TETRADS OF MÖBIUS TETRAHEDRA

SAHIB RAM MANDAN

(received 17 October 1960, revised 15 May 1962)

A tetrad of Möbius tetrahedra consists of a set of 4 mutually inscribed and therefore circumscribed tetrahedra whose 16 vertices and 16 faces form a Kummer's 16_6 configuration (5; 11; 12; 21). As pointed out by the referee, fundamental to all work on the 16_6 figure are the 10 quadrics, called fundamental for the associated Kummer's quartic surface (13). To every quadric F correspond a matrix scheme of the 16 points or planes, arranged in 4 rows or columns, such that the 8 Rosenhain tetrahedra (7) formed of the rows and columns are all self-polar for F. The rows form one and the columns another tetrad of Möbius tetrahedra. Nine new schemes can be derived from one such scheme to make the total ten as explained by Baker (3, p. 133) leading to 80 Rosenhain tetrahedra in all. The 16 nodes (5; 8) or tropes of the Kummer's quartic are the 16 common elements of the 10 schemes such that the nodes and tropes are poles and polars for any one of the 10 quadrics. Each trope touches the quartic along a singular conic through the 6 points of the figure lying therein. The lines tangent to the surface at its each node N generate a quadric cone which is enveloped by the 6 tropes through N (12).

The present paper is one in continuation of the two appeared earlier (15; 16), and deals with the 12 nets of quadrics circumscribing the 12 pairs of Möbius tetrahedra arising from one of the 10 said schemes. There the treatment is synthetic, while here it is analytic based mostly on the symmetrical algebra of Edge (8) for the net of quadrics associated with one such pair of tetrahedra. Six special webs of quadrics arise from the 12 nets, each web having a pair of generators common with 4 fundamental quadrics. Quadrics associated with Göpel tetrahedra (12), as related to the said nets, are also considered. The derivation of an allied pair of conjugate triads of desmic tetrahedra (1; 4—7; 9; 10; 12—14; 17; 19—21; 25) from a tetrad of Möbius tetrahedra too is indicated.

An attempt is made to give afresh an account of relevant known results for ready reference and necessary development of the subject.

1. Introduction

a. It is well known (11) that there exist sets of 4 tetrahedra such that every two of a set are Möbius in the same sense and thus form a *Möbius* 8_4 *configuration* (7). That is, the 2 transversals of the 4 joins of the corresponding vertices of every two tetrahedra of a set belong to the same system of generators of a quadric F for which the 4 tetrahedra of the set are all self-polar.

Let the vertices or faces of the 4 tetrahedra of one such set or tetrad be arranged in 4 columns of the scheme (cf. 2. p. 138, Ex. 15; 3, p. 133)

$$M \equiv \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \\ t_0 & t_1 & t_2 & t_3 \end{bmatrix}.$$

The rows of M are also seen to form a similar tetrad, called *conjugate* of the former.

b. In one formulation the elements of M are the linear functions of the coordinates (which may be taken to represent either planes or points) obtained from the product UV of the orthogonal matrices

$$U \equiv \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & -a & -b \\ d & -c & b & -a \end{bmatrix} \quad \text{and} \quad V \equiv \begin{bmatrix} x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \\ t & -x & -y & -z \end{bmatrix}.$$

Since U, V are orthogonal, UV is orthogonal. Consequently we have (cf. 12, pp. 30—31):

$$\text{(1)} \qquad \sum_p r_p^2 \equiv \sum_r r_p^2 \equiv \sum u^2 = \sum r^2 = 1$$

$$\text{(ii)} \qquad \sum_p r_p s_p \equiv 0 \equiv \sum_r r_p r_q$$

$$(u = a,\ b,\ c,\ d;\ r,\ s = x,\ y,\ z,\ t;\ p,\ q = 0,\ 1,\ 2,\ 3).$$

The relation (i) shows that the pair of conjugate tetrads $T_p \equiv x_p y_p z_p t_p$, $T_r \equiv r_0 r_1 r_2 r_3$ represented by M constitute 8 *Rosenhain tetrahedra*, all self-polar for the quadric $F \equiv \sum r^2 = 0$, and (ii) shows the Möbius character of the pairs of tetrahedra of either tetrad.

It may be observed here that the tetrahedron T of reference is distinct from the Rosenhain tetrahedra (see § 1e below).

c. In a second formulation, $x_0 y_0 z_0 t_0$ is taken to be the tetrahedron of reference and the conjugate scheme $\bar{M}$, interchanging the rows and columns of M, is introduced such that its elements are obtained from the product of the matrices W and $\bar{V}$ (cf. 2, p. 138, Ex. 14; 3, p. 141, Ex. 9; 5), where

$$W \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{11} & l_{12} & l_{13} \\ 0 & l_{21} & l_{22} & l_{23} \\ 0 & l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad \bar{V} \equiv \begin{bmatrix} x & y & z & t \\ t & z & -y & -x \\ -z & t & x & -y \\ y & -x & t & -z \end{bmatrix}$$

such that the submatrix $L = (l_{ij})$ of the matrix W and therefore W too are orthogonal. Hence the elements of rows or columns of L may be taken as the direction cosines of 3 mutually perpendicular lines referred to a rectangular axes (22, pp. 10—11; 23, p. 31; 24, pp. 37—38).

The scheme $\bar{M}$ too behaves like M to represent a pair of conjugate tetrads of Möbius tetrahedra such that the tetrahedron $T = T_0$ of reference is now one of the 8 Rosenhain tetrahedra, and the relations (i), (ii) hold true here too, $\sum u^2$ being replaced by unity as the value of the product $|W||\overline{W}|$ of the determinant $|W|$ of W and that of its conjugate $\overline{W}$.

d. The 4 joins of the corresponding vertices of a pair of Möbius tetrahedra of either tetrad T_p or T_r and the 4 joins of the complementary pair have 2 common transversals (3, p. 133) which are generators of the quadric F (§ 1b) or 8 Kummer lines of a group-set cut a pair of directrices (12, p. 77). Thus there are 3 pairs of such generators of one system of F for T_p and 3 pairs of the second system for T_r such that every pair of the former system meet a pair of opposite edges of every tetrahedron of T_r, and every pair of the later system meet likewise a pair of opposite edges of every tetrahedron of T_p. The 3 pairs of either system are mutually harmonic (21). Every two tetrahedra of T_p are *harmonically inverse* (15; 16) of each other w.r.t. one of the said 3 pairs of generators of the first system, the other two tetrahedra being likewise harmonically inverse w.r.t. the same pair of generators, and every two as well as the other two tetrahedra of T_r behave similarly w.r.t. a pair of generators of the other system.

Every one of the 3 pairs of generators of one system form a skew quadrilateral with every one of the 3 pairs of the other system. Thus there are 9 such quadrilaterals in all such that the 9 pairs of their skew diagonals form the 18 edges of an allied pair of conjugate, associated or related triads of tetrahedra (1; 4, pp. 99—102; 6; 7; 9; 10; 12; 13; 14; 17; 19; 20; 25) forming desmic systems as analysed analytically by Baker (5) using the scheme $\bar{M}$ (§ 1c) and synthetically by Rao [21] using the scheme M (§ 1b). In fact, they have derived T_p and T_r from every one of the 6 tetrahedra of the 2 desmic systems and another tetrahedron, all self-polar for the quadric F, by the well known operations of harmonic inversions (10; 18) only. If $a, b; c, d$ be 2 pairs of mutually harmonic generators of one system of a quadric F, $a', b'; c', d'$ 2 similar pairs of the other system, we may define or introduce *conjugate generator quadrilaterals* (cgqs) like $ab'ba'$, $cd'dc'$ or $ac'bd'$, $a'cb'd$ to prove the following Lemma required to establish the said results of Baker and Rao.

LEMMA. *The 2 pairs of diagonals of a pair of cgqs formed of the generators of a quadric F meet at the vertices of a tetrahedron self-polar for F.*

e. The 9 skew quadrilaterals of the preceding section may be recognised to form 9 of the 15 *fundamental tetrahedra,* the other six being those arising from the intersections of their 18 diagonals forming the 18 edges of the 6 tetrahedra of the pair of allied desmic systems derived there, thus giving rise to *Klein's* 60_{15} *configuration* (12). Thus the tetrahedron T of reference is fundamental or Rosenhain according as we follow the scheme M (§ 1b) or $\bar{M}$ (§ 1c).

f. On account of the perfect duality between point and plane, every theorem has its dual. Hence it is not necessary to state the dual in every case. By denoting the point and plane by the same symbol in both the schemes M and $\bar{M}$, no confusion arises, but on the contrary, the duality of the 16_6 figure is clearly brought out by this nomenclature. Thus the 6 elements of one kind incident with any element of the other kind are given by the row and column containing that element (12). Attention must always be drawn to this dual nomenclature to give the right meaning to the statements made hereafter.

g. The 10 fundamental quadrics arising from the scheme (§ 1b) M as its 10 non-zero elements by putting $a = x$, $b = y$, $c = z$, $d = t$ are well known (12). The same may be designated (as suggested by the referee) as

$$F \equiv x^2+y^2+z^2+t^2,$$

$$\begin{array}{lll} F_{11} \equiv x^2-y^2-z^2+t^2, & F_{12} \equiv 2(xy+zt), & F_{13} \equiv 2(zx-yt), \\ F_{21} \equiv 2(xy-zt), & F_{22} \equiv -x^2+y^2-z^2+t^2, & F_{23} \equiv 2(yz+xt), \\ F_{31} \equiv 2(zx+yt), & F_{32} \equiv 2(yz-xt), & F_{33} \equiv -x^2-y^2+z^2+t^2. \end{array}$$

They have the properties, which can be easily verified, as follows: (i) Each pair is mutually *apolar* such that one is outpolar as well as inpolar to the other (1; 2). (ii) Each is self-reciprocal w.r.t. to every other. (iii) They can be arranged in 15 sets of four which have a pair of common generators, polars w.r.t. each of the other six, viz.

$$F,\ F_{i1},\ F_{i2},\ F_{i3};\ \ F,\ F_{1f},\ F_{2f},\ F_{3f};\ F_{if},\ F_{ig},\ F_{jh},\ F_{kh}\ \ (i, j, k, f, g, h = 1, 2, 3).$$

The 15 pairs of these generators form the 30 edges of the 15 fundamental tetrahedra which split in 10 ways into 2 sets of 9 and 6 as in § 1e, 9 inscribed to each quadric for which the other six are self-polar.

h. As a property of an orthogonal matrix M or $\bar{M}$ of order 4, we have 18 new identities like $x_0y_1 - x_1y_0 \equiv z_2t_3 - z_3t_2 \ldots$ (iii) besides the 12 enumerated already as the relations (ii). Every one of these 30 *bilinear relations* represents a set of 8 *associated points* (12, pp. 32, 77) or planes which form a Möbius 8_4 configuration (§ 1a) giving rise to 4 pairs of Möbius tetrahedra having the same vertices and faces (15) but belonging to 4 different schemes. The 18 new identities suggest the formation of the 9 new schemes to give us 72 new Rosenhain tetrahedra arising from the same 16_6 configuration.

2. Interlocking quadrics and associated nets

a. From the identities (ii), we can at once deduce the 6 triads of *interlocking quadrics* containing respectively the pairs of skew quadrilaterals $y_pz_py_qz_q$, $x_pt_px_qt_q$, $z_px_pz_qx_q$ $y_pt_py_qt_q$; $x_py_px_qy_q$, $z_pt_pz_qt_q$ and designate them as follows (cf. 8; 16):

$$\begin{aligned} Q_{pq} &\equiv y_py_q + z_pz_q \equiv -x_px_q - t_pt_q, \\ Q'_{pq} &\equiv z_pz_q + x_px_q \equiv -y_py_q - t_pt_q, \\ Q''_{pq} &\equiv x_px_q + y_py_q \equiv -z_pz_q - t_pt_q. \end{aligned}$$

A triad of such quadrics ($p, q = 0, j$ in the scheme $\bar{M}$) is taken by Edge [8] as the basis of the *net of quadrics* circumscribing a pair of Möbius tetrahedra T_0, T_j to deduce a family of Kummer surfaces, each being the envelope of a set of quadrics belonging to the net, and many more results. Thus there arise 6 such nets $n(pq)$ for a tetrad T_p of Möbius tetrahedra, one net for each pair of them or for each pair of values of p, q.

b. Similarly we may deduce the 6 triads of interlocking quadrics containing respectively the pairs of skew quadrilaterals

$$r_0r_1s_0s_1,\quad r_2r_3s_2s_3;\quad r_0r_2s_0s_2,\quad r_3r_1s_3s_1;\quad r_0r_3s_0s_3,\quad r_1r_2s_1s_2$$

and designate them as follows:

$$\begin{aligned} Q_{rs} &\equiv r_0s_0 + r_1s_1 \equiv -r_2s_2 - r_3s_3, \\ Q'_{rs} &\equiv r_0s_0 + r_2s_2 \equiv -r_3s_3 - r_1s_1, \\ Q''_{rs} &\equiv r_0s_0 + r_3s_3 \equiv -r_1s_1 - r_2s_2. \end{aligned}$$

Each triad of such quadrics form a basis of the net of quadrics $n(rs)$ circumscribing a pair of Möbius tetrahedra T_r, T_s of the conjugate tetrad of T_p thus giving us 6 new nets, one for each such pair of tetrahedra or for each pair of values of r, s.

c. There are thus 36 interlocking quadrics, 18 forming in triads the basis of the 6 nets $n(pq)$ and 18 similarly related to the 6 nets $n(rs)$, associated with a scheme of a pair of conjugate tetrads of Möbius tetrahedra which obviously inscribe as well as circumscribe or *interlock* in pairs the basis triads of the corresponding nets as disclosed by their very construction Thus they are all apolar to the quadric F (§ 1b). It is now not difficult to prove that they are all self-reciprocal for F (8).

https://doi.org/10.1017/S1446788700027646 Published online by Cambridge University Press

d. Again we may observe that the 4 pairs of Möbius tetrahedra arising from a Möbius 8_4 configuration interlock simultaneously a triad of interlocking quadrics which then repeat for the 4 corresponding schemes (§ 1h). Hence there are in all $36 \times 10/4 = 90$ interlocking quadrics and $12 \times 10/4 = 30$ nets associated with a 16_6 configuration.

3. Special webs of quadrics

a. Following the scheme M (§ 1b), we observe that the 36 interlocking and 9 fundamental quadrics F_{ij} (§ 1g) are related as follows:

$$\begin{aligned}
2Q_{jk} &= F'_{k1}F_{j1} + F'_{j1}F_{k1}, & 2Q'_{jk} &= F'_{k2}F_{j2} + F'_{j2}F_{k2},\\
2Q_{0i} &= F'_{k1}F_{j1} - F'_{j1}F_{k1}, & 2Q'_{0i} &= F'_{k2}F_{j2} - F'_{j2}F_{k2},\\
2Q_{mn} &= F_{1n}F'_{1m} + F_{1m}F'_{1n}, & 2Q'_{mn} &= F_{2n}F'_{2m} + F_{2m}F'_{2n},\\
2Q_{lt} &= F_{1n}F'_{1m} - F_{1m}F'_{1n}, & 2Q'_{mn} &= F_{2n}F'_{2m} - F_{2m}F'_{2n},
\end{aligned}$$

$$\begin{aligned}
2Q''_{jk} &= F'_{k3}F_{j3} + F'_{j3}F_{k3},\\
2Q''_{0i} &= F'_{k3}F_{j3} - F'_{j3}F_{k3};\\
2Q''_{mn} &= F_{3n}F'_{3m} + F_{3m}F'_{3n},\\
2Q''_{mn} &= F_{3n}F'_{3m} - F_{3m}F'_{3n},
\end{aligned}$$

where i, j, k are the even permutations of 1, 2, 3 and l, m, n of x, y, z; $F_{ix} = F_{i1}$, $F_{iy} = F_{i2}$, $F_{iz} = F_{i3}$; F'_{ij} is obtained from F_{ij} by putting $x = a$, $y = b$, $z = c$, $t = d$.

If we follow the scheme $\bar{M}$ (§ 1c), we may establish the same relations replacing F by C, where C_{ij} are the 9 linear combinations of sets of three of the quadrics F_{ij} (as suggested by the referee) expressed as the 9 elements of the matrix $(C_{ik}) = (l_{ij})(F_{jk})$. Thus C_{ij} play the role of 9 fundamental quadrics arising from the scheme $\bar{M}$, the tenth being the same F whether we follow M or $\bar{M}$.

b. As an immediate consequence of the above relations, we may note that the 6 quadrics Q_{pq} are each expressible as a linear combination of two of the 3 quadrics F_{i1} or C_{i1}, and therefore belong to a special web w (say) having a pair of generators common with the quadric F by the property (iii) of § 1g and obviously a quadric common with each of the 6 nets $n(pq)$ (§ 2). Similarly are related the 5 hexads of quadrics Q'_{pq}, Q''_{pq}, Q_{rs}, Q'_{rs}, Q''_{rs} to the triads of F_{i2} or C_{i2}, F_{i3} or C_{i3}, F_{1i} or C_{1i}, F_{2i} or C_{2i}, F_{3i} or C_{3i} respectively, and therefore belong to 5 special webs w', w'', w_1, w_2, w_3 (say), each having a pair of generators common with F and a quadric common with each of the 6 nets $n(pq)$ or $n(rs)$. Hence follows

THEOREM 1. *With every tetrad of Möbius tetrahedra T_p are associated 6 nets and 3 special webs of quadrics such that every net has a quadric common with every web. The quadric F, for which T_p are self-polar, belongs to all the*

https://doi.org/10.1017/S1446788700027646 Published online by Cambridge University Press

3 webs. The 18 interlocking quadrics common to the 6 nets and the 3 webs are all apolar to and self-reciprocal for F such that the pair of Möbius tetrahedra determining a net interlock the 3 basis quadrics of the net (§ 2c). *The same is true of the conjugate tetrad too.*

c. From the construction (§ 2a) of the quadrics Q_{pq} and § 3b follows that the three of them through one of the 4 tetrahedra of the tetrad T_p have 2 pairs of generators of opposite systems common and are therefore linearly related or belong to a pencil. Thus the 6 quadrics Q_{pq} belong by threes to 4 pencils. Similar is then the situation for all the 6 such hexads of quadrics. From the relations of § 3b then follows

THEOREM 2. *The 6 interlocking quadrics belonging to a special web associated with a tetrad of Mobius tetrahedra T_p behave like the vertices of a quadrilateral whose 3 diagonal points represent 3 fundamental quadrics belonging to the web besides the one for which T_p are self-polar such that the 3 such quadrilaterals have each a diagonal point common with each of those arising similarly from the conjugate tetrad* (cf. 16).

d. Consequently from the Theorems 1—2 and §§ 1g, 2c, 3a follows

THEOREM 3. *With every 16_6 figure are associated 90 interlocking quadrics (Q) besides the 10 fundamental ones (F), 15 special webs (w), 30 nets (n) and 60 quadrilaterals (q) such that* (i) *every pair of (F) and the pair of (Q) linearly related to them belong to 2 (w) and are respectively represented by a pair of diagonal points and the pair of vertices on their join common to 4 (q)*; (ii) *the 2 triads of (Q) circumscribed to 2 complementary pairs of Möbius tetrahedra of a tetrad are apolar to and self-reciprocal for 4 (F) which belong to a (w) associated with the conjugate tetrad*; (iii) *12 (Q) belong to each (w) and are represented by the vertices of 4 (q) forming a desmic system whose one of the 3 'diagonal'-tetrahedra* (19) *represent the 4 (F) belonging to it*; (iv) *the pair of (n) circum scribed to 2 complementary pairs of Möbius tetrahedra of a tetrad have each a (Q) common with 6 (w), 3 associated with it and 3 having each a pair of (F) common with a (w) associated with its conjugate tetrad*; (v) *each (w) has a (Q) common with each of 12 (n)*; (vi) *each (F) belongs to 6 (w) and is therefore represented by the common diagonal point of 18 (q).*

The incidences of this theorem may be put down in the following self-explanatory table:

	Total No.	F	Q	q	w
F	10	—	—	18	6
Q	90	—	—	4	2
q	60	3	6	—	1
w	15	4	12	4	—

https://doi.org/10.1017/S1446788700027646 Published online by Cambridge University Press

4. Common generators of the special webs

a. The equations of the generators of the 2 systems of the quadric F (§ 1b) are given respectively by (15; 21)

$$x + Iy - u(z + It) = 0 = u(x - Iy) + z - It;$$
$$x + Iy - v(z - It) = 0 = v(x - Iy) + z + It,$$

where u, v are their respective parameters and $I^2 = -1$. The respective pairs of the common generators of the 3 webs w, w', w'' (§ 3b) are easily seen to be given by $u = \pm 1$; $\pm I$; 0, ∞ for both the schemes M and $\bar{M}$ (§§ 1b, 1c), and those of w_i by the same values of v if we follow the scheme M, but by $v = (\pm 1 - l_{i3})/(l_{i1} - Il_{i2})$, or, by the equations

$$Ex + Gz + Ht = Ey + Hz - Gt$$
$$(E = 1 - l_{i3},\ G = l_{i3}l_{i1} \mp Il_{i2},\ H = l_{i2}l_{i3} \pm Il_{i1})$$

If we follow the scheme $\bar{M}$. The statements made above (§§ 1d, 3b) can now be readily verified.

b. The pair of common generators of w_i for a particular value of i can be readily identified with the pair of transversals e, f, of Edge (8), of the 4 joins of the corresponding vertices of the pair of Möbius tetrahedra T_0, T_i. Hence they are the pair of transversals of the 4 joins of the corresponding vertices of the complementary pair of Mobius tetrahedra T_j, T_k too of the tetrad T_p (§ 1d). Therefore they form a pair of polar lines (8; 15) for every quadric of either net $n(0i)$, $n(jk)$. Again when harmonic inversion (13; 15; 18) is performed w.r.t. a pair of generators or polar lines of a quadric, it inverts into itself. Hence follows

THEOREM 4. *The pair of common generators e, f of a special web of quadrics associated with a tetrad of Möbius tetrahedra form a pair of polar lines for the quadrics through the vertices of either of 2 complementary pairs of tetrahedra of its conjugate tetrad such that all the said quadrics are harmonically self-inverse w.r.t. e, f* (cf. 8). It can also be deduced from §§ 1g (iii), 3a or Th. 3(ii).

5. Göpel tetrahedra and quadrics

a. There are 36 *Göpel tetrahedra of tropes*, associated with a scheme of 8 Rosenhain tetrahedra forming 2 conjugate tetrads (§ 1), each formed of the 4 common faces r_p, r_q, s_p, s_q of 2 pairs of Möbius tetrahedra $T_p, T_q; T_r, T_s$, one pair from each tetrad (12), such that the 4 singular conics of the associated Kummer's quartic lying in the faces of each Göpel tetrahedron lie on a *Göpel quadric* Q_{pqrs} passing through the 12 vertices T_p, T_q, T_r, T_s and therefore outpolar to the fundamental quadric F for which the tetrahedra

https://doi.org/10.1017/S1446788700027646 Published online by Cambridge University Press

of the scheme are self-polar (16). Obviously each Q_{pqrs} belong to the 2 nets $n(pq)$ and $n(rs)$ associated with the scheme.

Thus follows the following

THEOREM 5. *With every scheme of* 8 *Rosenhain tetrahedra forming* 2 *conjugate tetrads* T_p, T_r *are associated* 36 *Göpel tetrahedra of tropes and* 36 *Göpel quadrics, each belonging to* 2 *nets, one associated with* T_p *and the other with* T_r. *Thus: The* 6 *nets associated with* T_p *or* T_r, *having no quadric common with one another, have each a Göpel quadric common with each such net associated with the conjugate tetrad such that the* 36 *Göpel quadrics are all outpolar to the fundamental quadric F for which* T_p, T_r *are self-polar.*

b. Further we may observe that: If 3 of the 4 parameters p, q, r, s be fixed, the fourth varying one (say s) determine 3 quadrics Q_{pqrs} which form a pencil passing through the 2 conics (as their common degenerate quartic) lying in the 2 faces r_p, r_q of the third tetrahedron T_r common with the pair of Möbius tetrahedra T_p, T_q corresponding to the fixed parameters. Thus: If we fix one of the 2 pairs of parameters p, q; r, s (say p, q), we obtain 6 quadrics Q_{pqrs} belonging to the net $n(pq)$ with r, s varying such that they belong by threes to 4 pencils corresponding to the 4 values of r or s showing that each pencil has a Göpel quadric common with the other three (16). Therefore they behave like the 6 vertices of a quadrilateral whose diagonal points represent the 3 interlocking quadrics of $n(pq)$ by relations of § 3a.

Again if we pair the parameters differently, say as p, r; q, s, and fix one pair (say p, r), we obtain a pencil of 3 quadrics Q_{pqrs} for each value of q when s varies and another for each value of s when q varies. Thus: We have 2 triads of pencils, one for the 3 values of q and the other for s such that the pencils of either triad have no quadric common with one another and every pencil of one has a Göpel quadric common with every pencil of the other. Their 9 common quadrics are observed to have a singular conic lying in the face r_p common to the 2 tetrahedra T_p, T_r corresponding to the given values of p, r. Thus follows the following

THEOREM 6. *With every scheme M forming* 2 *conjugate tetrads of Möbius tetrahedra* T_p, T_r *are associated* 3 *Göpel tetrahedra of tropes having* 2 *faces common such that their* 3 *associated quadrics belong to a pencil, and* 9 *such Göpel tetrahedra having one face common such that their* 9 *associated quadrics can be represented by* 9 *points of a quadric R lying by threes on* 2 *triads of its generators of opposite systems representing the* 2 *triads of pencils to which they belong by threes. The* 6 *Göpel quadrics of a net associated with* T_p *or* T_r *common with the* 6 *nets associated with the conjugate tetrad can be represented by the* 6 *vertices of a quadrilateral* q' *whose diagonal points represent the* 3 *interlocking quadrics of the net. The* 36 *Göbel quadrics* (G) *thus associated with M are then represented by the vertices of* 2 *hexads of quadrilaterals* (q'), *the diagonal points of one hexad representing the* 18 *interlocking quadrics* (*Q*) *associated with* T_p *and of the other representing those associated with* T_r, *such that their* 48 *sides lie by sixes on* 16 *quadrics* (*R*) *as* 2 *triads of generators of opposite systems, the sides of one hexad belonging to one system and of the other to the second system, each side being common to* 2 (*R*) *and each* (*G*) *being represented by a point common to* 4 (*R*).

c. As an immediate consequence of what precedes follows the following

THEOREM 7. *With a* 16_6 *figure are associated* 60 *Göpel quadrics* (*G*), 30 *quadrilaterals* (*q'*) *and* 160 *quadrics* (*R*) *besides* 10 (*F*), 15 (*w*), 30 (*n*), 60 (*q*) *and* 90 (*Q*) *as in the Theorem* 3 *such that* (i) *the* 3 (*Q*) *of every* (*n*) *are represented by the diagonal points of a* (*q'*) *whose vertices represent* 6 (*G*); (ii) *the* 2 *hexads of* (*G*) *belonging to* 2 (*n*) *determined by* 2 *complementary pairs of Möbius tetrahedra of a tetrad are outpolar to* 4 (*F*) *beloning to a* (*w*) *associated with the conjugate tetrad*; (iii) *the sides of* (*q'*) *lie by sixes on* 160 (*R*) *as* 2 *triads of generators of opposite systems, each generator being common to* 8 (*R*); (iv) *each* (*G*) *is represented by a common vertex of* 3 (*q'*) *and by a point common to* 24 (*G*), *and is outpolar to* 6 (*F*); (v) *the vertices of* (*q*) *lie at the diagonal points of* (*q'*).

Thanks are due to the referee for the present form of the paper, and to Prof. B. R. Seth for his generous, kind and constant encouragement in my pure pursuits.

References

[1] Baker, H. F., Principles of geometry **2** (Cambridge, 1930).
[2] Baker, H. F., Principles of geometry **3** (Cambridge, 1934).
[3] Baker, H. F., Principles of geometry **4** (Cambridge, 1940).
[4] Baker, H. F., A locus with 25920 linear transformations (Cambridge, 1946).
[5] Baker, H. F., Note to the preceding paper by C. V. H. Rao, Proc. Camb. Phil. Soc. **42** (1946), 226—229.
[6] Court, N. A., Modern pure solid geometry (New York, 1935).
[7] Coxeter, H. S. M., The real projective plane (Cambridge, 1955).
[8] Edge, W. L., The net of quadrics associated with a pair of Möbius tetrads, Proc. Lond. Math. Soc. **41** (1936), 336—360.
[9] Edge, W. L., Geometry in three dimensions over GF(3), Proc. Roy. Soc. A **222** (1954), 262—286.
[10] Hamil, C. M., On a finite group of order 576, Proc. Camb. Phil. Soc. **44** (1948), 26—36.
[11] Hanumanta, C. V. and Baker H. F., On the generation of sets of four tetrahedra of which any two are mutually inscribed, Proc. Camb. Phil. Soc. **20** (1921), 155—157.
[12] Hudson, R. W. H. T., Kummer's quartic surface (Cambridge, 1905).
[13] Mandan, S. R., Properties of mutually self-polar tetrahedra, Bull. Cal. Math. Soc. **33** (1941), 147—155.
[14] Mandan, S. R., Umbilical projection, Proc. Ind. Acad. Sc. A **15** (1942), 16.
[15] Mandan, S. R., Möbius tetrads, Amer. Math. Mon. **64** (1957), 471—478.
[16] Mandan, S. R., Möbius tetrads, II, *ibid.* **65** (1958), 247—251.
[17] Mandan, S. R., An *S*-configuration in Euclidean and elliptic *n*-space, Can. J. Math. **10** (1958), 489—501.
[18] Mandan, S. R., Harmonic inversion, Math. Mag. **32** (1959), 71—78.

[19] Mandan, S. R., On four intersecting spheres, Jour. Ind. Math. Soc. 23 (1959), 151—167.
[20] Mandan, S. R., Semi-isodynamic and -isogonic tetrahedra, Rend. Mat. 19 (1960), 401—415.
[21] Rao, C. V. H., On the generation of sets of four tetrahedra of which any two are mutually inscribed, Proc. Camb. Phil. Soc. 42 (1946), 217—226.
[22] Salmon, G. Analytical geometry of three dimensions 1 (New York, 1927).
[23] Smith, C., Solid geometry (London, 1951).
[24] Sommerville, D. M. Y., Analytical geometry of three dimensions (Camb., 1959).
[25] Vaidyanathswamy, R., A memoir on the cubic transformations associated with a desmic system, Jour. Ind. Math. Soc. Suppl. (1927).

Indian Institute of Technology
Kharagpur