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Ziegler's Indecomposability Criterion

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Abstract. Ziegler's Indecomposability Criterion is used to prove that a totally transcendental, *i.e.*, Σ-pure injective, indecomposable left module over a left noetherian ring is a directed union of finitely generated indecomposable modules. The same criterion is also used to give a sufficient condition for a pure injective indecomposable module ${}_RU$ to have an indecomposable local dual $U^\sharp_R$.

Let R be a left noetherian ring and let ${}_RU$ be a totally transcendental, *i.e.*, Σ-pure injective indecomposable left R-module. One task of this article is to prove (Theorem 5) that ${}_RU$ is a directed union ${}_RU = \sum_i M_i$ of finitely generated indecomposable submodules ${}_RM_i$. A familiar example of this phenomenon is the case of an injective indecomposable left R-module ${}_RE$. Over a left noetherian ring, such a module is totally transcendental, and if we express it as a directed union ${}_RE = \sum_i M_i$ of finitely generated submodules, then each ${}_RM_i$ is uniform, hence indecomposable.

But a more interesting example is that of a generic module over an artin algebra. An *artin algebra* is a ring Λ whose center $C = C(\Lambda)$ is artinian and that is finitely generated as a module over C. A Λ-module G is *generic* if it is (1) indecomposable, (2) not finitely generated, and (3) of finite length as a module over its endomorphism ring. This last condition implies that G has a pp-composition series, and is therefore of finite Morley rank. The importance of generic modules arises from the work of Crawley-Boevey [1], who proved that an artin algebra has a generic module if and only if it satisfies the following conjecture.

The Brauer-Thrall Conjecture If an artin algebra Λ has infinitely many nonisomorphic indecomposable finitely generated left modules, then there is a natural number n and an infinite family of indecomposable left Λ-modules of length n.

Theorem 5, which implies that a generic module G is an amalgam of finitely generated *indecomposable* modules, may therefore be of some use if one is motivated to employ amalgamation techniques (*cf.* [4]) to construct such a G.

The other task of this article is to introduce several equivalent conditions (Theorem 4) for a pure injective indecomposable left R-module ${}_RU$ that ensure the local dual $U^\sharp_R$ be an indecomposable right R-module. Recall that a pure injective indecomposable left R-module ${}_RU$ has a local endomorphism ring $S = \mathrm{End}_R\, U$, and so obtains an R-S-bimodule structure. The top of S is a division ring Δ, and if we let $E_S = E(\Delta_S)$ be the injective envelope of the right S-module Δ_S, then the local dual of ${}_RU$ is defined to be

$$U^\sharp_R := \mathrm{Hom}_S({}_RU_S, E_S).$$

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It is a pure-injective right R-module, the right action being defined by $(\eta r)(u) := \eta(ru)$. A fundamental question in the study of pure-injective indecomposable modules over a ring R is whether the local dual $U_R^\sharp$ is itself indecomposable. If so, it yields a point in the right Ziegler spectrum of R, which is in some sense dual to ${}_RU$.

The proofs of these results rely on Ziegler's Indecomposability Criterion. To describe the criterion, we recall from [6, §1.1] that the language $\mathcal{L}(R)$ for left R-modules is the expansion of the language $\mathcal{L} = (+, -, 0)$ of abelian groups by a ring R of unary function symbols. The standard collection $T(R)$ of axioms for a left R-module are readily expressed in the language $\mathcal{L}(R)$. A formula of $\mathcal{L}(R)$ is said to be *positive-primitive* (pp) if it is built up from atomic fomulae using only conjunction and existential quantification. If ${}_RM$ is a left R-module and $\varphi(\overline{x}) = \varphi(x_1, \dots, x_n)$ is a pp-formula of $\mathcal{L}(R)$, then the subset of $({}_RM)^n$ defined by φ in M is a subgroup

$$\varphi(M) = \left\{ (u_1, \dots, u_n) \in ({}_RM)^n \mid M \models \varphi(\overline{u}) \right\}.$$

Such a subgroup of $({}_RM)^n$ is called *pp-definable* in ${}_RM$.

Suppose that $\varphi(\overline{x})$ and $\psi(\overline{x})$ are pp-formulae of $\mathcal{L}(R)$ in the same tuple of free variables. Evidently, the conjunction

$$(\varphi \wedge \psi)(\overline{x}) := \varphi(\overline{x}) \wedge \psi(\overline{x})$$

is itself a pp-formula, but so is the formula

$$(\varphi + \psi)(\overline{x}) := \exists\, \overline{y} \left[\varphi(\overline{y}) \wedge \psi(\overline{x} - \overline{y}) \right].$$

These two binary operations induce a modular lattice structure R-Latt$(\overline{x})$ on the classes of pp-formulae $\varphi(\overline{x})$ modulo equivalence relative to $T(R)$. There is an anti-isomorphism $\varphi(\overline{x}) \mapsto \varphi^*(\overline{x})$ between the lattice R-Latt$(\overline{x})$ and the similarly defined lattice R^{op}-Latt$(\overline{x})$ in the language $\mathcal{L}(R^{\mathrm{op}})$ of right R-modules. An explicit description of this anti-isomorphism can be found in [6, §1.3.1] or [5]; we will rely on the following two properties of this duality.

Fact 1 ([6, §1.3.2], [2]) *Let ${}_RM$ be a left R-module, N_R a right R-module, n a positive integer and suppose that a pair of n-tuples, $\overline{u} \in (N_R)^n$ and $\overline{v} \in ({}_RM)^n$, are given. Then*

$$\overline{u} \otimes \overline{v} := \sum_i u_i \otimes v_i = 0$$

in $N \otimes_R M$ if and only if there is a pp-formula $\varphi(\overline{x})$ in $\mathcal{L}(R)$ such that ${}_RM \models \varphi(\overline{v})$ and $N_R \models \varphi^*(\overline{u})$.

Fact 2 ([6, §1.3.], [8]) *Let ${}_RM_S$ be an R-S-bimodule, $E = E_S$ an injective cogenerator and $M_R^\sharp$ the right R-module* $\mathrm{Hom}_S({}_RM_S, E_S)$. *For every positive-primitive formula $\varphi(\overline{x})$ in the language $\mathcal{L}(R)$, $M_R^\sharp \models \varphi^*(\overline{\eta})$ if and only if $\overline{\eta}[\varphi(M)] = 0$. The convention here is that if $\overline{\eta} \in (M^\sharp)^n$ and $\overline{v} \in M^n$, then*

$$\overline{\eta}(\overline{v}) = \big(\eta_1(v_1), \dots, \eta_n(v_n) \big) \in E^n.$$

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A pp-type $p = p(\overline{x})$ is a collection of positive-primitive formulae in the variables $\overline{x}$, deductively closed relative to the axioms $T(R)$. Given a tuple $\overline{u} \in ({}_R M)^n$, its pp-type is given by

$$\text{pp-tp}_M(\overline{u}) = \left\{ \varphi(\overline{x}) \mid M \models \varphi(\overline{u}) \right\}.$$

If $\overline{u} \in M^n$ satisfies every formula in a pp-type $p(\overline{x})$, then it *realizes* $p(\overline{x})$ in M : $p(\overline{x}) \subseteq \text{pp-tp}_M(\overline{u})$.

Given a pp-type $p(\overline{x})$, the *pure-injective hull* $H(p)$ [6, §4.3.5] is a pure-injective left R-module with a specified tuple $\overline{u} \in ({}_R H(p))^n$ such that $\text{pp-tp}_{H(p)}(\overline{u}) = p(\overline{x})$. Furthermore,

(i) if M is a pure-injective module and $\overline{v} \in M^n$ realizes $p(\overline{x})$, then there is a morphism $f \colon H(p) \to M$ of left R-modules with $f(\overline{u}) = \overline{v}$; and
(ii) every R-endomorphism $g \colon H(p) \to H(p)$ satisfying $g(\overline{u}) = \overline{u}$ is an automorphism.

Fisher ([6, §4.3.5]) proved the existence of the pure-injective hull of a pp-type. Properties (i) and (ii) ensure that it is unique up to isomorphism over the specified realization $\overline{u}$ of $p(\overline{x})$. A pp-type $p(\overline{x})$ is called *indecomposable* if its pure-injective hull $H(p)$ is an indecomposable left R-module.

Ziegler's Indecomposability Criterion ([6, §4.3.6], [7]) A pp-type $p(\overline{x})$ is indecomposable if for every pair $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ of pp-formulae that do not belong to $p(\overline{x})$, there is a pp-formula $\varphi(\overline{x}) \in p(\overline{x})$ such that

$$\left[(\varphi \wedge \psi_1) + (\varphi + \psi_2) \right] (\overline{x}) \notin p(\overline{x}).$$

Let ${}_R M_S$ be an R-S-bimodule, where S is a local ring with top Δ. Let $E_S = E(\Delta)$ be the injective envelope of Δ considered as a right S-module. If $\overline{\eta}$ is an n-tuple of elements from the right R-module $M_R^\sharp = \text{Hom}_S({}_R M_S, E_S)$, then, trivially,

$$\text{Ker}\,\overline{\eta} \supseteq \sum \left\{ \varphi(M) \mid \overline{\eta}[\varphi(M)] = 0 \right\}.$$

If the equality holds, we consider that as a kind of *continuity condition* on $\overline{\eta}$.

Proposition 3 *Suppose that* $\text{Ker}\ \overline{\eta} = \sum \{\varphi(M) \mid \overline{\eta}[\varphi(M)] = 0\}$ *under the condition given above. Then the pp-type of* $\overline{\eta}$ *in* $M_R^\sharp$ *is indecomposable.*

Proof Suppose that $\psi_1^*(\overline{x})$, $\psi_2^*(\overline{x})$ do not belong to $\text{pp-tp}_{M^\sharp}(\overline{\eta})$. Because E_S is the minimal injective cogenerator in the category Mod-S of right S-modules, we may use Fact 2, which implies that both $\overline{\eta}(\psi_1(M))$ and $\overline{\eta}(\psi_2(M))$ are nonzero S-submodules of $E_S = E(\Delta)$. Thus, there are $\overline{u} \in \psi_1(M)$ and $\overline{v} \in \psi_2(M)$ such that $\overline{\eta}(\overline{u}) = \overline{\eta}(\overline{v}) = 1$, where $1 \in \Delta_S$ denotes the unit element of the top of S.

Because $\overline{\eta}(\overline{u} - \overline{v}) = 0$, the hypothesis implies that there is a pp-formula $\varphi(\overline{x})$ such that

$$\overline{u} - \overline{v} \in \varphi(M) \subseteq \text{Ker}\ \overline{\eta}.$$

Another application of Fact 2 implies that $\varphi^*(\overline{x}) \in \text{pp-tp}_{M^\sharp}(\overline{\eta})$, and it remains to verify that

$$(\varphi^* \wedge \psi_1^*) + (\varphi^* \wedge \psi_2^*) = \left[(\varphi + \psi_1) \wedge (\varphi + \psi_2) \right]^* \notin \text{pp-tp}_{M^\sharp}(\overline{\eta}).$$

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But $\overline{u} \in \psi_1(M) \subseteq (\varphi + \psi_1)(M)$ and $\overline{u} = (\overline{u} - \overline{v}) + \overline{v} \in (\varphi + \psi_2)(M)$. Thus $\overline{u} \in [(\varphi + \psi_1) \wedge (\varphi + \psi_2)](M)$, and because $\overline{\eta}(\overline{u})$ is nonzero, the claim is established. ■

Suppose that ${}_RM$ is a left R-module and S is the endomorphism ring $S = \mathrm{End}_R\ M$. If ${}_RM$ is totally transcendental, then every cyclic S-submodule $\overline{u}S$ of M^n is pp-definable in ${}_RM$. Therefore, every S-submodule is a sum of subgroups that are pp-definable in ${}_RM$, and the equality in the proposition is attained. Finitely presented left R-modules also enjoy this property; in fact, every locally pure projective module does. So if ${}_RM$ has a local endomorphism ring $S = \mathrm{End}_R\ M$, then, because the local dual $M^\sharp_R$ is a pure-injective right R-module realizing only indecomposable types, it must be indecomposable. More generally, we have the following.

Theorem 4 *Let ${}_RM_S$ be an R-S-bimodule and E_S an injective cogenerator with endomorphism ring $T = \mathrm{End}_S\ E$. The following are equivalent for the T-R-bimodule $M^\sharp = \mathrm{Hom}_S({}_RM_S, {}_TE_S)$:*

(i) *for every $n < \omega$, and every n-tuple $\overline{\eta} = (\eta_1, \dots, \eta_n) \in (M^\sharp_R)^n$,*

$$\mathrm{Ker}\ \overline{\eta} = \sum\{\varphi(M) \mid \overline{\eta}[\varphi(M)] = 0\};$$

(ii) *the evaluation map $\mathrm{Ev}\colon {}_TM^\sharp \otimes_R M_S \to E$, induced by $\eta \otimes u \mapsto \eta(u)$, is a monomorphism of T-S-bimodules;*

(iii) *the morphism of rings from T to $\mathrm{End}_R\ M^\sharp_R$ is onto.*

Suppose that the endomorphism ring of ${}_RM$ is local, and let $S = \mathrm{End}_R\ M$ and $E_S = E(\Delta_S)$, where Δ is the top of S. Because E_S is an injective indecomposable module, $T = \mathrm{End}_S\ E_S$ is a local ring. Condition (iii) then implies that the endomorphism ring $\mathrm{End}_R\ M^\sharp_R$ is a quotient of a local ring and is thus also local. Therefore, Theorem 4 subsumes the situation described just before its statement.

Proof (i) ⇒ (ii) Suppose that $\overline{\eta} \in (M^\sharp)^n$ and $\overline{u} \in M^n$ are such that

$$\mathrm{Ev}(\overline{\eta} \otimes \overline{u}) = \mathrm{Ev}\Big(\sum_i \eta_i \otimes u_i\Big) = \sum_i \eta_i(u_i) = 0.$$

By hypothesis, there is a positive-primitive formula $\varphi(\overline{x})$ such that

$$\overline{u} \in \varphi(M) \subseteq \mathrm{Ker}\ \overline{\eta}.$$

By Fact 2, $M^\sharp_R \models \varphi^*(\overline{\eta})$, and so Fact 1 implies that $\overline{\eta} \otimes \overline{u} = 0$ in $M^\sharp \otimes_R M$.

(ii) ⇒ (iii) Applying the exact functor $\mathrm{Hom}_S(-, E_S)$ to the monomorphism $\mathrm{Ev}\colon {}_TM^\sharp \otimes_R M_S \to E_S$, we get an epimorphism

$$\begin{aligned} T = \mathrm{End}_S\ E_S &\to \mathrm{Hom}_S(M^\sharp \otimes M_S, E_S) = \mathrm{Hom}_R(M^\sharp, \mathrm{Hom}_S(M_S, E_S)) \\ &= \mathrm{Hom}_R(M^\sharp, M^\sharp) = S. \end{aligned}$$

(iii) ⇒ (i) Let $\overline{\eta} \in (M^\sharp)^n$ and consider the inclusion

$$\Sigma = \sum\big\{\varphi(M) \mid \overline{\eta}[\varphi(M)] = 0\big\} \subseteq \mathrm{Ker}\ \overline{\eta}.$$

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To see that equality holds, suppose that $\overline{u} \notin \Sigma$. As E_S is an injective cogenerator for the category of right S-modules, there is an S-morphism $\overline{\gamma}\colon (M^n)_S \to E_S$ such that $\Sigma \subseteq \operatorname{Ker} \overline{\gamma}$, but $\overline{\gamma}(\overline{u}) \neq 0 \in E$. The n component morphisms $\gamma_i\colon M_S \to E_S$, $1 \leq i \leq n$, yield a tuple $\overline{\gamma} \in (M^\sharp)^n$ satisfying

$$\text{pp-tp}_{M^\sharp}(\overline{\eta}) \subseteq \text{pp-tp}_{M^\sharp}(\overline{\gamma}),$$

because if $\varphi^* \in \text{pp-tp}_{M^\sharp}(\overline{\eta})$, then $M^\sharp \models \overline{\eta}(\varphi^*)$, which is equivalent to $\overline{\eta}(\varphi(M)) = 0$. The assumption $\overline{\gamma}(\varphi(M)) = 0$ then implies that $\varphi^* \in \text{pp-tp}_{M^\sharp}(\overline{\gamma})$.

The right R-module $M^\sharp_R$ is pure injective, so that [7, Thm. 3.6] implies there is an R-morphism $f\colon M^\sharp_R \to M^\sharp_R$ such that $f(\overline{\eta}) = \overline{\gamma}$, that is, $f(\eta_i) = \gamma_i$, for each i. By hypothesis, f may be represented by the action of some $t \in \operatorname{End}_S(E_S)$. Because

$$t[\overline{\eta}(\overline{u})] = [t\overline{\eta}](\overline{u}) = [f(\overline{\eta})](\overline{u}) = \overline{\gamma}(\overline{u})$$

is nonzero, $\overline{\eta}(\overline{u}) \neq 0$, and so $\overline{u} \notin \operatorname{Ker} \overline{\eta}$. ■

If there exists an infinite family of finitely generated indecomposable modules over an artin algebra Λ of bounded endolength n, then ([6, §4.5.5], [3]) any point that belongs to the closure of this infinite family in the Ziegler Spectrum of Λ is a generic Λ-module. The next result uses Ziegler's Indecomposability Criterion to show that a generic module over Λ, if one exists, is necessarily an amalgam of finitely generated *indecomposable* Λ-modules, which cannot possibly be of bounded length.

Theorem 5 *Let R be a left noetherian ring and M a totally transcendental indecomposable left R-module. Then M is a directed union $M = \sum_i M_i$ of finitely generated indecomposable submodules M_i.*

Proof Let $u_1, \dots, u_n \in M$. To prove the theorem, we must produce a finitely generated indecomposable submodule $M' \subseteq M$ containing all the u_i. That will imply that the collection of finitely generated indecomposable submodules of M is directed and cofinal in the collection, partially ordered by inclusion, of finitely generated submodules of M.

Let $p(\overline{x}) = \text{pp-tp}_M(\overline{u})$ be the pp-type of $\overline{u}$ in M. Because $({}_RM)^n$ satisfies the descending chain condition on subgroups pp-definable in M, $p(\overline{x})$ is implied, relative to the complete theory of M, by a single pp-formula $\varphi(\overline{x})$,

$$M \models \text{pp-tp}_M(\overline{u}) \leftrightarrow \varphi(\overline{x}).$$

Because M is a pure injective indecomposable module, the type $p(\overline{x})$ satisfies Ziegler's Indecomposability Criterion, which implies that the collection of pp-formulae

$$\Psi(\overline{x}) = \{\psi(\overline{x}) : \psi(M) < \varphi(M)\}$$

forms an ideal in the lattice of pp-fomulae in $\overline{x}$, *i.e.,* it is downward closed and if $\psi_1(\overline{x}), \psi_2(\overline{x}) \in \Psi(\overline{x})$, then $(\psi_1 + \psi_2)(\overline{x}) \in \Psi(\overline{x})$.

The positive-primitive formula $\varphi(\overline{x})$ is equivalent, relative to $T(R)$, to an existentially quantified conjunction of atomic formulae, so if $K \subseteq M$ is a submodule

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generated by the u_i together with some witnesses to $M \models \varphi(\overline{u})$, then $K \models \varphi(\overline{u})$. Furthermore, $K \models \neg\psi(\overline{u})$, for every $\psi(\overline{x}) \in \Psi(\overline{x})$. As R is left noetherian, K is a finite direct sum $K = \oplus_j K_j$ of finitely generated indecomposable modules K_j. Decompose $\overline{u} = \sum_j \overline{u}_j$ in terms of its components, relative to this direct sum decomposition. Positive-primitive formulae respect direct sums, so that for every j, $K_j \models \varphi(\overline{u}_j)$, and hence $M \models \varphi(\overline{u}_j)$. As $\Psi(\overline{x})$ is an ideal of pp-formulae, there is a j, say $j = 1$, such that $M \models \neg\psi(\overline{u}_1)$, for every $\psi(\overline{x}) \in \Psi(\overline{x})$. Consequently, $\text{pp-tp}_M(\overline{u}) = \text{pp-tp}_M(\overline{u}_1)$. By [6, §4.3.5], there is an endomorphism f of M, necessarily an automorphism, such that $f\colon \overline{u}_1 \mapsto \overline{u}$. Then $M' = f(K_1)$ is a finitely generated indecomposable submodule of M that contains all the u_i. ■

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