ and thence that the terminal of G is λ.

1.4. Groups with Min-n. Suppose now that G is a group with Min-n, the minimal condition for normal subgroups. The subgroup K must be of finite index in G and consequently G/K is nilpotent. It follows that G/K' is a soluble group with Min-n. From a theorem of R. Baer [**1**; p. 389] we deduce that G/K' is locally finite.

In showing that (3) and (4) hold we may assume that $K' = 1$. Since $[N_p, {}_nG_p]$ is a normal subgroup of G for $n = 1, 2, \ldots$, it is clear that d_p is finite. Since G has Min-n, K can have only finitely many primary components. Therefore d is finite and (4) holds. To see that (3) holds, we may assume into the bargain that N is trivial. From this it follows easily that any finite homomorphic image of G is nilpotent. Hence K is the unique minimal normal subgroup of finite index in G. Because K is Abelian it must be divisible.

This shows that groups with Min-n have terminals less than $\omega 2$. We shall prove in § 3.5 that in such groups L is the same as M. Hence we may state the second

COROLLARY. *If G has* Min-n *and $G > K$ then the terminal of G is $\omega + d$ if $d \geqq 1$; if $d = 0$ the terminal is ω if $M = N$ and $\omega + 1$ if not.*

That both possibilities when $d = 0$ can occur is easily seen. First, if G is the direct product of an Abelian p^∞-group C and a cyclic group D of order p then $M = N = 1$, so that the terminal is ω. Contrariwise we may take G to be the wreath product $C \wr D$. Here $M = G' > N = 1$, so that the terminal is $\omega + 1$.

1.5. Lower central depths. The G-depth of a $\mathbf{Z}G$-module A is plainly related to the lower central depths of the extensions of A by G. For example, if E is the split extension then its lower central depth is precisely the maximum of the lower central depth of G and the G-depth of A. In particular, therefore, if G is a group as in Theorem 1 and A is a free module then the lower central depth of E is less than $\omega 2$. This is a consequence of the theorem together with the simple

LEMMA. *If G/K' is locally finite and K/N is divisible then G has lower central depth at most $\omega + 1$.*

Proof. As we have remarked, K' is contained in $[K, {}_nG]$ for all n. It is not difficult (cf. § 3.1) to see that $[K, G] \leqq M$, so that

$$K' \leqq [M, G, G] = \gamma_{\omega+2}(G).$$

Recalling that H was written for G/K' and that N/K' is the product $\prod_{p \neq q}[H_p, H_q]$, we can even go so far as to say that $N \leqq \gamma_{\omega+2}(G)$. For $[H_p, H_q] = [H_p, H_q, H]$ when p and q are different, so that N/K' is contained in the limit of the lower central series of H.

Now K/N is supposed divisible and therefore $N[K, G] = N[K, G, G]$. It follows that $[K, G] \leqq \gamma_{\omega+2}(G)$. However $M \leqq K$ and $\gamma_{\omega+1}(G) = [M, G]$ so that $\gamma_{\omega+1}(G) = \gamma_{\omega+2}(G)$.

That the bound $\omega + 1$ of the lemma cannot be improved is not hard to see. Let A_n be the Abelian p-group generated by elements $x_{1n}, x_{2n}, \ldots, x_{nn}$, each of order p^n. This group has an automorphism y_n such that $[x_{in}, y_n] = x_{i+1,n}$ $(1 \leqq i < n)$ and $[x_{nn}, y_n] = 1$. Let F_n be the split extension of A_n by $\langle y_n \rangle$ and F the direct product of the groups F_n, $n = 1, 2, 3, \ldots$. We write T for the subgroup of F generated by all the elements $x_{nn}{}^{-1}x_{n+1,n+1}{}^p$, $n = 1, 2, 3, \ldots$. The group $G = F/T$ has lower central depth precisely $\omega + 1$. Moreover, for this group G we have $K = M$ and this is an Abelian p^∞-group. The subgroup N is trivial.

2. Three Module Isomorphisms

2.1. Notation. Subsets of groups will be denoted with italic capitals. The corresponding small german letter will be used to denote the additive subgroup of the group ring generated by the elements of the set diminished by 1. For example, if X is a subgroup of G then $\mathfrak{x} = \langle x - 1; x \in X \rangle$, and this is clearly the augmentation ideal of $\mathbf{Z}X$. We shall use $\mathfrak{x}^-$ for the right ideal of $\mathbf{Z}G$ generated by $\mathfrak{x}$; thus $\mathfrak{x}^- = \mathfrak{x} + \mathfrak{x}\mathfrak{g}$. If X happens to be a normal subgroup of G then $\mathfrak{x}^-$ is a two-sided ideal of $\mathbf{Z}G$, to wit the kernel of the natural homomorphism of $\mathbf{Z}G$ onto $\mathbf{Z}(G/X)$. In particular, if X is normal in G then $\mathfrak{x} + \mathfrak{x}\mathfrak{g} = \mathfrak{x} + \mathfrak{g}\mathfrak{x}$.

When X is a normal subgroup of G we may view X/X' as a right G-module by $(xX')g = x^gX'$, and also as a left G-module by $g(xX') = x^{g^{-1}}X'$. If $Q = G/X$ then any Q-module may be considered as a G-module by means of the natural homomorphism of G onto Q. In particular $\mathbf{Z}Q$ is a two-sided G-module.

2.2. We prove first

LEMMA 1. *If $X \lhd G$ then the mapping*

$$(1 - x) + \mathfrak{x}\mathfrak{g} \mapsto xX' \quad (x \in X)$$

is an isomorphism of $\mathfrak{x}^-/\mathfrak{x}\mathfrak{g}$ onto X/X' as left G-modules.

Proof. Let T be a transversal to the cosets of X in G. The ideal $\mathfrak{x}^-$ is the direct sum $\bigoplus_{t \in T}\mathfrak{x}t$. The mapping of $\mathfrak{x}^-$ onto X/X' determined by

$$(1 - x)t \mapsto xX' \quad (x \in X, t \in T)$$

induces a homomorphism of $\mathfrak{x}^-/\mathfrak{x}\mathfrak{g}$ onto X/X', which has the obvious inverse

$$xX' \mapsto (1 - x) + \mathfrak{x}\mathfrak{g}.$$

Since

$$g(1 - x) + \mathfrak{x}\mathfrak{g} = g(1 - x)g^{-1} + \mathfrak{x}\mathfrak{g},$$

for g in G and x in X, the lemma follows.

As an obvious consequence we state the

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COROLLARY. *If* $X \lhd G$ *then* $\mathfrak{x}^- = \mathfrak{x}\mathfrak{g} + \mathfrak{g}\mathfrak{x}$ *if and only if* $X = [X, G]$.

The second useful result is

LEMMA 2. *If* $X \lhd G$ *and* $Q = G/X$ *then* $\mathfrak{x}^-/(\mathfrak{x}^-)^2$ *is isomorphic as a right G-module with* $X/X' \otimes \mathbf{Z}Q$.

Here G is supposed to act only on the right hand factor of the tensor product.

Proof. Again let T be a transversal to the cosets of X in G, so that $\mathfrak{x}^- = \bigoplus_{t\in T}\mathfrak{x}t$ and $(\mathfrak{x}^-)^2 = \bigoplus_{t\in T}\mathfrak{x}^2t$. By Lemma 1, $\mathfrak{x}/\mathfrak{x}^2$ is isomorphic to X/X' and so the mapping of $\mathfrak{x}^-$ onto $X/X' \otimes \mathbf{Z}Q$ determined by

$$(x - 1)t \mapsto xX' \otimes tX \quad (x \in X, t \in T)$$

induces an isomorphism of $\mathfrak{x}^-/(\mathfrak{x}^-)^2$ onto $X/X' \otimes ZQ$.

2.3. One last isomorphism of a more technical nature may just as well be stated here.

LEMMA 3. *Suppose G is the product of its normal subgroups U and V. If X is a normal subgroup of G contained in the centre of* $U \cap V$ *then* $\mathfrak{x}^-/(\mathfrak{u}\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{v})$ *and X are isomorphic as right U-modules.*

Proof. If $\mathfrak{x}^\sim = \mathfrak{x}U$, the right ideal of $\mathbf{Z}U$ generated by $\mathfrak{x}$, then by the right hand version of Lemma 1, $\mathfrak{x}^\sim/\mathfrak{u}\mathfrak{x}^\sim$ and X are isomorphic as right U-modules.

Let T be a transversal (containing 1) to the cosets of $U \cap V$ in V, and therefore to those of U in G. Then $\mathfrak{x}^- = \bigoplus_{t\in T}\mathfrak{x}^\sim t$ and there is a mapping ϕ of $\mathfrak{x}^-/\mathfrak{u}\mathfrak{x}^-$ into $\mathfrak{x}^\sim/\mathfrak{u}\mathfrak{x}^\sim$ determined by $\alpha t \mapsto \alpha$ for $\alpha \in \mathfrak{x}^\sim$ and $t \in T$. Now ϕ induces a mapping

$$\bar{\phi}: \mathfrak{x}^-/(\mathfrak{u}\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{v}) \to \mathfrak{x}^\sim/\mathfrak{u}\mathfrak{x}^\sim.$$

This is because $\mathfrak{x}^-\mathfrak{v} \leqq \mathfrak{u}\mathfrak{x}^- + \mathfrak{x}\mathfrak{t}$; for, if $x \in X$, $g \in G$, $v \in V$ and $g = ut$ with $u \in U$, $t \in T$ while $gv = uwt'$ with $w \in U \cap V$ and $t' \in T$, then

$$(x - 1)g(v - 1) \equiv (x^u - 1)(t' - t) \mod \mathfrak{u}\mathfrak{x}^-$$

follows from the hypothesis that $[X, U \cap V] = 1$. Clearly $\bar{\phi}$ has the inverse determined by the inclusion $\mathfrak{x}^\sim \to \mathfrak{x}^-$ and this is a U-homomorphism. Therefore so is $\bar{\phi}$.

3. $\mathfrak{g}^\omega$ in Periodic Groups

3.1. Let p be a prime. We shall write $G_{\omega p}$ for the set of all the "generalized p-elements" of Mal'cev [**5**, § 2]. Thus x is in $G_{\omega p}$ if and only if for every $n \geqq 1$ the element $\gamma_n(G)x$ of $G/\gamma_n(G)$ has finite p-power order. Of course $G_{\omega p}$ is a normal subgroup of G containing $M = \gamma_\omega(G)$.

We shall write $G(p)$ for the intersection of all the normal subgroups X of G such that G/X is a nilpotent p-group of finite exponent. Thus

$$G(p) = \bigcap_{m,n} G^{p^m}\gamma_n(G)$$

and, evidently (cf. § 1.2),

$$K = \bigcap_{q} G(q).$$

We note the following slight extension of Lemma 1 in Mal'cev's paper [**5**]. The proof is by an obvious induction on n.

LEMMA 4. *Let G be a group, V a normal subgroup and v an element of V. If $[v^m, G] = 1$, then $[v, G^{m^{n-1}}] \leqq [V, {}_nG]$ for all $n \geqq 1$.*

COROLLARY 1 [**5**, Theorem 1]. *If G is residually nilpotent, then $[G_{\omega p}, G(p)] = 1$ for all p.*

Proof. If v is in $G_{\omega p}$ and $v^{p^{r(n)}}$ is in $\gamma_n(G)$ then the lemma, with $V = G$, yields $[v, G^{p^s}] \leqq \gamma_n(G)$, where $s = r(n)(n-2)$. Hence $[v, G(p)] \leqq \gamma_n(G)$, and this is true for all $n \geqq 1$.

COROLLARY 2. *If K/K' is periodic then $K' \leqq [K, {}_nG]$ for all $n \geqq 1$.*

Proof. Taking $V = K$ in the lemma and working modulo $[K', G]$, we have

$$[v, G^{m^{n-1}}] \leqq [K', G][K, {}_nG]$$

for all v in K and $n \geqq 1$. But

$$[K, \gamma_n(G)] \leqq [K, {}_nG],$$

and so

$$[v, K] \leqq [v, \gamma_n(G)G^{m^{n-1}}] \leqq [K', G][K, {}_nG].$$

Thus

$$K' \leqq [K', G][K, {}_nG],$$

from which the result follows.

An immediate consequence of Corollaries 1 and 2 is

COROLLARY 3. *If G/K' is periodic then $K' \leqq M \leqq K$ and $[K, G] \leqq M$.*

3.2. We set

$$\Omega_G = \sum \mathfrak{g}_{\omega p_1} \cdots \mathfrak{g}^{-}_{\omega p_r},$$

the sum being taken over all $r \geqq 2$ and all r-tuples $p_1, \ldots, p_r$ of distinct primes. We note that if $G = \prod_p G_{\omega p}$ (which happens, for instance, if G/M is periodic) then $\mathfrak{g}^{-}_{\omega p_r}$ can be replaced by $\mathfrak{g}_{\omega p_r}$ in the definition of Ω_G.

The main result in this section is

THEOREM 2. *If G/K' is periodic then*

$$\mathfrak{k}\mathfrak{g} + \Omega_G \leqq \mathfrak{g}^{\omega} \leqq \mathfrak{k}^{-} + \Omega_G.$$

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By Lemma 1, every element α of $\mathfrak{k}^- + \Omega_G$ has the form $\beta + (x - 1)$ for some β in $\mathfrak{k}\mathfrak{g} + \Omega_G$ and x in K. If α is in $\mathfrak{g}^\omega$ the theorem shows that $x - 1$ will be in $\mathfrak{g}^\omega$ and conversely. Hence there is the

COROLLARY. *If G/K' is periodic then* $\mathfrak{g}^\omega = \mathfrak{k}\mathfrak{g} + \Omega_G + \mathfrak{l}$.

Again from Lemma 1, or directly, if X and Y are subgroups of G and $Z = [X, Y]$ then $\mathfrak{z}^- \leqq \mathfrak{x}^-\mathfrak{y}^- + \mathfrak{y}^-\mathfrak{x}^-$. It follows quickly by induction that if $T_n = \gamma_n(G)$, then $\mathfrak{t}_n \leqq \mathfrak{g}^n$. We deduce that $\mathfrak{m}^- \leqq \mathfrak{g}^\omega$. Thus Theorem 2 is really a result about periodic residually nilpotent groups. We observe that if G is periodic and residually nilpotent then $G_{\omega p} = G_p$ for all p.

3.3. For groups X and Y we shall write $X \otimes Y$ for the tensor product of the Abelian groups X/X' and Y/Y'. As in [**7**] we say that X and Y are *orthogonal* if and only if $X \otimes Y$ is trivial.

Suppose $X_1, \ldots, X_n$ are finitely many subgroups of a group G and that $P_i \lhd X_i$ $(1 \leqq i \leqq n)$. Let

$$\Phi_i = \mathfrak{x}_1\mathfrak{x}_2 \ldots \mathfrak{x}_{i-1}(\mathfrak{x}_i^2 + \mathfrak{p}_i)\mathfrak{x}_{i+1} \ldots \mathfrak{x}_n,$$

and $\Phi = \sum_{i=1}^n \Phi_i$. We need the simple

LEMMA 5. $\mathfrak{x}_1 \ldots \mathfrak{x}_n/\Phi$ *is a homomorphic image of* $X_1/P_1 \otimes \ldots \otimes X_n/P_n$.

Proof. The mapping

$$(x_1, x_2, \ldots, x_n) \mapsto \Phi + (x_1 - 1) \ldots (x_n - 1), \quad (x_i \in X_i, 1 \leqq i \leqq n)$$

is homomorphic in each component and therefore there is a homomorphism

$$X_1 \otimes \ldots \otimes X_n \to \mathfrak{x}_1 \ldots \mathfrak{x}_n/\Phi$$

in which

$$x_1 \otimes \ldots \otimes x_n \mapsto \Phi + (x_1 - 1) \ldots (x_n - 1).$$

If x_i is in P_i then $(x_1 - 1) \ldots (x_n - 1)$ is in Φ, so that the result follows.

We shall use this lemma only for rather special choices of $X_1, \ldots, X_n$. As a first example, suppose X and Y are orthogonal subgroups of G. With $n = i + j$, $P_1, \ldots, P_n$ trivial and

$$X_1 = X_2 = \ldots = X_i = X, X_{i+1} = X_{i+2} = \ldots = X_n = Y,$$

we deduce that $\mathfrak{x}^i\mathfrak{y}^j = \mathfrak{x}^{i+1}\mathfrak{y}^j + \mathfrak{x}^i\mathfrak{y}^{j+1}$. Hence we have the

COROLLARY. *If X and Y are orthogonal subgroups of G then*

$$\mathfrak{x}\mathfrak{y} = \mathfrak{x}^n\mathfrak{y} + \mathfrak{x}\mathfrak{y}^n \quad (n \geqq 1)$$

and (therefore) $\mathfrak{x}\mathfrak{y} \leqq \mathfrak{g}^\omega$.

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LEMMA 6. *If G is the direct product of its subgroups X_i $(i \in I)$, and if X_i and X_j are orthogonal whenever $i \neq j$, then for all ordinals* λ

$$\mathfrak{g}^\lambda = \bigoplus_i \mathfrak{x}_i^\lambda \oplus \sum_{r \geqq 2} \mathfrak{x}_{i_1} \dots \mathfrak{x}_{i_r},$$

where $\sum$ is taken over all r-tuples $i_1, \dots, i_r$ of distinct suffixes.

Proof. Since X_i and X_j commute when $i \neq j$, the above corollary shows that $\mathfrak{x}_i\mathfrak{x}_j \leqq \mathfrak{x}_i\mathfrak{x}_j\mathfrak{g}$. Consequently

$$\sum \mathfrak{x}_{i_1} \dots \mathfrak{x}_{i_r} \leqq \mathfrak{g}^\lambda,$$

for all λ. But

$$\mathfrak{g} = \bigoplus_i \mathfrak{x}_i \oplus \sum \mathfrak{x}_{i_1} \dots \mathfrak{x}_{i_r},$$

so that the result follows.

If G is a finite nilpotent group then G_p is a finite p-group. As we remarked earlier, this implies $\mathfrak{g}_p^\omega = 0$. It follows from the lemma that $\mathfrak{g}^\omega = \mathfrak{g}^{\omega+1} = \Omega_G$. Therefore, if G is non-trivial, its terminal is ω.

Using the deeper theorem of Hartley [**4**] that $\mathfrak{g}^\omega = 0$ if $G(p)$ is trivial, a correspondingly deeper statement can be made. We write this as a

COROLLARY. *If G is periodic and if $G > K = 1$, then the terminal of G is ω and $\mathfrak{g}^\omega = \Omega_G$.*

Proof. Since G is residually nilpotent, it is the direct product of its subgroups G_p; and because K is trivial, $G_p(p)$ is trivial for each prime p. Hence $\mathfrak{g}_p^\omega = 0$ and the result follows from the lemma.

The corollary shows that if G is any group for which G/M is periodic then $\mathfrak{g}^\omega \leqq \mathfrak{k}^- + \Omega_G$. For then $G_{\omega p}/M = (G/M)_p$ and this maps onto $(G/K)_p$ under the natural homomorphism of G/M onto G/K. Hence $\Omega_{G/K} = (\Omega_G + \mathfrak{k}^-)/\mathfrak{k}^-$. By the corollary, $(\Omega_G + \mathfrak{k}^-)/\mathfrak{k}^-$ is precisely the ωth power of the augmentation ideal of G/K. This proves half of Theorem 2.

Suppose now that G is any group for which G/K' is periodic; we assert that $\Omega_G \leqq \mathfrak{g}^\omega$. For suppose that p and q are any distinct primes and write $A = G_{\omega p}$ and $B = G_{\omega q}$. By Corollary 2 to Lemma 4, $K' \leqq M$ and so A/M and B/M are orthogonal. Since $\mathfrak{m}^- \leqq \mathfrak{g}^\omega$, the corollary to Lemma 5 yields $\mathfrak{a}\mathfrak{b} \leqq \mathfrak{g}^\omega$. The inclusion $\Omega_G \leqq \mathfrak{g}^\omega$ now follows.

To complete the proof of Theorem 2 it remains only to show that $\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^\omega$.

3.4. To show that $\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^\omega$ we first observe

LEMMA 7. *For any group G,*

$$\mathfrak{k} \leqq \bigcap_{n,m} (\mathfrak{g}^n + m\mathfrak{g}).$$

Proof. For $n \geqq 1$ let $V_{n,m}$ be the normal subgroup $(1 + m\mathfrak{g} + \mathfrak{g}^n) \cap G$ of G. Then, clearly, $G/V_{n,m}$ is nilpotent of class at most $n - 1$. Let $G^* = G/V_{n,m}$. Now $G^*/G^{*\prime}$ is isomorphic with $\mathfrak{g}^*/\mathfrak{g}^{*2}$ which, being an image of $\mathfrak{g}/\mathfrak{g}^2 + m\mathfrak{g}$,

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has exponent dividing m. Now as Robinson [**6**] has remarked, $\gamma_i(G^*)/\gamma_{i+1}(G^*)$ is an image of

$$\underbrace{G^* \otimes \ldots \otimes G^*}_{i}.$$

Hence every lower central factor of G^* has exponent dividing m. Hence G^* has exponent dividing m^{n-1}. From the definition of K, it follows that $K \leqq V_{n,m}$. This gives the result.

As a matter of fact, if G is periodic, the intersection in Lemma 7 equals $\mathfrak{k}^- + \Omega_G$. This may be seen using Hartley's theorem. We make no use of this fact however so that the proof can be omitted.

It is another application of Lemma 5 which is now needed to finish the proof of Theorem 2. We state it in a slightly more general form than is strictly necessary for this purpose because we shall later need some consequences of it.

LEMMA 8. *Suppose* $P \lhd X \leqq G$. *If* X/P *has exponent dividing* m *then* $\mathfrak{x}\mathfrak{g}/(\mathfrak{x}\mathfrak{g}^{n+1} + \mathfrak{p}\mathfrak{g})$ *has exponent dividing* m^n *for all* $n \geqq 1$.

Proof. In Lemma 5 we take $X_2 = \ldots = X_{n+1} = G$, $P_2 = \ldots = P_{n+1} = 1$ and $n \geqq 1$. It follows that $\mathfrak{x}\mathfrak{g}^n/(\mathfrak{x}\mathfrak{g}^{n+1} + \mathfrak{p}\mathfrak{g}^n)$ is an image of

$$X/P \otimes \underbrace{G \otimes \ldots \otimes G}_{n}.$$

When X/P has exponent dividing m, so does this tensor product. We deduce that $(\mathfrak{x}\mathfrak{g}^n + \mathfrak{p}\mathfrak{g})/(\mathfrak{x}\mathfrak{g}^{n+1} + \mathfrak{p}\mathfrak{g})$ has exponent dividing m for each $n \geqq 1$. The lemma now follows.

Another way of putting Lemma 8 is to say that $\mathfrak{x}(\mathfrak{g}^{n+1} + m^n\mathfrak{g}) \leqq \mathfrak{x}\mathfrak{g}^{n+1} + \mathfrak{p}\mathfrak{g}$ when X/P has exponent m. From Lemma 7 we may deduce that

$$\mathfrak{x}\mathfrak{k} \leqq \mathfrak{x}\mathfrak{g}^n + \mathfrak{p}\mathfrak{g} \quad (n \geqq 1). \tag{5}$$

Now suppose that K/K' is periodic and set $P = K'$. Inclusion (5) holds for any X such that X/K' is finite. Since $\mathfrak{p} \leqq \mathfrak{k}^2$, we deduce that $\mathfrak{k}^2 \leqq \mathfrak{k}\mathfrak{g}^n + \mathfrak{k}^2\mathfrak{g}$ for all $n \geqq 1$. It follows, of course, that $\mathfrak{k}^2 \leqq \mathfrak{k}\mathfrak{g}^n$ for all $n \geqq 1$. Hence $\mathfrak{k}^2 \leqq \mathfrak{g}^\omega$.

The left hand version

$$\mathfrak{k}\mathfrak{x} \leqq \mathfrak{g}^n\mathfrak{x} + \mathfrak{g}\mathfrak{p} \quad (n \geqq 1)$$

of (5) will hold, similarly, when X/P is of finite exponent. Again with $P = K'$ we may deduce that

$$\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^{n+1} + \mathfrak{g}\mathfrak{k}^2$$

whenever G/K' is periodic. Since $\mathfrak{k}^2 \leqq \mathfrak{g}^\omega$ in these circumstances, it follows that $\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^\omega$.

This completes the proof of Theorem 2. For future reference however we state the

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COROLLARY. (i) *If* K/K' *is periodic then* $\mathfrak{k}^2 \leqq \mathfrak{k}\mathfrak{g}^n$ *for all* $n \geqq 1$.
(ii) *If* G/K' *is periodic then* $\mathfrak{k}^2 \leqq \mathfrak{g}^{\omega 2}$.

To see (ii), we merely observe that for all $n \geqq 0$,

$$\mathfrak{k}^2 \leqq \mathfrak{k}\mathfrak{g}^{n+1} \leqq \mathfrak{g}^{\omega}\mathfrak{g}^n,$$

by (i) and Theorem 2.

3.5. We pause briefly to prove

LEMMA 9. *If* G *has* Min-n *then* $L = M$.

We may assume that $M = 1$. Then G is actually nilpotent and locally finite. The groups $G(p)$ have finite index in G, so that there is a finite supplement to $G(p)$ in G. The lemma will therefore follow from the rather more general

LEMMA 10. *Suppose* G *is periodic and residually nilpotent. If* $G_p(p)$ *has a finite supplement in* G_p *for each prime* p, *then* $L = 1$.

Proof. Since G is the direct product of the subgroups G_p, it follows from Lemma 6 that $\mathfrak{g}^{\omega} \cap \mathfrak{g}_p = \mathfrak{g}_p^{\omega}$. Thus $L_p = L \cap G_p$. We may therefore assume that G is a p-group.

Let T be a transversal to the cosets of K (which is now $G(p)$) in G such that $F = \langle T \rangle$ is finite. The intersection $D = K \cap F$ has exponent p^r, say. Since D and $\mathfrak{d}/\mathfrak{d}^2$ are isomorphic, it follows that $p^r\mathfrak{d} \leqq \mathfrak{d}^2$. Now, F being a finite p-group, $\mathfrak{f}/p^r\mathfrak{f}$ lies in the radical of the finite ring $(\mathbf{Z}/p^r\mathbf{Z})F$. Hence $\mathfrak{f}^s \leqq p^r\mathfrak{f}$ for some s. Since $F = DT$, we may write $\mathfrak{f} \leqq \mathfrak{d}^- + \mathfrak{t}$. It follows that $\mathfrak{f}^s \leqq (\mathfrak{d}^-)^2 + \mathfrak{t}$.

Since $\mathfrak{g} = \mathfrak{k}^- + \mathfrak{f}$ and K is central in G (by Corollary 3 to Lemma 4), $\mathfrak{g}^s \leqq \mathfrak{k}\mathfrak{g} + \mathfrak{t}$. Therefore $\mathfrak{k}^- \cap \mathfrak{g}^s \leqq \mathfrak{k}\mathfrak{g} + \mathfrak{t} \cap \mathfrak{k}^-$. But $\mathfrak{t} \cap \mathfrak{k}^- = 0$, and we already know that $\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^{\omega}$. Thus $\mathfrak{k}^- \cap \mathfrak{g}^s = \mathfrak{k}\mathfrak{g}$. Now G/K is a finite p-group so that $\mathfrak{g}^{\omega} \leqq \mathfrak{k}^-$. Consequently $\mathfrak{g}^{\omega} = \mathfrak{k}\mathfrak{g}$ and the result follows from Lemma 1 with K replacing X.

4. Proof of Theorem 1: Reductions

4.1. Suppose that G/K' is periodic. From the corollary to Lemma 8 and because $K' \leqq (1 + \mathfrak{k}^2) \cap G$, we deduce that the terminal of G is less than $\omega 2$ if and only if that of G/K' is less than $\omega 2$, and that when this happens the two terminals are equal. Accordingly *we shall assume throughout this section that* $K' = 1$ *and that* G *is locally finite.*

As a matter of fact, in proving Theorem 1 we may even assume, whenever it is convenient, that O is trivial. This follows from

LEMMA 11. $\mathfrak{o}^- \leqq \mathfrak{g}^{\omega 2}$.

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Proof. Suppose that p, q, r are distinct primes. From the corollary to Lemma 5, $\mathfrak{g}_q\mathfrak{g}_r \leqq \bigcap_n \mathfrak{g}_q\mathfrak{g}^n$. Hence $\mathfrak{g}_p\mathfrak{g}_q\mathfrak{g}_r \leqq \bigcap_n \mathfrak{g}_p\mathfrak{g}_q\mathfrak{g}^n$. But $\mathfrak{g}_p\mathfrak{g}_q \leqq \mathfrak{g}^\omega$, so that $\mathfrak{g}_p\mathfrak{g}_q\mathfrak{g}_r \leqq \mathfrak{g}^{\omega 2}$. Now O is the product of all subgroups of the form $[G_p, G_q, G_r]$, so that $\mathfrak{o}^-$ is contained in the sum of all ideals $\mathfrak{g}_p{}^-\mathfrak{g}_q{}^-\mathfrak{g}_r{}^-$. The result follows.

4.2. To prove the necessity of condition (3) in Theorem 1 (cf. Lemma 13 below) we first establish the elementary

LEMMA 12. *For every prime p, $N_p = [N_p, G_{p'}]$ and $G_p \cap G_{p'} \leqq N$.*

Proof. Suppose that F is a finite subgroup of G and that $X = \gamma_\omega(F)$. Because X is contained in K it is Abelian. For any prime q, therefore, $X_q = [X_q, F_{q'}]$. Hence $X \leqq N$. Since $F_p \cap F_{p'} \leqq X$, this shows that $F_p \cap F_{p'} \leqq N$. Any element of $G_p \cap G_{p'}$ is contained in $F_p \cap F_{p'}$ for some finite subgroup F of G. It follows that $G_p \cap G_{p'} \leqq N$.

Now if x is in N then x lies in $\prod_q[G_q, G_{q'}]$. There is therefore some finite subgroup F with x in $\prod_q[F_q, F_{q'}]$. Then x will lie in X. If x happens to be in N_p, then x will be in X_p and so in $[X_p, F_{p'}]$. Therefore $N_p \leqq [N_p, G_{p'}]$. Since the opposite inclusion is obvious, the lemma is proven.

LEMMA 13. *If G has terminal less than $\omega 2$ then K/N is divisible.*

Proof. Since $N \leqq M$ and $\mathfrak{m}^- \leqq \mathfrak{g}^\omega$, the terminal of G/N is also less than $\omega 2$. We therefore assume that N is trivial. From Lemma 12, G is the direct product of the subgroups G_p. It follows that K is the direct product of its subgroups $G_p(p)$. From Lemma 6, $\mathfrak{g}^\lambda$ equals $\oplus_p \mathfrak{g}_p{}^\lambda \oplus \sum_{r \geqq 2} \mathfrak{g}_{p1} \ldots \mathfrak{g}_{pr}$. Therefore each G_p has terminal less than $\omega 2$. Since K is divisible if and only if each $G_p(p)$ is, we may assume that G is a p-group.

If K is not divisible, there is a homomorphic image C of K of order p. Let $Q = G/K$. From the definition of K, it follows that Q is infinite. Let $B = C \otimes \mathbf{Z}Q$ be viewed as a G-module by the action of G on the right hand factor; thus B is isomorphic to the group algebra of Q over the field of p elements. Since Q is residually a nilpotent p-group of finite exponent, we deduce from [**5**, Theorem 2] (cf. also [**4**]), that $\bigcap_n B\mathfrak{g}^n$ is trivial. Since Q is infinite, G has no fixed points on B. Hence B has infinite G-depth.

Now $K \otimes \mathbf{Z}Q$ has B as an image and therefore it too has infinite G-depth. From Lemma 2, however, $K \otimes \mathbf{Z}Q$ is isomorphic with $\mathfrak{k}^-/(\mathfrak{k}^-)^2$. Therefore $\mathfrak{k}^-$ has infinite G-depth. However, Theorem 2 asserts that $\mathfrak{k}\mathfrak{g} \leqq \mathfrak{g}^\omega \leqq \mathfrak{k}^-$. Since G has terminal less than $\omega 2$, it follows that $\mathfrak{k}^-$ has finite G-depth. This contradiction proves the lemma.

4.3. In our present situation (i.e., $K' = 1$ and G locally finite) we can replace the ideal Ω_G appearing in Theorem 2 by another that is easier to handle. We define

$$\Lambda_G = \sum \mathfrak{g}_{p1} \ldots \mathfrak{g}_{pr},$$

where the sum is taken over all $r \geqq 2$ and r-tuples $p_1, \ldots, p_r$ of distinct primes. We always have $\Lambda_G \leqq \Omega_G$; and, since $G_{\omega p} = G_p M$, $\mathfrak{k}^- + \Lambda_G = \mathfrak{k}^- + \Omega_G$. Consequently Theorem 2 remains true here with Λ_G replacing Ω_G. In what follows we shall always use the theorem in this form.

For the remainder of the proof of Theorem 1 we are concerned, in view of Lemma 13, with groups in which K/N is divisible. We prove

LEMMA 14. *If K/N is divisible then*

$$\mathfrak{n}^- + \mathfrak{g}^{\omega+1} = \mathfrak{n}^- + \mathfrak{g}^{\omega+2}.$$

Proof. Since G is periodic it is orthogonal to K/N. It follows from Lemma 5 (with $n = 2$, $X_1 = K$, $P_1 = N$ and $X_2 = G$, $P_2 = 1$) that

$$\mathfrak{k}\mathfrak{g} = \mathfrak{k}\mathfrak{g}^2 + \mathfrak{n}\mathfrak{g}. \tag{6}$$

From Theorem 2,

$$\mathfrak{k}\mathfrak{g}^2 + \Lambda_G\mathfrak{g} \leqq \mathfrak{g}^{\omega+1} \leqq \mathfrak{k}\mathfrak{g} + \Lambda_G\mathfrak{g}.$$

Since $\mathfrak{n} \leqq \Lambda_G$, the left hand term here contains $\mathfrak{k}\mathfrak{g}^2 + \mathfrak{n}\mathfrak{g}$ which, from (6), equals $\mathfrak{k}\mathfrak{g}$. Therefore

$$\mathfrak{g}^{\omega+1} = (\mathfrak{k} + \Lambda_G)\mathfrak{g}. \tag{7}$$

Suppose that p and q are distinct primes. Since $[G_p, G_q] \leqq N$, it follows that

$$\mathfrak{n}^- + \mathfrak{g}_p\mathfrak{g}_q = \mathfrak{n}^- + \mathfrak{g}_q\mathfrak{g}_p.$$

Because G_p and G_q are orthogonal, we may deduce from the corollary to Lemma 5 that

$$\mathfrak{n}^- + \mathfrak{g}_p\mathfrak{g}_q \leqq \mathfrak{n}^- + \mathfrak{g}_p\mathfrak{g}_q\mathfrak{g}^n$$

for all n. Since Λ_G is generated as a right ideal by all such $\mathfrak{g}_p\mathfrak{g}_q$ and $\mathfrak{n}^- \leqq \Lambda_G$, we may write

$$\Lambda_G = \mathfrak{n}^- + \Lambda_G\mathfrak{g}^n \quad (n \geqq 0). \tag{8}$$

From (7),

$$\mathfrak{n}^- + \mathfrak{g}^{\omega+1} = \mathfrak{n}^- + \mathfrak{k}\mathfrak{g} + \Lambda_G\mathfrak{g}.$$

From (6) and (8), it follows that

$$\mathfrak{n}^- + \mathfrak{g}^{\omega+1} = \mathfrak{n}^- + \mathfrak{k}\mathfrak{g}^2 + \Lambda_G\mathfrak{g}^2.$$

From (7) again we obtain the required result.

4.4. Now suppose that K/N is divisible. From Lemma 14, $\mathfrak{g}^{\omega+n} = \mathfrak{g}^{\omega+n+1}$ for $n \geqq 1$ if and only if $\mathfrak{n}^- \cap \mathfrak{g}^{\omega+n} = \mathfrak{n}^- \cap \mathfrak{g}^{\omega+n+1}$. From (7), therefore, $\mathfrak{g}^{\omega+n} = \mathfrak{g}^{\omega+n+1}$ for $n \geqq 1$ is equivalent with

$$\mathfrak{n}^- \cap (\mathfrak{k} + \Lambda_G)\mathfrak{g}^n = \mathfrak{n}^- \cap (\mathfrak{k} + \Lambda_G)\mathfrak{g}^{n+1}.$$

Now in the final section we shall prove

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LEMMA 15. *If* $O = 1$, *then for* $n \geqq 0$
(i) $\mathfrak{k}^- \cap (\Lambda_G \mathfrak{g}^n) \leqq \mathfrak{k}^- \mathfrak{g}^n + \Lambda_G \mathfrak{n}^-$, *and*
(ii) $\mathfrak{n}^- \cap (\mathfrak{k}^- + \Lambda_G)\mathfrak{g}^n = \Lambda_G \mathfrak{n}^- + \mathfrak{n}^- \mathfrak{g}^n$.

It follows from (ii) that $\mathfrak{g}^{\omega+n} = \mathfrak{g}^{\omega+n+1}$ for $n \geqq 1$ is equivalent with

$$\Lambda_G \mathfrak{n}^- + \mathfrak{n}^- \mathfrak{g}^n = \Lambda_G \mathfrak{n}^- + \mathfrak{n}^- \mathfrak{g}^{n+1}.$$

Thus it is $d(\mathfrak{n}^-/\Lambda_G \mathfrak{n}^-, G)$ which is significant.

LEMMA 16. *If* $O = 1$, *then* $d = d(\mathfrak{n}^-/\Lambda_G \mathfrak{n}^-, G)$.

Here d is the number defined in equation (2) in § 1.2.

We shall prove Lemma 16 in the next section. The qualitative aspect of Theorem 1 is, of course, a direct consequence of it. As for the quantitative part, we may assume that $G > K$ and that d is finite. It is clear that the terminal of G will be $\omega + d$ provided that d is positive and $\mathfrak{g}^\omega > \mathfrak{g}^{\omega+1}$. We need only discuss the exceptional cases.

If $\mathfrak{g}^\omega = \mathfrak{g}^{\omega+1}$ then $\mathfrak{n}^- \leqq \mathfrak{g}^{\omega+1}$; from (7) it follows that $\mathfrak{n}^- = \mathfrak{n}^- \cap (\mathfrak{k} + \Lambda_G)\mathfrak{g}$. From Lemma 15 (ii) we deduce that $\mathfrak{n}^- = \Lambda_G \mathfrak{n}^- + \mathfrak{n}^- \mathfrak{g}$ and from Lemma 16 that $d = 0$. Now $\mathfrak{l}$ must also be contained in $\mathfrak{g}^{\omega+1}$. Therefore

$$\mathfrak{l} \leqq (\mathfrak{k} + \Lambda_G)\mathfrak{g} \cap \mathfrak{k}^- = \mathfrak{k}\mathfrak{g} + (\Lambda_G \mathfrak{g}) \cap \mathfrak{k}^-.$$

From Lemma 15 (i) it follows that $\mathfrak{l} \leqq \mathfrak{k}\mathfrak{g} + \Lambda_G \mathfrak{n}^-$. Hence $\mathfrak{l} \leqq \mathfrak{k}\mathfrak{g} + \mathfrak{n}$. From Lemma 1, therefore, $L = N$.

Conversely, if $d = 0$, then Lemma 16 gives $\mathfrak{n}^- = \Lambda_G \mathfrak{n}^- + \mathfrak{n}^- \mathfrak{g}^2$. From Lemma 15 (ii) we deduce that $\mathfrak{n}^- \leqq (\mathfrak{k} + \Lambda_G)\mathfrak{g}^2$ and from (7) that $\mathfrak{n}^- \leqq \mathfrak{g}^{\omega+2}$. Lemma 14 now shows that $\mathfrak{g}^{\omega+1} = \mathfrak{g}^{\omega+2}$. If it happens that $L = N$, then from the corollary to Theorem 2,

$$\mathfrak{g}^\omega = \mathfrak{k}\mathfrak{g} + \Lambda_G + \mathfrak{n} = \mathfrak{k}\mathfrak{g} + \Lambda_G.$$

From (8), this gives $\mathfrak{g}^\omega = \mathfrak{k}\mathfrak{g} + \mathfrak{n}^- + \Lambda_G \mathfrak{g}$. Hence $\mathfrak{g}^\omega = \mathfrak{g}^{\omega+1}$, as required.

We have reduced the proof of Theorem 1 to proving Lemmas 15 and 16.

5. Proof of Theorem 1: Conclusion

Throughout this section we assume that O is trivial and that G is locally finite.

5.1. Proof of Lemma 16. Since $\mathfrak{n}^- \leqq \Lambda_G$, the module in question is an image of $\mathfrak{n}^-/(\mathfrak{n}^-)^2$. By Lemma 2 and because N is Abelian, this is isomorphic with $N \otimes \mathbf{Z}(G/N)$. Hence

$$\mathfrak{n}^-/(\mathfrak{n}^-)^2 = \bigoplus_p ((\mathfrak{n}^-)^2 + \mathfrak{n}_p^-)/(\mathfrak{n}^-)^2.$$

The submodule $\Lambda_G \mathfrak{n}^-/\mathfrak{n}^{-2}$ therefore splits up into the direct sum

$$\bigoplus_p ((\mathfrak{n}^-)^2 + \Lambda_G \mathfrak{n}_p^-)/(\mathfrak{n}^-)^2.$$

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From this we deduce that the G-depth of $\mathfrak{n}^-/\Lambda_G\mathfrak{n}^-$ is the maximum of the G-depths of the components $((\mathfrak{n}^-)^2 + \mathfrak{n}_p)/((\mathfrak{n}^-)^2 + \Lambda_G\mathfrak{n}_p^-)$. Again from Lemma 2, $\mathfrak{n}_p^- \cap (\mathfrak{n}^-)^2 = \mathfrak{n}_p^-\mathfrak{n}^-$. Therefore

$$((\mathfrak{n}^-)^2 + \mathfrak{n}_p^-)/((\mathfrak{n}^-)^2 + \Lambda_G\mathfrak{n}_p^-) \cong \mathfrak{n}_p^-/(\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{n}^-).$$

Accordingly, to prove Lemma 16, we should show for each prime p that

$$d_p = d(\mathfrak{n}_p^-/(\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{n}^-), G). \tag{9}$$

From the corollary to Lemma 5, $\mathfrak{n}_p\mathfrak{g}_{p'} \leqq \bigcap_n \mathfrak{n}_p\mathfrak{g}^n$. Therefore

$$\mathfrak{n}_p^-/(\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{n}^-)$$

has the same G-depth as $\mathfrak{n}_p^-/(\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{n}^- + \mathfrak{n}_p^-\mathfrak{g}_{p'})$. Now $\mathfrak{n}_p\mathfrak{n}_q \leqq \mathfrak{n}_p\mathfrak{g}_{p'}$, if $p \neq q$, and $\mathfrak{n}_p^2 \leqq \Lambda_G\mathfrak{n}_p^-$ because $\mathfrak{n}^- \leqq \Lambda_G$. Hence $\mathfrak{n}_p^-\mathfrak{n}^- \leqq \Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{g}_{p'}$. For (9), therefore, it will be good enough to prove

$$d_p = d(\mathfrak{n}_p^-/(\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{g}_{p'}), G_p). \tag{10}$$

From Lemma 12, $N_p = [N_p, G_{p'}] \leqq G_{p'}$. From the corollary to Lemma 1 we deduce that $\mathfrak{n}_p \leqq \mathfrak{n}_p\mathfrak{g}_{p'} + \mathfrak{g}_{p'}\mathfrak{n}_p$. Therefore

$$\mathfrak{n}_p^-\mathfrak{g}_{p'} + \mathfrak{g}_p\mathfrak{n}_p^- = \mathfrak{n}_p^-\mathfrak{g}_{p'} + \mathfrak{g}_p\mathfrak{g}_{p'}\mathfrak{n}_p^-. \tag{11}$$

Further, from Lemma 1 itself with $N_pG_{p'}$ replacing G and N_p replacing X, it follows that $(\mathfrak{n}_p + \mathfrak{n}_p\mathfrak{g}_{p'})/\mathfrak{n}_p\mathfrak{g}_{p'}$ is isomorphic with N_p as a left $G_{p'}$-module. For different primes q and r, both distinct from p, $[G_p, G_q, G_r]$ is trivial. Therefore $[N_p, G_q, G_r] = 1$ and we deduce

$$\mathfrak{g}_q\mathfrak{g}_r\mathfrak{n}_p \leqq \mathfrak{n}_p\mathfrak{g}_{p'}. \tag{12}$$

Now $\Lambda_G\mathfrak{n}_p^-$ is the right ideal generated by all $\mathfrak{g}_q\mathfrak{g}_r\mathfrak{n}_p$ with $q \neq r$. From (11) and (12) it follows that

$$\Lambda_G\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{g}_{p'} = \mathfrak{g}_p\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{g}_{p'}. \tag{13}$$

From Lemma 12, $G_p \cap G_{p'} \leqq N$, so that N_p is certainly in the centre of $G_p \cap G_{p'}$. Moreover $G = G_pG_{p'}$ because G is periodic. We may therefore appeal to Lemma 3 with N for X, G_p for U and $G_{p'}$ for V, to deduce that

$$\mathfrak{n}_p^-/(\mathfrak{g}_p\mathfrak{n}_p^- + \mathfrak{n}_p^-\mathfrak{g}_{p'}) \text{ and } N_p$$

are isomorphic as right G_p-modules. Together with (13) and the definition (1) of d_p, this gives (10). The proof of Lemma 16 is thereby completed.

5.2. Proof of Lemma 15. Since $K' = 1$, the limit X of the lower central series of any finite subgroup F of G must be Abelian. According to a theorem of Schenkman [**8**], this has as a consequence that there is a complement to X in F. We need a result about such finite groups.

LEMMA 17. *Suppose F is finite and $X = \gamma_\omega(F)$ is Abelian. Suppose that C is a complement to X in F. If $X \leqq Y \leqq F$ then*

(i) $\mathfrak{y}^-\mathfrak{f}^n \cap \mathfrak{x}^- \leqq \mathfrak{y}^-\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{f}^n$,
(ii) $\mathfrak{c}^\omega \cap \mathfrak{y}^- \leqq (\mathfrak{y}^-)^2 + \sum_p \mathfrak{y}_p\mathfrak{f}_{p'}$,
(iii) $\mathfrak{f}^{\omega+n} = \mathfrak{x}^-\mathfrak{f}^n + \mathfrak{c}^\omega\mathfrak{x}^- + \mathfrak{c}^\omega$,
(iv) $\mathfrak{f}^{\omega+n} \cap \mathfrak{y}^- \leqq (\mathfrak{y}^-)^2 + \mathfrak{x}^-\mathfrak{f}^n + \Lambda_F\mathfrak{x}^- + \sum_p \mathfrak{y}_p\mathfrak{f}_{p'}$
for all $n \geqq 0$.

Here, of course, $\mathfrak{x}^-$ and $\mathfrak{y}^-$ refer to the ideals of $\mathbf{Z}F$ generated by $\mathfrak{x}$ and $\mathfrak{y}$.

Proof. Since $\mathfrak{x}^- \leqq \mathfrak{f}^\omega$ and $\mathfrak{f} = \mathfrak{x}^- + \mathfrak{c}$ it follows that $\mathfrak{f}^n = \mathfrak{x}^- + \mathfrak{c}^n$. Writing $Z = Y \cap C$ we have $Y = XZ$, so that $\mathfrak{y} \leqq \mathfrak{x}^- + \mathfrak{z}$. Hence,

$$\mathfrak{y}\mathfrak{f}^n \leqq (\mathfrak{x}^- + \mathfrak{z})(\mathfrak{x}^- + \mathfrak{c}^n) \leqq (\mathfrak{x}^-)^2 + \mathfrak{z}\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{c}^n + \mathfrak{z}\mathfrak{c}^n \leqq \mathfrak{z}^-\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{f}^n + \mathfrak{c},$$

Since $\mathfrak{x}^- \cap \mathfrak{c} = 0$, this gives (i) if n is positive. For $n = 0$, (i) is obvious, so that we may proceed to (ii).

Let T_p be a transversal to the cosets of Z_p in C_p and write T for the product $\prod_p T_p$. Because C is nilpotent, this set T is a transversal to the cosets of Z in C and moreover

$$\prod_p \mathfrak{t}_p \leqq \mathfrak{t}. \tag{14}$$

From the corollary to Lemma 6, $\mathfrak{c}^\omega$ equals $\sum \mathfrak{c}_{p_1}\mathfrak{c}_{p_1} \ldots \mathfrak{c}_{p_r}$. Now $C_p = Z_pT_p$, so that $\mathfrak{c}_p = \mathfrak{z}_p + \mathfrak{z}_p\mathfrak{t}_p + \mathfrak{t}_p$. On substituting this expression for $\mathfrak{c}_p$ in $\mathfrak{c}^\omega$ we obtain

$$\mathfrak{c}^\omega \leqq (\mathfrak{z}^-)^2 + \sum_p \mathfrak{z}_p^-\mathfrak{c}_{p'} + \prod_p \mathfrak{t}_p. \tag{15}$$

Inclusions (14) and (15), together with $\mathfrak{y}^- \cap \mathfrak{t} = 0$, now show that (ii) is true.

Now $\mathfrak{f}^\omega = \mathfrak{x}^- + \mathfrak{c}^\omega$. Moreover

$$(\mathfrak{c}^\omega\mathfrak{x}^- + \mathfrak{c}^\omega)\mathfrak{f} = (\mathfrak{c}^\omega\mathfrak{x}^- + \mathfrak{c}^\omega)(\mathfrak{x}^- + \mathfrak{c}) = \mathfrak{c}^\omega\mathfrak{x}^- + \mathfrak{c}^\omega,$$

since, by the corollary to Lemma 6, C has terminal ω. Now (iii) follows by induction on n. Finally (iv) follows from (ii) and (iii) and the fact that $\mathfrak{c}^\omega = \Lambda_C \leqq \Lambda_F$.

With the help of Lemma 17 we may prove Lemma 15. Suppose first that ξ is in $\mathfrak{k}^- \cap (\Lambda_G\mathfrak{g}^n)$. There is some finitely generated subgroup F_1 of G such that ξ is in $\Lambda_{F_1}\mathfrak{f}_1{}^n$. Since $\Lambda_{F_1} \leqq \mathfrak{f}_1{}^\omega$, this shows that ξ is in $\mathfrak{f}_1{}^{\omega+n}$. There is some finitely generated subgroup F_2 of G such that if P is $K \cap F_2$ then ξ belongs to $\mathfrak{p} + \mathfrak{p}\mathfrak{f}_2$. Let $F = \langle F_1, F_2\rangle$ and $Y = K \cap F$. Since G is locally finite, F is finite. Let $X = \gamma_\omega(F)$. From part (iv) of Lemma 17 it follows that ξ is in

$$(\mathfrak{y}^-)^2 + \mathfrak{x}^-\mathfrak{f}^n + \Lambda_F\mathfrak{x}^- + \sum_p \mathfrak{y}_p\mathfrak{f}_{p'}.$$

Now X is contained in N, as we remarked in the proof of Lemma 12, and Y is contained in K. From the corollary to Lemma 8, $(\mathfrak{k}^-)^2 \leqq \mathfrak{k}^-\mathfrak{g}^n$ and from the corollary to Lemma 5, $\mathfrak{k}_p\mathfrak{g}_{p'} \leqq \mathfrak{k}_p^-\mathfrak{g}^n$. Therefore ξ is in $\mathfrak{k}^-\mathfrak{g}^n + \Lambda_G\mathfrak{n}^-$. This proves part (i) of Lemma 15.

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To prove part (ii) we notice first that $\mathfrak{n}^- \leqq \mathfrak{k}^-$, so that

$$\mathfrak{n}^- \cap (\mathfrak{k}^- + \Lambda_G)\mathfrak{g}^n = \mathfrak{n}^- \cap \{\mathfrak{k}^- \cap (\mathfrak{k}^-\mathfrak{g}^n + \Lambda_G\mathfrak{g}^n)\} =$$
$$\mathfrak{n}^- \cap \{\mathfrak{k}^-\mathfrak{g}^n + (\mathfrak{k}^- \cap \Lambda_G\mathfrak{g}^n)\} \leqq \mathfrak{n}^- \cap (\mathfrak{k}^-\mathfrak{g}^n + \Lambda_G\mathfrak{n}^-),$$

by part (i). Hence

$$\mathfrak{n}^- \cap (\mathfrak{k}^- + \Lambda_G)\mathfrak{g}^n \leqq \Lambda_G\mathfrak{n}^- + (\mathfrak{n}^- \cap \mathfrak{k}^-\mathfrak{g}^n). \tag{16}$$

We prove

$$\mathfrak{n}^- \cap \mathfrak{k}^-\mathfrak{g}^n \leqq \Lambda_G\mathfrak{n}^- + \mathfrak{n}^-\mathfrak{g}^n. \tag{17}$$

This will be sufficient, for (16) and (17) together give

$$\mathfrak{n}^- \cap (\mathfrak{k}^- + \Lambda_G)\mathfrak{g}^n \leqq \Lambda_G\mathfrak{n}^- + \mathfrak{n}^-\mathfrak{g}^n.$$

On the other hand, since $\mathfrak{n}^- \leqq \mathfrak{g}^\omega$, it is obvious that

$$\Lambda_G\mathfrak{n}^- + \mathfrak{n}^-\mathfrak{g}^n \leqq \mathfrak{n}^- \cap (\mathfrak{k}^- + \Lambda_G)\mathfrak{g}^n.$$

Suppose that η is in $\mathfrak{n}^- \cap \mathfrak{k}^-\mathfrak{g}^n$. As before, there is some finitely generated subgroup F_3 of G such that if $Q = K \cap F_3$ then η lies in $(\mathfrak{q} + \mathfrak{q}\mathfrak{f}_3)\mathfrak{f}_3{}^n$. Moreover, since N is the product of the subgroups $[G_p, G_{p'}]$, there is some finitely generated subgroup F_4 of G such that if

$$R = \prod_p \, [F_{4,p}, F_{4,p'}]$$

then η is in $\mathfrak{r} + \mathfrak{r}\mathfrak{f}_4$. Now write $F = \langle F_3, F_4\rangle$, $X = \gamma_\omega(F)$ and $Y = K \cap F$. It follows that η lies in $(\mathfrak{x} + \mathfrak{x}\mathfrak{f}) \cap (\mathfrak{y} + \mathfrak{y}\mathfrak{f})\mathfrak{f}^n$. From (i) of Lemma 17 we deduce that η lies in $\mathfrak{y}^-\mathfrak{x}^- + \mathfrak{x}^-\mathfrak{f}^n$. As before, this shows that η is in $\mathfrak{k}^-\mathfrak{n}^- + \mathfrak{n}^-\mathfrak{g}^n$.

However, K is Abelian so that if p and q are distinct primes then

$$\mathfrak{k}_p\mathfrak{n}_q = \mathfrak{n}_q\mathfrak{k}_p.$$

This lies in $\mathfrak{n}^-\mathfrak{g}^n$ by the corollary to Lemma 5. However

$$\mathfrak{k}_p\mathfrak{n}_p \leqq \mathfrak{g}_p\mathfrak{n}_p{}^- \leqq \mathfrak{n}_p{}^-\mathfrak{g}_{p'} + \Lambda_G\mathfrak{n}^-$$

by (11). It follows that η is in $\Lambda_G\mathfrak{n}^- + \mathfrak{n}^-\mathfrak{g}^n$. This then gives (17) and the proof of Lemma 15 is finished.

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