

SUMMATION AND UNCOUNTABLE BI-ORTHOGONAL SYSTEMS IN LOCALLY CONVEX SPACES

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1. Introduction

The purpose of this paper is to extend to locally convex spaces and to uncountable systems several well-known results concerning infinite series, bi-orthogonal sequences, and Schauder bases. Section 2 gives three extensions of the theorem of Orlicz (10) and Pettis (11) and some lemmas that will be needed later. The third section introduces the notions of a summability basis and a summability basis of subspaces, and two main theorems are proved, including a simplification of Retherford and McArthur's proof (12) of a theorem of Nikol'skii (9). Section 4 investigates the positive cone of an uncountable bi-orthogonal system, particularly conditions equivalent to the regularity of this cone.

Throughout this paper, we adopt the following notation: E will denote a topological vector space and E' its topological dual. A locally convex space will be assumed Hausdorff. Let A denote an arbitrary index set and \mathcal{F} denote the set of all nonempty finite subsets of A .

2. Summability and the Orlicz-Pettis Theorem

Let $(x_\alpha)_{\alpha \in A}$ be an indexed family in E . The series, denoted by $\Sigma_\alpha x_\alpha$, is summable (unconditionally Cauchy) provided the net $(\Sigma_{\alpha \in a} x_\alpha)_{a \in \mathcal{F}}$ converges (is Cauchy) in E where \mathcal{F} is directed by set inclusion. We write $\Sigma_\alpha x_\alpha = x$ if $x = \lim_a \Sigma_{\alpha \in a} x_\alpha$. For countable families A , summability of the series $\Sigma_\alpha x_\alpha$ is equivalent to unordered or unconditional convergence of the series. We shall need the following:

Lemma 1. *In a topological vector space, the series $\Sigma_\alpha x_\alpha$ is unconditionally Cauchy if and only if for every countable sub-family $\{\alpha_1, \alpha_2, \dots\}$ of A , the series $\sum_{i=1}^\infty x_{\alpha_i}$ is Cauchy.*

Proof. Suppose that $\Sigma_\alpha x_\alpha$ is not unconditionally Cauchy. We now construct a non-Cauchy countable series. Let U be a neighbourhood of θ such that for every $a \in \mathcal{F}$, there exists $b \in \mathcal{F}$ with $a \cap b = \emptyset$ and $\Sigma_{\alpha \in b} x_\alpha \notin U$. Let $a \in \mathcal{F}$, and choose $b_1 \in \mathcal{F}$ such that $a \cap b_1 = \emptyset$ and $\Sigma_{\alpha \in b_1} x_\alpha \notin U$. Now choose $b_2 \in \mathcal{F}$ such that $b_1 \cap b_2 = \emptyset$ and $\Sigma_{\alpha \in b_2} x_\alpha \notin U$. Similarly choose $b_n \in \mathcal{F}$ such that $b_n \cap (\bigcup_{i=1}^{n-1} b_i) = \emptyset$ and $\Sigma_{\alpha \in b_n} x_\alpha \notin U$. Clearly, $B = \bigcup_{i=1}^\infty b_i$ is countable and $\Sigma_{\alpha \in B} x_\alpha$ is not Cauchy. The converse is obvious.

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The following lemma is due to Robertson (13, see his remark on p. 152 concerning Theorem 1).

Lemma 2. *In a topological vector space, a series $\sum_{\alpha} x_{\alpha}$ is Cauchy if and only if the set $\{\sum_{\alpha \in a} x_{\alpha} : a \in \mathcal{F}\}$ is precompact. Hence, the set of finite partial sums is bounded.*

For locally convex spaces, we have the following extension by adapting McArthur’s argument (8, Lemma 4):

Lemma 3. *If $\sum_{\alpha} x_{\alpha}$ is a Cauchy series in a locally convex space E , then for each bounded set B of the Banach space m_A of bounded scalar-valued functions on A with supremum norm, the set*

$$S(B) = \{\sum_{\alpha \in a} b_{\alpha} x_{\alpha} : b = (b_{\alpha}) \in B \text{ and } a \in \mathcal{F}\}$$

is precompact.

The following remarkable theorem was first proved in Banach spaces by Orlicz (10) and Pettis (11) and extended to the following form by Grothendieck (2), McArthur (6) and Robertson (13):

Theorem. (Orlicz-Pettis) *In a locally convex space E , each subseries of a (countable) series converges with respect to the initial topology on E if and only if each subseries of the series converges with respect to the $\sigma(E, E')$ topology.*

For countable series, subseries convergence implies unconditional convergence (summability), and for sequentially complete spaces, the two notions are equivalent.

Theorem 1. *If E is a complete locally convex space, then $\sum_{\alpha} x_{\alpha}$ is summable whenever every countable subseries $\sum_{i=1}^{\infty} x_{\alpha_i}$ is $\sigma(E, E')$ -convergent.*

Proof. This is immediate from the completeness since Robertson’s proof (13) of the Orlicz-Pettis Theorem implies the series is unconditionally Cauchy, although not necessarily summable as the following example illustrates.

Let E be the subspace of all countably nonzero elements in an uncountable product $\prod_{\alpha \in A} \mathbf{R}$ of reals with the product topology. Let e_{α} be the element of E with 1 as the α coordinate and 0 elsewhere. Then every countable subfamily $(e_{\alpha_i})_{i=1}^{\infty}$ sums to the element with 1 at $\alpha_i (i = 1, 2, \dots)$ and 0 elsewhere, but the whole family sums to the element of $\prod_{\alpha \in A} \mathbf{R}$ with 1 in every coordinate, which, of course, is not in E .

Theorem 2. *If E is a $\sigma(E, E')$ -sequentially complete locally convex space then $\sum_{\alpha} x_{\alpha}$ is strongly summable if and only if $\sum_{\alpha} x_{\alpha}$ is $\sigma(E, E')$ -summable.*

Proof. It is necessary only to show that $\sum_{\alpha} x_{\alpha}$ is strongly summable when it is $\sigma(E, E')$ -summable. If $\sum_{\alpha} x_{\alpha}$ is $\sigma(E, E')$ -summable to x , then every countable subseries $\sum_{i=1}^{\infty} x_{\alpha_i}$ is $\sigma(E, E')$ -Cauchy by Lemma 1. The sequential completeness of E implies that $\sum_{i=1}^{\infty} x_{\alpha_i}$ $\sigma(E, E')$ -converges, and hence strongly converges

by the Orlicz-Pettis Theorem. From Lemma 1, we conclude that the series $\Sigma_\alpha x_\alpha$ is Cauchy in the original topology of E . Since $\Sigma_\alpha x_\alpha$ is $\sigma(E, E')$ -summable to x and strongly Cauchy, $\Sigma_\alpha x_\alpha$ is strongly summable to x .

A third variation is

Theorem 3. *Let E be a locally convex space. If $\Sigma_\alpha x_\alpha$ is $\sigma(E, E')$ -summable and every countable subseries $\sum_{i=1}^\infty x_{\alpha_i}$ $\sigma(E, E')$ -converges, then $\Sigma_\alpha x_\alpha$ is strongly summable.*

The proof is straightforward and will be omitted.

Remark. For uncountable index sets, the Orlicz-Pettis Theorem may be proved by applying Theorem 3 (which requires only the countable form of the Orlicz-Pettis Theorem in its proof).

In a locally convex space, every subseries of a series is summable if and only if every subseries is $\sigma(E, E')$ -summable.

3. Summability bases

In this section we shall state and prove the theorems in very general terms and restate them in more useful and concrete terms as corollaries. The following definitions are extensions of the well-known concepts for Banach spaces of a Schauder basis and a Schauder basis of subspaces (because of summability we must restrict our attention essentially to unconditional bases). The system $(x_\alpha; f_\alpha)_{\alpha \in A}$ is *bi-orthogonal* provided $\{x_\alpha: \alpha \in A\} \subset E$ and $\{f_\alpha: \alpha \in A\} \subset E'$ and $f_\alpha(x_\beta) = \delta_{\alpha\beta}$, the Kronecker symbol. An (*unconditional*) *Schauder basis* for E is a countable bi-orthogonal system $(x_n; f_n)$ such that $\sum_{n=1}^\infty f_n(x)x_n$ (unconditionally) converges to x for every $x \in E$. A bi-orthogonal system $(x_\alpha; f_\alpha)_{\alpha \in A}$ is a *summability basis* for E provided $\Sigma_\alpha f_\alpha(x)x_\alpha$ is summable to x for each $x \in E$. (For countable systems, a summability basis is an unconditional Schauder basis and vice versa.) For a bi-orthogonal system $(x_\alpha; f_\alpha)_{\alpha \in A}$, define

$$S_a(x) = \Sigma_{\alpha \in a} f_\alpha(x)x_\alpha$$

for each $x \in E$ and each $a \in \mathcal{F}$. Then S_a is the *a-partial sum operator* of $(x_\alpha; f_\alpha)$. A summability basis $(x_\alpha; f_\alpha)$ is *equicontinuous* if $\{S_a: a \in \mathcal{F}\}$ is an equicontinuous family of operators on E . Summability bases have received recent attention by Knowles (4) and Marti (5).

Let $(E_\alpha)_{\alpha \in A}$ be a family of subspaces of E such that there is a unique family $(P_\alpha)_{\alpha \in A}$ of continuous projections on E with $P_\alpha(E) = E_\alpha$ and $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$ and $\Sigma_\alpha P_\alpha(x)$ is summable to x for each $x \in E$. Then $(E_\alpha; P_\alpha)$ is a *summability basis of subspaces* for E . The *a-partial sum operator* for $(E_\alpha; P_\alpha)$ is $S_a = \Sigma_{\alpha \in a} P_\alpha$ for each $a \in \mathcal{F}$. The basis $(E_\alpha; P_\alpha)_{\alpha \in A}$ is *equicontinuous* if $\{S_a: a \in \mathcal{F}\}$ is an equicontinuous family of operators on E . Every summability basis $(x_\alpha; f_\alpha)_{\alpha \in A}$ corresponds obviously to the summability basis of subspaces $(E_\alpha; P_\alpha)_{\alpha \in A}$ where $E_\alpha = \text{span}(x_\alpha)$ and $P_\alpha(x) = f_\alpha(x)x_\alpha$ for each x and each α .

Remark. A summability basis in a barrelled space is necessarily equicontinuous (by Lemma 2 and (14, Theorem 3, p. 69)).

We shall use the following lemma whose proof is indicated since we could find no direct reference.

Lemma 4. *Let E and F be topological vector spaces with F complete. Let S be a family of continuous linear functions from E into F and let S^* denote the family of continuous extensions of the functions in S to the completion E^* of E . Then S is equicontinuous if and only if S^* is equicontinuous.*

Proof. This is clear since the closed neighbourhoods of E^* are exactly the closures (in E^*) of neighbourhoods in E (1, p. 62) and the closed neighbourhoods of F form a neighbourhood base.

One common example of a summability basis of subspaces occurs in the product E of the spaces $\{E_\alpha: \alpha \in A\}$; for, if P_α is the canonical projection of E on to E_α and if E has the product topology, then $(E_\alpha; P_\alpha)_{\alpha \in A}$ is a summability basis of subspaces. This will be an equicontinuous basis when the spaces $E_\alpha (\alpha \in A)$ are barrelled since E will be barrelled (14, p. 94).

The equicontinuity condition is important because it allows us to extend the basis, if necessary, to the completion.

Theorem 4. *Let E be a Hausdorff topological vector space and let $(E_\alpha)_{\alpha \in A}$ be a family of complete subspaces with $M = \text{span}(\cup_{\alpha \in A} E_\alpha)$ and M' , the closure of M in E . If there exist a family of continuous projections $(P_\alpha)_{\alpha \in A}$ on M such that $(E_\alpha; P_\alpha)_{\alpha \in A}$ is an equicontinuous summability basis of subspaces for M , and if $P'_\alpha (\alpha \in A)$ is the continuous extension of P_α to M' and $S'_a = \sum_{\alpha \in a} P'_\alpha (a \in \mathcal{F})$, then $(E_\alpha; P'_\alpha)$ is an equicontinuous summability basis for M' .*

Proof. Clearly, the image of P'_α is E_α ; P'_α is a continuous projection; and $P'_\alpha P'_\beta = 0$ for $\alpha \neq \beta$. Let $x \in M'$. We complete the proof by showing that

$$\lim_a S'_a(x) = x.$$

Let V be a neighbourhood of θ in M' . There is a balanced neighbourhood U of θ such that $U + U + U \subset V$. Since $\{S'_a: a \in \mathcal{F}\}$ is equicontinuous by Lemma 4, there is a balanced neighbourhood W of θ such that $S'_a(W) \subset U$ for all $a \in \mathcal{F}$. Now there is $y \in M$ such that $y - x \in W \cap U$. Hence,

$$S'_a(y - x) = S'_a(y) - S'_a(x) \in U$$

for every $a \in \mathcal{F}$. Since (E_α, P_α) is a summability basis for M , there exists $a \in \mathcal{F}$ such that

$$y - S'_b(y) = y - S_b(y) \in U$$

for every $b \in \mathcal{F}$ with $a \subset b$. Let $a \subset b \in \mathcal{F}$. Then

$$x - S'_b(x) = (x - y) + (y - S'_b(y)) + (S'_b(y) - S'_b(x)) \in U + U + U \subset V.$$

Thus, $\lim_a S'_a(x) = x$. for every $x \in M'$.

We have the following obvious corollaries:

Corollary 4.1. *Let E be a Hausdorff topological vector space and let $(x_\alpha; f_\alpha)_{\alpha \in A}$ be a bi-orthogonal system which is a summability basis for the span M of $(x_\alpha)_{\alpha \in A}$. If $S_a(x) = \sum_{\alpha \in a} f_\alpha(x)x_\alpha$ for $x \in E$ and $a \in \mathcal{F}$ and if $\{S_a: a \in \mathcal{F}\}$ is equicontinuous, then $(x_\alpha; f_\alpha)$ is a summability basis for M' , the closure of M .*

Corollary 4.2. *Let E be a Hausdorff topological vector space and let $(x_n; f_n)$ be an equicontinuous Schauder basis for the span M of (x_n) . Then $(x_n; f_n)$ is an equicontinuous Schauder basis for M -closure.*

The following theorem (7, Theorem 1) is due to McArthur and is stated here for the sake of completeness.

Theorem (McArthur). *Let $\{S_i\}_{i \in I}$ be a family of linear functions from a vector space E into a topological vector space (F, \mathcal{T}) and suppose that*

$$\{S_i(x): i \in I\}$$

is bounded for each $x \in E$. Then there exists a weakest vector topology \mathcal{T}' for E such that $\{S_i\}_{i \in I}$ is $\mathcal{T}' - \mathcal{T}$ equicontinuous. If \mathcal{T} is locally convex, then so is \mathcal{T}' . If (F, \mathcal{T}) is Hausdorff, then (E, \mathcal{T}') is Hausdorff if and only if $S_i(x) = \theta$ for all $i \in I$ implies $x = \theta$. If (F, \mathcal{T}) is locally convex and \mathcal{T} is generated by a family $\Gamma = \{p\}$ of seminorms then \mathcal{T}' is generated by the family $\Gamma' = \{p'\}$ of seminorms where for each $p \in \Gamma$, $p'(x) = \sup_{i \in I} p(S_i(x))$.

The topology \mathcal{T}' defined in McArthur's Theorem is called the projective equicontinuous topology on E .

The following extends the theorem of Nikol'skiĭ (9).

Theorem 5. *Let (E, \mathcal{T}) be a locally convex space and let $(E_\alpha)_{\alpha \in A}$ be a family of complete subspaces of E such that the span of $\cup_{\alpha \in A} E_\alpha$ is dense in E and $E_\alpha \cap E_\beta = \{\theta\}$ for all $\alpha \neq \beta$. Let Γ be a family of seminorms generating the topology \mathcal{T} . If for each $p \in \Gamma$, there exist $q \in \Gamma$ and a positive constant $K = K(p)$ such that*

$$p(\sum_{\alpha \in a} x_\alpha) \leq Kq(\sum_{\alpha \in b} x_\alpha) \tag{1}$$

for arbitrary $a, b \in \mathcal{F}$ with $a \subset b$ and arbitrary $x_\alpha \in E_\alpha$ ($\alpha \in b$), then (E_α) is a summability basis of subspaces for E .

Proof. Let M be the span of $\cup_\alpha E_\alpha$. If $\sum_\alpha x_\alpha = \theta$ and $x_\alpha \in E_\alpha$, then for $p \in \Gamma$ and fixed $\alpha_0 \in A$, we have $p(x_{\alpha_0}) \leq 2Kq(\sum_{\alpha \in a} x_\alpha)$ for all $a \in \mathcal{F}$ with $\alpha_0 \in a$. Since E is Hausdorff and the right-hand side converges to 0 on \mathcal{F} , we have $x_{\alpha_0} = \theta$. Define $P_\alpha(x) = x_\alpha$ for $x \in M$ and $\alpha \in A$. For $a \in \mathcal{F}$, let $S_a = \sum_{\alpha \in a} P_\alpha$. Let \mathcal{T}' be the projective equicontinuous topology on M defined as above using the family $\{S_a: a \in \mathcal{F}\}$ and the topology \mathcal{T} on E . Then $\{S_a(x): a \in \mathcal{F}\}$ and finite, and hence bounded, for each $x \in M$; and we may apply McArthur's Theorem. Thus, \mathcal{T}' is defined by the family $\Gamma' = \{p': p \in \Gamma\}$ of seminorms

where $p'(x) = \sup_{a \in \mathcal{F}} p(\sum_{\alpha \in a} x_\alpha)$ for each $x \in M$ and $p \in \Gamma$; and $\{S_a: a \in \mathcal{F}\}$ is $\mathcal{T}' - \mathcal{T}$ equicontinuous. It is easy to see that (1) implies $p'(x) \leq Kq(x)$, and of course, $p(x) \leq p'(x)$ for all $x \in M$ and $p \in \Gamma$. Hence, \mathcal{T}' is equivalent to the topology induced on M by \mathcal{T} and $\{S_a\}$ is equicontinuous for \mathcal{T} . Thus $(E_\alpha; P_\alpha)$ is an equicontinuous summability basis for M . By Theorem 4, $(E_\alpha; P'_\alpha)$ is an equicontinuous summability basis for E , the closure of M .

Corollary 5.1. *Let (E, \mathcal{T}) be a locally convex space and let $(x_\alpha)_{\alpha \in A}$ be a family in E with $x_\alpha \neq \theta$ and the span of (x_α) dense in E . Let Γ be a family of seminorms generating the topology \mathcal{T} . If for each $p \in \Gamma$, there exist $q \in \Gamma$ and a positive constant $K = K(p)$ such that*

$$p(\sum_{\alpha \in a} \lambda_\alpha x_\alpha) \leq Kq(\sum_{\alpha \in b} \lambda_\alpha x_\alpha)$$

for arbitrary scalars $\lambda_\alpha (\alpha \in b)$ and for arbitrary $a, b \in \mathcal{F}$ with $a \subset b$, then $(x_\alpha)_{\alpha \in A}$ is a summability basis for E .

The next corollary removes the completeness requirement in the theorem (12, Theorem 3.1) of Retherford and McArthur for locally convex spaces.

Corollary 5.2. *Let (E, \mathcal{T}) be a locally convex space and let $(x_n)_{n=1}^\infty$ be a sequence in E with $x_n \neq \theta$ for all n and the span of (x_n) dense in E . Let Γ be a family of seminorms generating the topology \mathcal{T} . If for each $p \in \Gamma$, there exist $q \in \Gamma$ and a positive constant $K = K(p)$ such that*

$$p\left(\sum_{n=1}^k \lambda_n x_n\right) \leq Kq\left(\sum_{n=1}^{k+m} \lambda_n x_n\right)$$

for arbitrary scalars (λ_n) and positive integers k and m , then (x_n) is a Schauder basis for E .

Proof. Note that the basis in this case may be a conditional Schauder basis. The above proof is modified by using the family $\{S_n\}$ of operators defined by

$$S_n\left(\sum_{i=1}^k \lambda_i x_i\right) = \begin{cases} \sum_{i=1}^n \lambda_i x_i, & \text{for } n \leq k, \\ \sum_{i=1}^k \lambda_i x_i, & \text{for } k < n, \end{cases}$$

instead of the unordered partial sums $\{S_a\}$.

A corresponding result for a basis of complete subspaces also holds.

Since condition (1) of Theorem 5 is essentially the equicontinuity of the finite partial sum operators, we have the following partial converse which includes all Banach, Fréchet, and reflexive spaces:

Theorem 6. *If E is barrelled and has a summability basis (of subspaces), then condition (1) holds.*

4. Positive cones of bi-orthogonal systems

Let E be a real vector space. A set K in E is a *cone* provided:

- (i) $K + K \subset K$,
- (ii) $\lambda K \subset K$ for all $\lambda \geq 0$, and
- (iii) $K \cap (-K) = \{\theta\}$.

A set satisfying (i) and (ii) is called a *wedge*. A cone K in E induces a partial ordering \leq on E by $x \leq y$ if and only if $y - x \in K$. If E is ordered in this way by a cone, we call E an *ordered vector space*. Clearly, $K = \{x: \theta \leq x\}$.

Let $(x_\alpha; f_\alpha)_{\alpha \in A}$ be a bi-orthogonal system on E . Let

$$K = \{x \in E: f_\alpha(x) \geq 0 \text{ for all } \alpha \in A\}.$$

Clearly, K is a wedge in E and K will be a cone if and only if $\{f_\alpha: \alpha \in A\}$ is total over E , i.e. $f_\alpha(x) = 0$ for $\alpha \in A$ implies $x = \theta$. We call K the *positive cone* of the bi-orthogonal system $(x_\alpha; f_\alpha)$ (when $\{f_\alpha\}$ is total).

Let E be ordered by the cone K , and let $[x, y]$ denote the order interval

$$\{z: x \leq z \leq y\} = (x + K) \cap (y - K).$$

Then K is *regular* (*sequentially regular*) provided every increasing net (sequence) in K which is contained in an order interval converges to an element of K . The following is an extension to general bi-orthogonal systems of a recent theorem of McArthur (8).

Theorem 7. *Let E be a locally convex space and $(x_\alpha; f_\alpha)_{\alpha \in A}$ a bi-orthogonal system on E with $\{f_\alpha\}$ total over E . Let E be ordered by the positive cone K of $(x_\alpha; f_\alpha)$. The following statements are equivalent:*

- (i) *For each $x \in K$ and each bounded non-negative real-valued function $(\lambda_\alpha)_{\alpha \in A}$, the series $\sum_\alpha \lambda_\alpha f_\alpha(x) x_\alpha$ is summable,*
- (ii) *$[\theta, x]$ is compact for each $x \in K$,*
- (iii) *$[\theta, x]$ is weakly compact for each $x \in K$,*
- (iv) *K is weakly regular,*
- (v) *K is regular,*
- (vi) *$[\theta, x]$ is complete, $\sigma(E, E')$ -sequentially complete and bounded for each $x \in K$.*

The above conditions imply the following condition and are equivalent to it if $\sum_\alpha f_\alpha(x) x_\alpha$ converges for every $x \in K$:

- (vii) *The supremum of each order bounded subset of K exists and is an element of K .*

If K is complete, then (i)-(vi) are also equivalent to the following statements:

- (viii) *K is sequentially regular,*
- (ix) *K is weakly sequentially regular.*

Proof. With obvious modifications, based on Lemma 3 for (i)⇒(ii) McArthur’s proof extends for the following implications:

(i)⇒(ii)⇒(iii)⇒(iv)⇒(ix)⇔(viii) and (ii)⇒(v)⇒(i)⇒(vii) and (v)⇒(iv), and clearly, ((ii) and (iii))⇒(vi). Thus, to complete the proof we need only show that (iv)⇒(i), (vi)⇒(i), (viii)⇒(i) (if K is complete) and (vii)⇒(i) (if $\sum_a f_a(x)x_a$ is summable for $x \in K$). Let $(\lambda_a)_{a \in A}$ be a bounded non-negative real-valued function on A and $x \in K$. In each case we must show that $\sum_a \lambda_a f_a(x)x_a$ is summable.

Suppose (iv) holds. If $\lambda = \sup_a \lambda_a$, then $\theta \leq \sum_{a \in \mathcal{F}} \lambda_a f_a(x)x_a \leq \lambda x$ for all $a \in \mathcal{F}$. In addition, if $a \subset b$ and $a, b \in \mathcal{F}$, then $\sum_{a \in a} \lambda_a f_a(x)x_a \leq \sum_{a \in b} \lambda_a f_a(x)x_a$. Hence, $\sum_a \lambda_a f_a(x)x_a$ and every subseries of it are $\sigma(E, E')$ -summable by (iv). By Theorem 3, $\sum_a \lambda_a f_a(x)x_a$ is strongly summable.

Assume (vi) holds. Then $\{\sum_{a \in a} \lambda_a f_a(x)x_a : a \in \mathcal{F}\}$ is bounded since it is a subset of $[\theta, \lambda x]$, which is bounded. Hence, for each

$$f \in E', \sum_a |\lambda_a f_a(x)f(x_a)| < +\infty.$$

It follows that each countable subseries is $\sigma(E, E')$ -Cauchy and by hypothesis, $\sigma(E, E')$ -convergent. Hence, as in the proof of Theorem 2, $\sum_a \lambda_a f_a(x)x_a$ is strongly Cauchy and by the assumed completeness of $[\theta, \lambda x]$, is summable.

Next, assume K is complete and sequentially regular. Then every countable subseries $\sum_{i=1}^\infty \lambda_{a_i} f_{a_i}(x)x_{a_i}$ is increasing and bounded above by λx , and hence convergent. Now by Lemma 1, the series $\sum_a \lambda_a f_a(x)x_a$ is Cauchy and by the completeness of K , is summable.

We conclude the proof of the theorem by noting that McArthur’s proof for (vii)⇒(i) (our numbers) requires the added hypothesis that $\sum_a f_a(x)x_a$ converge for all $x \in K$ (even in the countable case) and that this is sufficient by the extension of (8, Lemma 5). Consider the space m of bounded sequences with the bi-orthogonal sequence $(e_n; f_n)$ where e_n is the sequence with 1 in the n th coordinate and 0 elsewhere. Then K is the usual positive cone of m which has property (vii). The sequence $\left(\sum_{n=1}^k e_n\right)_k$ (determined by $(1) \in K$) does not converge in m (contradicting (i)).

The order interval $[\theta, x]$ will be metrisable if and only if $\{\alpha; f_a(x) \neq 0\}$ is countable (3).

Corollary 7.1. *Let E be a locally convex space, $(x_a; f_a)_{a \in A}$ a bi-orthogonal system on E with $\{f_a\}$ total over E , and K the positive cone of $(x_a; f_a)$. If K is regular and $E = K - K$, then $(x_a; f_a)$ is a summability basis for E . If K is complete, the converse is true.*

Proof. For the first part, we need only show that $\sum_a f_a(x)x_a$ is summable to x for every $x \in E$. Since $E = K - K$, it suffices to consider only $x \in K$. Let $x \in K$. By (i) in Theorem 7, $\sum_a f_a(x)x_a$ is summable. By the bi-orthogonality and totality of $\{f_a\}$, the sum must be x . Conversely, suppose that K is complete and $(x_a; f_a)$

is a summability basis. Let $x \in K$ and $(\lambda_\alpha)_{\alpha \in A}$ be a bounded non-negative real-valued function on A . Now $\sum_\alpha f_\alpha(x)x_\alpha$ is summable, and thus, by Lemma 3, $\{\sum_{\alpha \in a} \lambda_\alpha f_\alpha(x)x_\alpha : a \in \mathcal{F}\}$ is precompact. Lemma 2 and the completeness of K imply that $\sum_\alpha \lambda_\alpha f_\alpha(x)x_\alpha$ is summable. Hence, condition (i) of Theorem 7 is satisfied and K is regular. Now let $x \in E$. Let $B = \{\alpha \in A : f_\alpha(x) \geq 0\}$ and $C = \{\alpha \in A : f_\alpha(x) < 0\}$. By condition (i) also, $\sum_{\alpha \in B} f_\alpha(x)x_\alpha$ is summable to, say, y and $\sum_{\alpha \in C} f_\alpha(x)x_\alpha$ is summable to, say, $-z$. Clearly, $y \in K$ and $z \in K$ and $x = y - z$. Hence, $K - K = E$.

We now consider two examples. Let A be uncountable and consider the product $\prod_{\alpha \in A} \mathbf{R}$ of reals with the product topology. Let E be the countably non-zero elements and $(e_\alpha; f_\alpha)_{\alpha \in A}$ the "unit vector" basis mentioned earlier. The positive cone of $(e_\alpha; f_\alpha)$ is not complete but is regular (by condition (i) of Theorem 7). Let F represent the countable nonconstant elements of the product. The system $(e_\alpha; f_\alpha)$ is again a summability basis, but the positive cone is not complete and not regular (because condition (i) fails), but is sequentially regular (since the coordinatewise supremum is in F and is clearly the limit of an increasing sequence).

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