Bull. Aust. Math. Soc. **90** (2014), 427–443
doi:10.1017/S0004972714000562

REMARKS ON VALUE SHARING OF CERTAIN DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

XIAO-MIN LI[✉] and HONG-XUN YI

(Received 1 February 2014; accepted 20 March 2014; first published online 28 August 2014)

Abstract

We use Zalcman's lemma to study a uniqueness question for meromorphic functions where certain associated nonlinear differential polynomials share a nonzero value. The results in this paper extend Theorem 1 in Yang and Hua ['Uniqueness and value-sharing of meromorphic functions', *Ann. Acad. Sci. Fenn. Math.* **22** (1997), 395–406] and Theorem 1 in Fang ['Uniqueness and value sharing of entire functions', *Comput. Math. Appl.* **44** (2002), 823–831]. Our reasoning in this paper also corrects a defect in the reasoning in the proof of Theorem 4 in Bhoosnurmath and Dyavanal ['Uniqueness and value sharing of meromorphic functions', *Comput. Math. Appl.* **53** (2007), 1191–1205].

2010 *Mathematics subject classification*: primary 30D35; secondary 30D30.

Keywords and phrases: meromorphic functions, shared values, differential polynomials, uniqueness theorems.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notation in the Nevanlinna theory of meromorphic functions as explained in the references [7, 10, 15, 16]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$, as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros, with the same multiplicities. Similarly, we say that f and g share a IM if $f - a$ and $g - a$ have the same zeros, ignoring multiplicities. In addition, we say that f and g share ∞ CM if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM if $1/f$ and $1/g$

This work is supported by the NSFC (No. 11171184), the NSFC & RFBR (Joint Project) (No. 10911120056), the NSF of Shandong Province, China (No. Z2008A01), and the NSF of Shandong Province, China (No. ZR2009AM008).

share 0 IM. These definitions can be found, for example, in the book by Yang and Yi [15]. We say that a is a small function of f if a is a meromorphic function satisfying $T(r, a) = S(r, f)$. This definition can also be found, for example, in [15]. Throughout this paper, we denote by $\mu(f)$, $\rho(f)$ and $\lambda(f)$ the lower order of f, the order of f and the exponent of convergence of zeros of f, respectively (see, for example, [7, 10, 15, 16]). Throughout this paper, we denote by $\rho(f)$ and $\lambda(f)$ the order of f and the exponent of convergence of zeros of f, respectively (see, for example, [7, 10, 15, 16]). In addition, we need the following three definitions.

Definition 1.1 [9, Definition 1]. Let p be a positive integer and $a \in C \cup \{\infty\}$. Then we denote by $N_{p)}(r, 1/(f-a))$ the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not greater than p, and by $\overline{N}_{p)}(r, 1/(f-a))$ the corresponding reduced counting function (ignoring multiplicities). We denote by $N_{(p}(r, 1/(f-a))$ the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not less than p, and by $\overline{N}_{(p}(r, 1/(f-a))$ the corresponding reduced counting function (ignoring multiplicities).

Definition 1.2. Let a be an any value in the extended complex plane and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a, f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N_k(r, (1/f - a))}{T(r, f)}, \tag{1.1}$$

where

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \cdots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big). \tag{1.2}$$

Remark 1.3. From (1.1) and (1.2) we have $0 \le \delta_k(a, f) \le \delta_{k-1}(a, f) \le \delta_1(a, f) \le \Theta(a, f) \le 1$.

Definition 1.4 [5, Definition 1.1] or [6, Definition 3.1]. A nonconstant monic polynomial $P(w)$ is called a uniqueness polynomial for meromorphic functions (or entire functions) in a broad sense if $P(f) = P(g)$ implies $f = g$ for two nonconstant meromorphic functions (or entire functions) f and g.

In 1997, Lahiri [8] posed the following question. What can be said about the relationship between two meromorphic functions f, g when two differential polynomials, generated by f and g respectively, share certain values? In this direction, Fang [3], Yang and Hua [17] and Bhoosnurmath and Dyavanal [1] respectively proved the following results.

Theorem 1.5 [3, Theorem 2]. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers satisfying $n \ge 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Theorem 1.6 [1, Theorem 4]. *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty, f) > 3/(n + 1)$, and let n, k be two positive integers satisfying $n \ge 3k + 13$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Yang and Hua [17] and Bhoosnurmath and Dyavanal [1] respectively, proved Theorem 1.6 for $k = 1$ and for $k \ge 1$, respectively. However, there is a defect in the proof of Theorem 4 in [1] which is discussed in Section 4 of the present paper. As far as we know, the following question is still open by now.

QUESTION 1.7. What can be said about the relationship between two meromorphic functions f and g, when $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, where n and $k \geq 1$ are two positive integers satisfying $n \geq 3k+13$?

We will prove the following results to deal with Question 1.7.

THEOREM 1.8. *Let f and g be nonconstant meromorphic functions such that $\rho(f) > 2$. Suppose that $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, where n, k are two positive integers satisfying $n > 3k+11$. If $\Theta(\infty, f) > 2/n$, then $f = g$.*

Proceeding as in the proof of Theorem 1.8 and using Lemma 2.3 in Section 2 below, we get the following result which is an IM analogue of Theorem 1.8.

THEOREM 1.9. *Let f and g be nonconstant meromorphic functions such that $\rho(f) > 2$. Suppose that $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 IM, where n, k are two positive integers satisfying $n > 9k+20$. If $\Theta(\infty, f) > 2/n$, then $f = g$.*

The following example shows that the assumption that $\Theta(\infty, f) > 2/n$ in Theorems 1.8 and 1.9 is necessary.

EXAMPLE 1.10. Let

$$f(z) = g(z)e^{z^3}, \quad g(z) = \frac{1 + e^{z^3} + e^{2z^3} + \cdots + e^{(n-1)z^3}}{1 + e^{z^3} + e^{2z^3} + \cdots + e^{nz^3}},$$

where n is any positive integer. Then

$$f^n(f-1) = g^n(g-1), \tag{1.3}$$

$$f(z) - 1 = -\frac{1}{1 + e^{z^3} + e^{2z^3} + \cdots + e^{nz^3}} \tag{1.4}$$

and

$$g(z) - 1 = -\frac{e^{nz^3}}{1 + e^{z^3} + e^{2z^3} + \cdots + e^{nz^3}}. \tag{1.5}$$

From (1.3)–(1.5) we have $\Theta(\infty, f) = \Theta(\infty, g) = 0$ and $\rho(f) = \rho(g) = 3$. Moreover, $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, where n and k are positive integers. But $f \not\equiv g$.

Finally, we will prove the following result to complement Theorems 1.8 and 1.9.

THEOREM 1.11. *Let f and g be nonconstant meromorphic functions such that $\rho(f) > 2$. Suppose that $(f(f^n-1))^{(k)}$ and $(g(g^n-1))^{(k)}$ share 1 CM, that 0 is a Picard exceptional value of f and g, and that every pole of f and g is of multiplicity greater than or equal to $2k+1$. If*

$$(n+1)(\delta(0, f^{n+1} - f) + \delta(0, g^{n+1} - g)) + (k+2)(\Theta(\infty, f) + \Theta(\infty, g)) > n + 2k + 5, \tag{1.6}$$

where $n \geq 4$ and k are two positive integers, then $f = g$.

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Proceeding as in the proof of Theorem 1.11 and using Lemmas 2.12 and 2.15, we get the following result which is an IM-analogue of Theorem 1.11.

Theorem 1.12. *Let f and g be nonconstant meromorphic functions such that $\rho(f) > 2$. Suppose that $(f(f^n - 1))^{(k)}$ and $(g(g^n - 1))^{(k)}$ share 1 IM, that 0 is a Picard exceptional value of f and g, and that every pole of f and g is of multiplicity $\geq 2k + 1$. If*

$$3(n+1)(\delta(0, f^{n+1} - f) + \delta(0, g^{n+1} - g)) + (2k+4)(\Theta(\infty, f) + \Theta(\infty, g)) > 5n + 4k + 13,$$

where $n \geq 4$ and k are two positive integers, then $f = g$.

From Theorems 1.11 and 1.12 and their proof in Section 3 below we get the following results, respectively.

Corollary 1.13. *Let f and g be nonconstant entire functions such that $\rho(f) > 2$. Suppose that $(f(f^n - 1))^{(k)}$ and $(g(g^n - 1))^{(k)}$ share 1 CM, and that 0 is a Picard exceptional value of f and g. If $\delta(0, f^{n+1} - f) + \delta(0, g^{n+1} - g) > 1$, where $n \geq 3$ and k are two positive integers, then $f = g$.*

Corollary 1.14. *Let f and g be nonconstant entire functions such that $\rho(f) > 2$. Suppose that $(f(f^n - 1))^{(k)}$ and $(g(g^n - 1))^{(k)}$ share 1 IM, and that 0 is a Picard exceptional value of f and g. If $\delta(0, f^{n+1} - f) + \delta(0, g^{n+1} - g) > 5/3$, where $n \geq 3$ and k are two positive integers, then $f = g$.*

2. Preliminaries

In this section, we introduce some important lemmas to prove the main results in this paper. First of all, we introduce the following result from Valiron–Mokhon'ko.

Lemma 2.1 (Valiron–Mokhon'ko lemma [12]). *Let f be a nonconstant meromorphic function, and let*

$$F = \frac{\sum_{k=0}^{p} a_k f^k}{\sum_{j=0}^{q} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r, F) = d\,T(r, f) + O(1)$, where $d = \max\{p, q\}$.

The following two results were established by Li and Yi [11], which is used to prove Theorems 1.8 and 1.9 respectively.

Lemma 2.2 [11, Lemma 2.5]. *Let F and G be two nonconstant meromorphic functions such that $F^{(k)} - P$ and $G^{(k)} - P$ share 0 CM, where $k \geq 1$ is a positive integer and P is a nonzero polynomial. If*

$$\Delta_1 = (k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G) > k + 7$$

and

$$\Delta_2 = (k+2)\Theta(\infty, G) + 2\Theta(\infty, F) + \Theta(0, G) + \Theta(0, F) + \delta_{k+1}(0, G) + \delta_{k+1}(0, F) > k + 7,$$

then $F^{(k)}G^{(k)} = P^2$ or $F = G$.

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

LEMMA 2.3 [11, Lemma 2.4]. *Let F and G be two transcendental meromorphic functions such that $F^{(k)} - P$ and $G^{(k)} - P$ share 0 IM, where $k \geq 1$ is a positive integer and P is a nonzero polynomial. If*

$$\begin{aligned}\Delta_3 = (2k+3)\Theta(\infty, F) + (2k+4)\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G)\\ + 2\delta_{k+1}(0, F) + 3\delta_{k+1}(0, G) > 4k + 13\end{aligned}$$

and

$$\begin{aligned}\Delta_4 = (2k+3)\Theta(\infty, G) + (2k+4)\Theta(\infty, F) + \Theta(0, G) + \Theta(0, F)\\ + 2\delta_{k+1}(0, G) + 3\delta_{k+1}(0, F) > 4k + 13,\end{aligned}$$

then $F^{(k)}G^{(k)} = P^2$ or $F = G$.

To prove Lemma 2.11 below, we need the following result.

LEMMA 2.4 [7, Lemma 3.5]. *Suppose that F is meromorphic in a domain D and set $f = F'/F$. Then, for $n \geq 1$,*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} f'^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$, and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Next we introduce some other results related to Zalcman's lemma, which can be found, for example, in [7, 16]. We will use Zalcman's lemma to prove our Lemma 2.10 which plays an important role in the proof of the main results of this paper.

First, we introduce the notation of the spherical derivative. Let f be a nonconstant meromorphic function. The spherical derivative of f at $z \in \mathbb{C}$ is given as $f^{\#}(z) = (|f'(z)|/1 + |f(z)|^2)$, and the order of f is defined as $\rho(f) = \limsup_{r\to\infty} (\log T(r, f)/\log r)$ (see, for example, [7, 16]).

LEMMA 2.5 [2, Lemma 1]. *Let f be a meromorphic function on $\mathbb{C}$. If f has bounded spherical derivative on $\mathbb{C}$, f is of order at most 2. If, in addition, f is entire, then the order of f is at most 1.*

LEMMA 2.6 [15, Theorem 1.24]. *Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then*

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leq N\Big(r, \frac{1}{f}\Big) + k\overline{N}(r, f) + O(\log T(r, f) + \log r),$$

as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

LEMMA 2.7 [15, Theorem 2.11]. *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f)$ is a positive integer or ∞.*

LEMMA 2.8 (Zalcman's Lemma [13, 19]). *Let F be a family of meromorphic functions in the unit disc $\triangle$ and α be a real number satisfying $-1 < \alpha < 1$. Then if F is not normal at a point $z_0 \in \triangle$, there exist, for each α with $-1 < \alpha < 1$:*

(i) *points $z_n \in \triangle$, $z_n \to z_0$;*
(ii) *positive numbers ρ_n, $\rho_n \to 0^+$; and*
(iii) *functions $f_n \in F$;*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically uniformly on compact subset of $\mathbb{C}$, where g is a nonconstant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \le g^{\#}(0) = 1$.

REMARK 2.9. Suppose additionally in Lemma 2.8 that F is a family of zero-free meromorphic functions in the domain D. Then the real number α in Lemma 2.8 can be such that $-1 < \alpha < \infty$.

LEMMA 2.10. *Let f and g be two nonconstant meromorphic functions and let n and k be positive integers such that $n > 2k$ and $k \ge 1$. If $(f^n)^{(k)}(g^n)^{(k)} = 1$, then f and g are transcendental entire functions such that $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are nonzero constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.*

PROOF. Suppose that $z_0 \in \mathbb{C}$ is a zero of f with multiplicity m. Then from the assumption $(f^n)^{(k)}(g^n)^{(k)} = 1$, we see that z_0 is a pole of g with multiplicity, say p, such that $mn - k = np + k$, and so $(m - p)n = 2k$, which contradicts the assumptions that $n > 2k$ and that m, p are positive integers. Therefore 0 is a Picard exceptional value of f. Similarly, we can prove that 0 is a Picard exceptional value of g.

Suppose that f and g are nonconstant rational functions. Then $f = 1/P$ and $g = 1/Q$, where P, Q are some two nonconstant polynomials. Combining this with the assumption that $(f^n)^{(k)}(g^n)^{(k)} = 1$, we have

$$\frac{(f^n)^{(k)}}{f^n} \cdot \frac{(g^n)^{(k)}}{g^n} = (PQ)^n. \tag{2.1}$$

By Lemma 2.4, $f = 1/P$ and $g = 1/Q$, we see that

$$\lim_{|z|\to\infty} \frac{(f^n(z))^{(k)}}{f^n(z)} \cdot \frac{(g^n(z))^{(k)}}{g^n(z)} = 0. \tag{2.2}$$

On the other hand, we have

$$\lim_{|z|\to\infty} (P(z)Q(z))^n = \infty. \tag{2.3}$$

From (2.1)–(2.3) we have a contradiction.

Next we suppose that f and g are transcendental meromorphic functions. To complete the proof of Lemma 2.10, we next set $F = \{f_\omega\}$ and $G = \{g_\omega\}$, where $f_\omega(z) = f(z + \omega)$ and $g_\omega(z) = g(z + \omega)$, $z \in \mathbb{C}$. Evidently, F and G are two families of meromorphic functions defined on $\mathbb{C}$. We discuss the following two cases.

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Case 1. Suppose that one of the families F and G, say F, is normal on $\mathbb{C}$. Then, by Marty's theorem, $f^{\#}(\omega) = f^{\#}_{\omega}(0) \le M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence, it follows from Lemma 2.5 that f is of order at most 2. This, together with the reasoning in Whittaker [14, page 82] and the assumption $(f^n)^{(k)}(g^n)^{(k)} = 1$, gives

$$\rho(f) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)}) = \rho(g^n) = \rho(g) \le 2. \tag{2.4}$$

Noting that 0 is a Picard exceptional value of g, we get from (2.4), Lemma 2.6 and the assumption $(f^n)^{(k)}(g^n)^{(k)} = 1$ that

$$\begin{aligned} (n+k)\overline{N}(r,f) &\le N(r,(f^n)^{(k)}) = N\Big(r, \frac{1}{(g^n)^{(k)}}\Big) \\ &\le N\Big(r, \frac{1}{g^n}\Big) + k\overline{N}(r,g^n) + O(\log r) \\ &\le k\overline{N}(r,g) + O(\log r), \end{aligned} \tag{2.5}$$

as $r \to \infty$. Similarly,

$$(n+k)\overline{N}(r,g) \le k\overline{N}(r,f) + O(\log r), \tag{2.6}$$

as $r \to \infty$. From (2.5) and (2.6) we have

$$\overline{N}(r,f) + \overline{N}(r,g) \le O(\log r),$$

as $r \to \infty$. This implies that f and g have at most finitely many poles. Noting that f and g are transcendental meromorphic functions, we can see from (2.4) and Lemma 2.7 that $\mu(f) = \rho(f) = 1$ or $\mu(f) = \rho(f) = 2$. Next we set

$$f = \frac{1}{P}e^{\alpha}, \quad g = \frac{1}{Q}e^{\beta}, \tag{2.7}$$

where P and Q are nonzero polynomials and α and β are nonconstant polynomials with degree at most 2. From (2.7) we have

$$(f^n)^{(k)} = \bigg(\Big(\alpha' - \frac{P'}{P}\Big)^k + P_{k-1}\Big(\alpha' - \frac{P'}{P}\Big)\bigg)\frac{A}{P^n}e^{n\alpha} \tag{2.8}$$

and

$$(g^n)^{(k)} = \bigg(\Big(\beta' - \frac{Q'}{Q}\Big)^k + P_{k-1}\Big(\beta' - \frac{Q'}{Q}\Big)\bigg)\frac{B}{Q^n}e^{n\beta}, \tag{2.9}$$

where A, B are nonzero constants and $P_{k-1}(\alpha' - P'/P)$ $(P_{k-1}(\beta' - Q'/Q))$ is a differential polynomial of degree at most $k-1$ in $\alpha' - P'/P$ $(\beta' - Q'/Q)$. By substituting (2.8) and (2.9) into $(f^n)^{(k)}(g^n)^{(k)} = 1$ we have

$$AB\bigg(\Big(\alpha' - \frac{P'}{P}\Big)^k + P_{k-1}\Big(\alpha' - \frac{P'}{P}\Big)\bigg)\bigg(\Big(\beta' - \frac{Q'}{Q}\Big)^k + P_{k-1}\Big(\beta' - \frac{Q'}{Q}\Big)\bigg)e^{n(\alpha+\beta)} = (PQ)^n. \tag{2.10}$$

Noting that P, Q, α, β are nonconstant polynomials, we deduce from (2.10) that

$$\alpha + \beta = c, \tag{2.11}$$

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

where c is some constant. Therefore, (2.10) can be rewritten as

$$AB\bigg(\Big(\alpha' - \frac{P'}{P}\Big)^k + P_{k-1}\Big(\alpha' - \frac{P'}{P}\Big)\bigg)\bigg(\Big(\beta' - \frac{Q'}{Q}\Big)^k + P_{k-1}\Big(\beta' - \frac{Q'}{Q}\Big)\bigg)e^{nc} = (PQ)^n. \quad (2.12)$$

From (2.11) we have $\alpha' + \beta' = 0$. Combining this with (2.12) and letting $|z| \to \infty$, we see that

$$2k\deg(\alpha') = n\deg(PQ). \quad (2.13)$$

Noting that $\deg(\alpha') \le 1$, we can deduce from (2.13) and the assumption $n > 2k$ that P and Q reduce to constants. This, together with (2.7), implies that ∞ is a Picard exceptional value of f and g. Combining this with Theorem 1 in Fang [3] and the assumption $n > 2k$, we get the conclusion of Lemma 2.10.

Case 2. Suppose that one of the families F and G, say F, is not normal on $\mathbb{C}$. Then, by Marty's theorem, we find that there exists a sequence of meromorphic functions $f_j(z) \subset F$, where $f_j(z) = f(\omega_j + z)$, $z \in \{z : |z| < 1\}$, and $\{\omega_j\} \subset \mathbb{C}$ is some infinite sequence of complex values, such that

$$f_j^{\#}(0) = f^{\#}(\omega_j) \to \infty,$$

as $|\omega_j| \to \infty$. By Lemma 2.8 we see that there exist:

(i) points $z_j \to 0$, $|z_j| < 1$;
(ii) positive numbers ρ_j, $\rho_j \to 0^+$;
(iii) a subsequence of functions $f_j(z_j + \rho_j\zeta) = f(\omega_j + z_j + \rho_j\zeta)$ of $\{f(\omega_j + z)\}$;

such that

$$\rho_j^{-k/n} f_j(z_j + \rho_j\zeta) =: h_j(\zeta) \to h(\zeta) \quad (2.14)$$

spherical uniformly on compact subsets of $\mathbb{C}$, where $h(\zeta)$ is some nonconstant meromorphic function such that $h^{\#}(\zeta) \le h^{\#}(0) = 1$. Moreover, from Lemma 2.5 we can see that $\rho(h) \le 2$, ρ_j is a positive number satisfying

$$\rho_j = \frac{1}{f_j^{\#}(z_j)} = \frac{1}{f^{\#}(b_j)} \quad (2.15)$$

and

$$f^{\#}(b_j) = f_j^{\#}(z_j) \ge f_j^{\#}(0) = f^{\#}(\omega_j), \quad (2.16)$$

where $b_j = \omega_j + z_j$, and (2.15), (2.16) can be found in the proof of Zalcman's lemma (see [13, 19]). From (2.14) we find that

$$(h_j^n(\zeta))^{(k)} = (f_j^n(z_j + \rho_j\zeta))^{(k)} \to (h^n(\zeta))^{(k)} \quad (2.17)$$

spherical uniformly on compact subsets of $\mathbb{C} \setminus h^{-1}(\infty)$ with respect to the spherical metric.

We claim that $(g^n)^{(k)}$ is not a constant. In fact, if $(g^n)^{(k)}$ is a constant, then $g^n = P_k$, where P_k is a nonconstant polynomial with degree at most $\le k$, which contradicts the assumption that $n > 2k$. Next we set

$$\tilde{h}_j(\zeta) = \rho_j^{-k/n} g_j(z_j + \rho_j\zeta). \quad (2.18)$$

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Then

$$(\tilde{h}_j^n(\zeta))^{(k)} = (g_j^n(z_j + \rho_j\zeta))^{(k)}. \tag{2.19}$$

From (2.17), (2.19) and the assumption $(f^n)^{(k)}(g^n)^{(k)} = 1$ we get

$$(h_j^n(\zeta))^{(k)}(\tilde{h}_j^n(\zeta))^{(k)} = 1. \tag{2.20}$$

From (2.17), (2.20) and the formula of higher derivatives we can deduce that

$$\tilde{h}_j^n(\zeta) \to \hat{h}(\zeta), \tag{2.21}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where $\hat{h}(\zeta)$ is some nonconstant meromorphic function in the complex plane. Moreover, by Hurwitz's theorem we can see that the multiplicity of every pole of $\hat{h}(\zeta)$ is a multiple of n. Combining (2.14), (2.18), (2.21) and Hurwitz's theorem, we find that 0 is a Picard exceptional value of f and we can deduce $\hat{h} = \tilde{h}^n$, where $\tilde{h}$ is some nonconstant meromorphic function in the complex plane. Therefore (2.21) can be rewritten as

$$\tilde{h}_j^n(\zeta) \to \tilde{h}^n(\zeta),$$

spherical uniformly on compact subsets of $\mathbb{C}$, and so

$$(\tilde{h}_j^n(\zeta))^{(k)} \to (\tilde{h}^n(\zeta))^{(k)}, \tag{2.22}$$

spherical uniformly on compact subsets of $\mathbb{C} \setminus \tilde{h}^{-1}(\infty)$ with respect to the spherical metric. From (2.17), (2.20) and (2.22) we get

$$(h^n(\zeta))^{(k)}(\tilde{h}^n(\zeta))^{(k)} = 1 \tag{2.23}$$

for all $\zeta \in \mathbb{C} \setminus \{h^{-1}(\infty) \cup \tilde{h}^{-1}(\infty)\}$. Proceeding as in the proof of (2.4), we get from (2.23) and $\rho(h) \le 2$ that

$$\rho(h) = \rho(\tilde{h}) \le 2. \tag{2.24}$$

Next, in the same manner as in Case 1, we get from (2.23) and (2.24) that

$$h(z) = \tilde{c}_1 e^{\tilde{c}z}, \quad \tilde{h}(z) = \tilde{c}_2 e^{-\tilde{c}z}, \tag{2.25}$$

where $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}$ are nonzero constants satisfying $(-1)^k(\tilde{c}_1\tilde{c}_2)^n(n\tilde{c})^{2k} = 1$. On the other hand, from (2.14) and the left equality of (2.25) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \frac{\rho_j f'(\omega_j + z_j + \rho_j\zeta)}{f(\omega_j + z_j + \rho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = \tilde{c}, \tag{2.26}$$

spherical uniformly on compact subsets of $\mathbb{C}$. From (2.15) and (2.26) we get

$$\rho_j\left|\frac{f'(\omega_j + z_j)}{f(\omega_j + z_j)}\right| = \frac{1 + |f(\omega_j + z_j)|^2}{|f'(\omega_j + z_j)|} \cdot \frac{|f'(\omega_j + z_j)|}{|f(\omega_j + z_j)|} = \frac{1 + |f(\omega_j + z_j)|^2}{|f(\omega_j + z_j)|} \to \left|\frac{h'(0)}{h(0)}\right| = |\tilde{c}|,$$

which implies that

$$\lim_{j\to\infty} f(\omega_j + z_j) \neq 0, \infty. \tag{2.27}$$

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

From (2.14) and (2.27) we deduce that

$$h_j(0) = \rho_j^{-k/n} f_j(z_j) = \rho_j^{-k/n} f(\omega_j + z_j) \to \infty. \tag{2.28}$$

Again from (2.14) and the left equality of (2.25) we have

$$h_j(0) \to h(0) = \tilde{c}_1. \tag{2.29}$$

From (2.28) and (2.29) we have a contradiction. This completes the proof of Lemma 2.10. □

Lemma 2.11. *Let f and g be two nonconstant meromorphic functions, and let n, k be positive integers such that $n > 2k$. If*

$$(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = 1, \tag{2.30}$$

then f and g are of order at most 2.

Proof. Suppose that one of f and g is rational function. Then from (2.30) we have $\rho(f) = \rho(g) = 0$. Next we suppose that f and g are transcendental meromorphic functions. To complete the proof of this lemma, we set $F = \{f_\omega\}$ and $G = \{g_\omega\}$, where $f_\omega(z) = f(z+\omega)$ and $g_\omega(z) = g(z+\omega)$, $z \in \mathbb{C}$. Evidently, F and G are two families of meromorphic functions defined on $\mathbb{C}$. We discuss the following two cases.

Case 1. Suppose that one of the families F and G, say F, is normal on $\mathbb{C}$. Then, by Marty's theorem, $f^\#(\omega) = f_\omega^\#(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence, it follows from Lemma 2.5 that f is of order at most 2.

Case 2. Suppose that one of the families F and G, say F, is not normal on $\mathbb{C}$. Then, by Marty's theorem, we find that there exists a sequence of meromorphic functions $f_j(z) \subset F$, where $f_j(z) = f(\omega_j + z)$, $z \in \{z : |z| < 1\}$, and $\{\omega_j\} \subset \mathbb{C}$ is some infinite sequence of complex values, such that

$$f_j^\#(0) = f^\#(\omega_j) \to \infty,$$

as $|\omega_j| \to \infty$. By Lemma 2.8 we find that there exist:

(i) points $z_j \to 0$, $|z_j| < 1$;
(ii) positive numbers ρ_j, $\rho_j \to 0^+$;
(iii) a subsequence of functions $f_j(z_j + \rho_j\zeta) = f(\omega_j + z_j + \rho_j\zeta)$ of $f(\omega_j + z)$;

such that

$$\rho_j^{-k/n} f_j(z_j + \rho_j\zeta) =: h_j(\zeta) \to h(\zeta) \tag{2.31}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where $h(\zeta)$ is some nonconstant meromorphic function such that $h^\#(\zeta) \leq h^\#(0) = 1$. Moreover, from Lemma 2.5 we can see that $\rho(h) \leq 2$, ρ_j is a positive number satisfying

$$\rho_j = \frac{1}{f_j^\#(z_j)} = \frac{1}{f^\#(b_j)} \tag{2.32}$$

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

and

$$f^{\#}(b_j) = f_j^{\#}(z_j) \geq f_j^{\#}(0) = f^{\#}(\omega_j), \tag{2.33}$$

where $b_j = \omega_j + z_j$, and (2.32), (2.33) can be found in the proof of Zalcman's lemma (see [13, 19]). From (2.31) we find that

$$(\rho_j^{k/n} h_j^{n+1}(\zeta) - h_j^n(\zeta))^{(k)} = (f_j^{n+1}(z_j + \rho_j\zeta) - f_j^n(z_j + \rho_j\zeta))^{(k)} \to -(h^n(\zeta))^{(k)} \tag{2.34}$$

spherical uniformly on compact subsets of $\mathbb{C} \setminus h^{-1}(\infty)$ with respect to the spherical metric.

We claim that $(g^{n+1} - g^n)^{(k)}$ is not a constant. In fact, if $(g^{n+1} - g^n)^{(k)}$ is a constant, then $g^{n+1} - g^n = P_k$, where P_k is a nonconstant polynomial with degree at most k, which contradicts the assumption $n > 2k$. Next we set

$$\tilde{h}_j(\zeta) = \rho_j^{-k/n} g_j(z_j + \rho_j\zeta).$$

Then

$$(\rho_j^{k/n} \tilde{h}_j^{n+1}(\zeta) - \tilde{h}_j^n(\zeta))^{(k)} = (g_j^{n+1}(z_j + \rho_j\zeta) - g_j^n(z_j + \rho_j\zeta))^{(k)}. \tag{2.35}$$

From (2.30), (2.34) and (2.35) we get

$$(\rho_j^{k/n} h_j^{n+1}(\zeta) - h_j^n(\zeta))^{(k)} (\rho_j^{k/n} \tilde{h}_j^{n+1}(\zeta) - \tilde{h}_j^n(\zeta))^{(k)} = 1. \tag{2.36}$$

Letting $j \to \infty$, we get from (2.34) and (2.36) that

$$(h^n(\zeta))^{(k)} (\hat{h}(\zeta))^{(k)} = 1$$

for all $\zeta \in \mathbb{C} \setminus \{h^{-1}(\infty) \cup \hat{h}^{-1}(\infty)\}$, where $\hat{h}$ is a meromorphic function such that

$$\rho_j^{k/n} \tilde{h}_j^{n+1}(\zeta) - \tilde{h}_j^n(\zeta) \to -\hat{h}(\zeta), \tag{2.37}$$

spherical uniformly on compact subsets of $\mathbb{C} \setminus \hat{h}^{-1}(\infty)$, which is deduced by the formula for higher derivatives, and $\hat{h}(\zeta)$ is some nonconstant meromorphic function in the complex plane. Moreover, by Hurwitz's theorem we can see that the multiplicity of every zero and every pole of $\hat{h}(\zeta)$ is a multipler of n. Hence $\hat{h} = \tilde{h}^n$, where $\tilde{h}$ is some nonconstant meromorphic function in the complex plane. Therefore (2.37) can be rewritten as

$$\rho_j^{k/n} \tilde{h}_j^{n+1}(\zeta) - \tilde{h}_j^n(\zeta) \to -\tilde{h}^n(\zeta), \tag{2.38}$$

spherical uniformly on compact subsets of $\mathbb{C} \setminus \tilde{h}^{-1}(\infty)$. From (2.34), (2.38) and (2.36) we get

$$(h^n(\zeta))^{(k)} (\tilde{h}^n(\zeta))^{(k)} = 1 \tag{2.39}$$

for all $\zeta \in \mathbb{C} \setminus \{h^{-1}(\infty) \cup \tilde{h}^{-1}(\infty)\}$. Next, in the same manner as in Case 2 of the proof of Lemma 2.10, we get a contradiction from (2.39). This completes the proof of Lemma 2.11. □

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Proceeding as in the proof of Lemma 2.11, we get the following result by Remark 2.9.

LEMMA 2.12. *Let f and g be two nonconstant meromorphic functions, and let n, k be positive integers. Suppose that 0 is a Picard exceptional value of f and g, and that every pole of f and g is of multiplicity $2k+1$ or greater. If*

$$(f(f^n-1))^{(k)}(g(g^n-1))^{(k)}=1,$$

then f and g are of order at most 2.

LEMMA 2.13 [20]. *If $s\,(>0)$ and t are relatively prime integers, and if c is a finite complex number such that $c^s=1$, then there exists one and only one common zero of ω^s-1 and ω^t-c.*

The following results are due to Yi [18], and will be used to prove Theorems 1.11 and 1.12 above.

LEMMA 2.14 [18, Theorem 2]. *Let f and g be two nonconstant meromorphic functions such that $f^{(k)}$ and $g^{(k)}$ share the value 1 CM, where $k\geq 1$ is a positive integer. If $\delta(0,f)+\delta(0,g)+(k+2)(\Theta(\infty,f)+\Theta(\infty,g))>2k+5$, then either $f=g$ or $f^{(k)}g^{(k)}=1$.*

LEMMA 2.15 [18, Theorem 4]. *Let f and g be two nonconstant meromorphic functions such that $f^{(k)}$ and $g^{(k)}$ share the value 1 IM, where $k\geq 1$ is a positive integer. If $3(\delta(0,f)+\delta(0,g))+(2k+4)(\Theta(\infty,f)+\Theta(\infty,g))>4k+13$, then either $f=g$ or $f^{(k)}g^{(k)}=1$.*

3. Proofs of the theorems

PROOF OF THEOREM 1.8. First of all, we set

$$F_1=f^n(f-1),\qquad G_1=g^n(g-1). \tag{3.1}$$

Then, from (3.1), Lemma 2.1 and the assumption $n>3k+11$,

$$\begin{aligned}\Theta(\infty,F_1)&=1-\limsup_{r\to\infty}\frac{\overline{N}(r,F_1)}{T(r,F_1)}=1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{(n+1)T(r,f)+O(1)}\\&\geq 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)+O(1)}=1-\frac{1}{n+1},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\Theta(0,F_1)&=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{F_1})}{T(r,F_1)}=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-1})}{(n+1)T(r,f)+O(1)}\\&\geq 1-\limsup_{r\to\infty}\frac{2T(r,f)+O(1)}{(n+1)T(r,f)+O(1)}=1-\frac{2}{n+1}\end{aligned} \tag{3.3}$$

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

and

$$\begin{aligned}\delta_{k+1}(0,F_1) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r,(1/F_1))}{T(r,F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{(k+1)\overline{N}(r,(1/f)) + N(r,(1/f-1))}{(n+1)T(r,f)+O(1)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(k+2)T(r,f)+O(1)}{(n+1)T(r,f)+O(1)} \\ &= 1 - \frac{k+2}{n+1}. \end{aligned} \tag{3.4}$$

Similarly, from the second equality of (3.1),

$$\Theta(\infty,G_1) \geq 1 - \frac{1}{n+1}, \quad \Theta(0,G_1) \geq 1 - \frac{2}{n+1}, \quad \delta_{k+1}(0,G_1) \geq 1 - \frac{k+2}{n+1}. \tag{3.5}$$

Next we set

$$\Delta_1 = (k+2)\Theta(\infty,F_1) + 2\Theta(\infty,G_1) + \Theta(0,F_1) + \Theta(0,G_1) + \delta_{k+1}(0,F_1) + \delta_{k+1}(0,G_1) \tag{3.6}$$

and

$$\Delta_2 = (k+2)\Theta(\infty,G_1) + 2\Theta(\infty,F_1) + \Theta(0,G_1) + \Theta(0,F_1) + \delta_{k+1}(0,G_1) + \delta_{k+1}(0,F_1). \tag{3.7}$$

Then, from (3.2)–(3.7),

$$\Delta_1 \geq (k+4)\Big(1 - \frac{1}{n+1}\Big) + 2\Big(1 - \frac{2}{n+1}\Big) + 2\Big(1 - \frac{k+2}{n+1}\Big) \tag{3.8}$$

and

$$\Delta_2 \geq (k+4)\Big(1 - \frac{1}{n+1}\Big) + 2\Big(1 - \frac{2}{n+1}\Big) + 2\Big(1 - \frac{k+2}{n+1}\Big). \tag{3.9}$$

By (3.8), (3.9) and the assumption $n > 3k+11$ we get $\Delta_1 > k+7$ and $\Delta_2 > k+7$. Combining this with Lemma 2.2 and the assumption that $F_1^{(k)}$ and $G_1^{(k)}$ share 1 CM, we get $F_1^{(k)}G_1^{(k)} = 1$ or $F_1 = G_1$. We consider the following two cases.

Case 1. Suppose that $F_1^{(k)}G_1^{(k)} = 1$. Then, from (3.1), we have (2.30). Combining this with Lemma 2.11, we have $\rho(f) \leq 2$, which contradicts the assumption $\rho(f) > 2$ of Theorem 1.8.

Case 2. Suppose that $F_1 = G_1$. Then, from (3.1), we have

$$f^n(f-1) = g^n(g-1). \tag{3.10}$$

Let

$$H = \frac{f}{g}. \tag{3.11}$$

We discuss the following two subcases.

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Subcase 2.1. Suppose that H is a nonconstant meromorphic function. Then, from (3.10) and (3.11), we get

$$g = \frac{1 - H^n}{1 - H^{n+1}}. \tag{3.12}$$

Noting that n and $n + 1$ are two relatively prime integers, we know that $\omega = 1$ is the only common zero of $\omega^n - 1$ and $\omega^{n+1} - 1$. Therefore, from (3.11), (3.12), Lemma 2.1 and Lemma 2.13 we get

$$T(r, f) = T(r, Hg) = nT(r, H) + O(1). \tag{3.13}$$

From (3.11)–(3.13) and the second fundamental theorem we get

$$\overline{N}(r, f) = \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{H - \lambda_j}\right) \geq (n - 2)T(r, H) + S(r, H), \tag{3.14}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are finite complex numbers satisfying $\lambda_j \neq 1$ and $\lambda_j^{n+1} = 1$ for $1 \leq j \leq n$. From (3.13) and (3.14) we get

$$\begin{aligned}\Theta(\infty, f) &= 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 - \limsup_{r\longrightarrow\infty} \frac{(n - 2)T(r, H) + S(r, H)}{nT(r, H)} \\ &\leq 1 - \frac{n - 2}{n} = \frac{2}{n},\end{aligned}$$

which contradicts the condition $\Theta(\infty, f) > 2/n$.

Subcase 2.2. Suppose that H is a constant. If $H^{n+1} \neq 1$, then from (3.10) and (3.11) we get (3.12). From (3.12) we know that g is a constant, which is impossible. Thus $H^{n+1} = 1$. From (3.10) and (3.11) we get

$$(H^{n+1} - 1)g = H^n - 1. \tag{3.15}$$

From (3.15) and $H^{n+1} = 1$ we get $H^{n+1} = H^n = 1$, which implies $H = 1$. This, together with (3.11), completes the proof of Theorem 1.8. □

Proof of Theorem 1.11. First of all, we set

$$F_2 = f(f^n - 1), \quad G_2 = g(g^n - 1). \tag{3.16}$$

Then, from (3.16) and Lemma 2.1,

$$\begin{aligned}\Theta(\infty, F_2) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, F_2)}{T(r, F_2)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{(n + 1)T(r, f) + O(1)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{(n + 1)T(r, f) + O(1)} = 1 - \frac{1}{n + 1}\end{aligned} \tag{3.17}$$

and

$$\Theta(\infty, G_2) \geq 1 - \frac{1}{n + 1}. \tag{3.18}$$

Next we set

$$\Delta_3 = \delta(0, F_2) + \delta(0, G_2) + (k + 2)(\Theta(\infty, F_2) + \Theta(\infty, G_2)). \tag{3.19}$$

Then, from (3.16)–(3.19) and Lemma 2.1, we find that the assumption (1.6) implies $\Delta_3 > 2k + 5$. Combining this with Lemma 2.14, we have $f(f^n - 1) = g(g^n - 1)$ or $(f(f^n - 1))^{(k)}(g(g^n - 1))^{(k)} = 1$. We consider the following two cases.

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

Case 1. Suppose that $(f(f^n - 1))^{(k)}(g(g^n - 1))^{(k)} = 1$. Then, from Lemma 2.12, we have $\rho(f) \le 2$, which contradicts the assumption $\rho(f) > 2$.

Case 2. Suppose that $f(f^n - 1) = g(g^n - 1)$. Then $P(f) = P(g)$, where $P(z) = z^{n+1} - z$. From the assumption $n \ge 4$ and Fujimoto [4, Theorem 4.1] we can deduce that $P(z) = z^{n+1} - z$ is a uniqueness polynomial. Therefore, $P(f) = P(g)$ implies that $f = g$, and so we get the conclusion of Theorem 1.11. An alternate demonstration that $P(f) = P(g)$ implies that $f = g$ goes as follows. Suppose that $f \not\equiv g$, and let H be defined as in (3.11). We consider the following two subcases.

Subcase 2.1. Suppose that H is a nonconstant meromorphic function. From the fact that 1 and $n + 1$ are two relatively prime integers, we know from Lemma 2.13 that $\omega = 1$ is the only common zero of $\omega^{n+1} - 1$ and $\omega - 1$. This, together with (3.11) and the assumption $f(f^n - 1) = g(g^n - 1)$, gives

$$g^n = \frac{H-1}{H^{n+1}-1} = \frac{1}{H^n + H^{n-1} + \cdots + H + 1}. \tag{3.20}$$

From (3.20) we find that every zero of $H - \omega_j$ is of multiplicity $n \ge 4$ for $1 \le j \le n$, where $\omega_1, \omega_2, \ldots, \omega_n$ are n distinct roots of $\omega^{n+1} = 1$ such that $\omega_j \ne 1$ for $1 \le j \le n$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n} \Theta(\omega_j, H) &= \sum_{j=1}^{n}\Big(1 - \limsup \frac{\overline{N}(r, (1/H - \omega_j))}{T(r,H)}\Big)\\ &\ge \sum_{j=1}^{n}\Big(1 - \limsup \frac{N(r, (1/H - \omega_j))}{nT(r,H)}\Big)\\ &\ge \sum_{j=1}^{n}\Big(1 - \frac{1}{n}\Big) \ge \sum_{j=1}^{4}\Big(1 - \frac{1}{4}\Big) = 3,\end{aligned}$$

which is impossible.

Subcase 2.2. Suppose that H is a constant. If $H^{n+1} \ne 1$, from (3.11) and the assumption $f(f^n - 1) = g(g^n - 1)$ we have (3.20), which implies that g is a constant, which is impossible. Therefore,

$$H^{n+1} = 1. \tag{3.21}$$

Again from (3.11) and the assumption $f(f^n - 1) = g(g^n - 1)$ we have $(H^{n+1} - 1)g^n = H - 1$. Combining this with (3.21), we have $H = 1$. This together with (3.11) completes the proof of Theorem 1.11. □

4. Comments on the proof of Theorem 4 [1]

There are two errors in the proof of Theorem 4 in the paper by Bhoosnurmath and Dyavarna [1, page 1203]. We comment on these points as follows.

Bhoosnurmath and Dyavanal wrote: 'Suppose that f, g are two nonconstant meromorphic functions satisfying $(f^n(f - 1))^{(k)}(g^n(g - 1))^{(k)} = 1$, where n, k are two

https://doi.org/10.1017/S0004972714000562 Published online by Cambridge University Press

positive integers such that $n \geq 3k + 13$. Then any zero z_1 of $f - 1$ of order p_1 is a zero of $(f^n(f-1))^{(k)}$ of order $p_1 - k$.'

Indeed, if z_1 is a zero of $f - 1$ of order $p_1 \geq k$, then z_1 is a zero of $(f^n(f-1))^{(k)}$ of order $p_1 - k$, and this reasoning is correct. But if z_1 is a zero of $f - 1$ of order at most $k - 1$, then z_1 is possibly not a zero of $(f^n(f-1))^{(k)}$. Moreover, even if z_1 is a zero of $(f^n(f-1))^{(k)}$, the order of z_1 as a zero of $(f^n(f-1))^{(k)}$ may not be equal to $p_1 - k$. Hence the reasoning of Bhoosnurmath and Dyavanal is defective for $p_1 \leq k - 1$.

Bhoosnurmath and Dyavanal also wrote: 'Let z_2 be a zero of f' of order p_2 that is not a zero of $f(f-1)$, then z_2 is a zero of $(f^n(f-1))^{(k)}$ of order $p_2 - (k-1)$.'

Obviously, if z_2 is a zero of f' of multiplicity $p_2 \geq k - 1$, then z_2 is a zero of $(f^n(f-1))^{(k)}$ of order $p_2 - (k-1)$, and this is correct. But if z_2 is a zero of f' of order $\leq k - 2$, then z_2 is possibly not a zero of $(f^n(f-1))^{(k)}$. Moreover, even if z_2 is a zero of $(f^n(f-1))^{(k)}$, the order of z_2 as a zero of $(f^n(f-1))^{(k)}$ may not be equal to $p_2 - (k-1)$. Hence the above reasoning of Bhoosnurmath and Dyavanal is defective for $p_2 \leq k - 2$.

5. Concluding remarks

Regarding Theorems 1.8 and 1.9, we propose the following question.

QUESTION 5.1. What can be said about the relationship between f and g, if the assumption that $\rho(f) > 2$ in Theorems 1.8–1.12 is replaced with $\rho(f) \leq 2$?

Acknowledgements

The authors wish to express their thanks to the referee for his/her valuable suggestions and comments. The authors also wish to express their thanks to the language copy editor for his/her valuable suggestions and comments for the authors of the present paper to complete the final revisions.

References

[1] S. S. Bhoosnurmath and R. S. Dyavanal, 'Uniqueness and value sharing of meromorphic functions', *Comput. Math. Appl.* **53** (2007), 1191–1205.

[2] J. M. Chang and L. Zalcman, 'Meromorphic functions that share a set with their derivatives', *J. Math. Anal. Appl.* **338** (2008), 1020–1028.

[3] M. L. Fang, 'Uniqueness and value sharing of entire functions', *Comput. Math. Appl.* **44** (2002), 823–831.

[4] H. Fujimoto, 'On uniqueness of meromorphic functions sharing finite sets', *Amer. J. Math.* **122** (2000), 1175–1203.

[5] H. Fujimoto, 'On uniqueness polynomials for meromorphic functions', *Nagoya Math. J.* **170** (2003), 33–46.

[6] H. Fujimoto, 'Finiteness of entire functions sharing a finite set', *Nagoya Math. J.* **185** (2007), 111–122.

[7] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).

[8] I. Lahiri, 'Uniqueness of meromorphic functions as governed by their differential polynomials', *Yokohama Math. J.* **44** (1997), 147–156.

[9] I. Lahiri, 'Weighted sharing of three values and uniqueness of meromorphic functions', *Kodai. Math. J.* **24** (2001), 421–435.
[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations* (Walter de Gruyter, Berlin, 1993).
[11] X. M. Li and H. X. Yi, 'Uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a polynomial', *Comput. Math. Appl.* **62** (2011), 539–550.
[12] A. Z. Mokhon'ko, 'On the Nevanlinna characteristics of some meromorphic functions', in: *Theory of Functions*, Functional Analysis and Their Applications, 14 (Izd-vo Khar'kovsk. Un-ta, 1971), 83–87.
[13] X. C. Pang, 'Normality conditions for differential polynomials', *Kexue Tongbao* **33** (1988), 1690–1693 (in Chinese).
[14] J. M. Whittaker, 'The order of the derivative of a meromorphic function', *J. Lond. Math. Soc. (2)* **11** (1936), 82–87.
[15] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions* (Kluwer Academic Publishers, Dordrecht, 2003).
[16] L. Yang, *Value Distribution Theory* (Springer, Berlin, 1993).
[17] C. C. Yang and X. H. Hua, 'Uniqueness and value-sharing of meromorphic functions', *Ann. Acad. Sci. Fenn. Math.* **22** (1997), 395–406.
[18] H. X. Yi, 'Uniqueness theorems for meromorphic functions whose n-th derivatives share the same 1-points', *Complex Var. Theory Appl.* **34** (1997), 421–436.
[19] L. Zalcman, 'A heuristic principle in complex function theory', *Amer. Math. Monthly* **82** (1975), 813–817.
[20] Q. C. Zhang, 'Meromorphic functions sharing three values', *Indian J. Pure Appl. Math.* **30** (1999), 667–682.

XIAO-MIN LI, Department of Mathematics, Ocean University of China,
Qingdao, Shandong 266100, PR China
e-mail: lixiaomin@ouc.edu.cn

HONG-XUN YI, Department of Mathematics, Shandong University,
Jinan, Shandong 250100, PR China
e-mail: hxyi@sdu.edu.cn