Ergod. Th. & Dynam. Sys., (2023), **43**, 794–826 © The Author(s), 2022. Published by Cambridge University Press.
doi:10.1017/etds.2021.158

Actions of symplectic homeomorphisms/diffeomorphisms on foliations by curves in dimension 2

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(*Received* 18 *October* 2020 *and accepted in revised form* 2 *November* 2021)

Abstract. The two main results in this paper concern the regularity of the invariant foliation of a C^0-integrable symplectic twist diffeomorphism of the two-dimensional annulus, namely that (i) the generating function of such a foliation is C^1, and (ii) the foliation is Hölder with exponent $\frac{1}{2}$. We also characterize foliations by graphs that are straightenable via a symplectic homeomorphism and prove that every symplectic homeomorphism that leaves invariant all the leaves of a straightenable foliation has Arnol'd–Liouville coordinates, in which the dynamics restricted to the leaves is conjugate to a rotation. We deduce that every Lipschitz integrable symplectic twist diffeomorphisms of the two-dimensional annulus has Arnol'd–Liouville coordinates and then provide examples of 'strange' Lipschitz foliations by smooth curves that cannot be straightened by a symplectic homeomorphism and cannot be invariant by a symplectic twist diffeomorphism.

Key words: twist maps, symplectic homeomorphisms, generating functions, integrability
2020 Mathematics Subject Classification: 37E40, 37J11, 37J70 (Primary); 37E10, 37J30 (Secondary)

1. *Introduction and main results*

1.1. *Main results.* This paper deals with foliations by curves in a two-dimensional symplectic setting. The questions we raise for such a foliation are as follows.

- When is it (locally or globally) symplectically homeomorphic to the straight foliation? (This will be defined precisely later.)

- What can be said on the foliation when it is invariant by a symplectic twist? (In this paper, we will say that a foliation is invariant by f if every leaf is (globally) invariant.)
- What can be said on a symplectic dynamics that preserves such a foliation?

Before going into detail, let us explain our motivations.

The first problem in which we were interested is the possible extension of the Arnol'd–Liouville theorem (see, for example, [6]). This classical theorem concerns Hamiltonian dynamics associated to a C^2 Hamiltonian function endowed with a complete system of independent commuting C^2 integrals. Then there exists an invariant C^2 foliation by Lagrangian submanifolds, and in the neighbourhood of every compact leaf of this foliation there exist symplectic C^1 angle-action coordinates $H : (q, p) \in \mathcal{U} \subset M \mapsto (\theta, I) \in U \subset \mathbb{T}^n \times \mathbb{R}^n$ such that at these coordinates:

- the invariant foliation is the straight foliation $I =$ constant;
- the flow is $(\theta, I) \mapsto (\theta + t\nabla h(I), I)$ where h is a C^2 function.

In fact, there are two steps in this result.

- The first step consists in symplectically straightening the foliation via the chart H. The diffeomorphism H is defined via its generating function $S(q, I)$. We recall that

$$H(q, p) = (\theta, I) \Longleftrightarrow \theta = \frac{\partial S}{\partial I}(q, I) \quad \text{and} \quad p = \frac{\partial S}{\partial q}(q, I).$$

  A priori this generating function S is only C^2 as the foliation was and the diffeomorphism H is only C^1, but because the invariant foliation is C^2, we can say a little more: when I is fixed, $\Phi = H^{-1}$ is C^2 in the θ-variables.
- Then the second step consists in noticing that a symplectic flow that preserves every leaf of the straight foliation has to be a flow of rotations on every leaf.

In [3], the hypothesis concerning the regularity of the invariant foliation was relaxed and the invariant foliation was just assumed to be C^1. In this case, when the Hamiltonian satisfies the so-called A-non degeneracy condition (which contains the case of Tonelli Hamiltonians), the authors proved the existence of a symplectic homeomorphism H straightening the invariant Lagrangian foliation, such that H^{-1} is C^1 in the θ variable and such that the flow is written in the chart $(\theta, I) \mapsto (\theta + t\nabla h(I), I)$ where h is a C^1 function.

Here we raise the problem of invariant C^0 foliation by invariant C^0-Lagrangian tori. In high dimension, the first problem is to define what is a C^0-foliation by Lagrangian tori. An interesting discussion on this topic is provided in the appendix to [3], but here we will consider the simplest case: in dimension 2, any foliation by curves can be seen as Lagrangian. Also we will assume that the foliations that we consider are not too complicated, because they are (at least locally in C^1 charts) foliations by graphs.

The dynamics will be that of symplectic twist diffeomorphisms. Let us give the definition here; it will be detailed and explained later in the text.

Definition. A *symplectic twist diffeomorphism* $f : \mathbb{A} \to \mathbb{A}$ is a C^1 diffeomorphism such that:

- f is isotopic to the identity;
- f is symplectic, that is, $f^*\omega = \omega$ where ω is the standard symplectic form on $\mathbb{A}$;

- f has the *twist property*, that is, if $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f, for any $\tilde{\theta} \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde{\theta}, r) \in \mathbb{R}$ is an increasing C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Even in this setting and for a symplectic twist diffeomorphism of the two-dimensional annulus $\mathbb{A} = \mathbb{T} \times \mathbb{R}$, there exist results in which the authors are able to prove the existence of such an invariant continuous foliation by curves that are graphs (see, for example, [5] or [8]) but not able to say more (e.g. to describe the dynamics or prove that the foliation is symplectically straightenable). Observe too that the case of a Tonelli Hamiltonian with two degrees of freedom corresponds to the case of twist maps by using a Poincaré section close to some invariant torus in an energy surface, and that in this setting also the same questions are open and relevant (see, for example, [15]).

Definition. A map $f : \mathbb{A} \to \mathbb{A}$ is C^0*-integrable* if f has an invariant C^0-foliation by graphs.

For such a foliation by graphs of $\theta \in \mathbb{T} \mapsto \eta_c(\theta) \in \mathbb{R}$ where $\int_{\mathbb{T}} \eta_c(\theta) = c$, we introduce what we call its generating function, that is, $u : \mathbb{A} \to \mathbb{R}$ defined by

$$u(\theta, c) = \int_0^\theta (\eta_c(t) - c)\, dt.$$

(This terminology will be better understood when we will introduce the generating functions of a large class of symplectic homeomorphisms. Let us mention that for a general foliation, the generating function of the foliation is not necessarily the generating function of a symplectic homeomorphism. We will later give conditions for this to be true.)

Definition. Let $\mathcal{F}$ be a continuous foliation of $\mathbb{A}$ into graphs. Then the unique continuous function $u : \mathbb{A} \to \mathbb{R}$ that is C^1 with respect to the $\mathbb{T}$ variable such that

- for all $c \in \mathbb{R}$, $u(0, c) = 0$,
- for all $c \in \mathbb{R}$, the graph of $c + (\partial u/\partial \theta)(\cdot, c)$ is a leaf of $\mathcal{F}$,

is called the *generating function* of $\mathcal{F}$.

Our first result proves that in the C^0 integrable case of twist diffeomorphisms, there is more regularity of the generating function giving the foliation than we should expect.

THEOREM 1.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a C^1 symplectic twist diffeomorphism. When f is C^0 integrable, the generating function u of its invariant foliation is C^1.*

Moreover, in this case, we have (see the notation π_1 at the beginning of §1.2), for all $c \in \mathbb{R}$, that:

- *the graph of $c + (\partial u/\partial \theta)(\cdot, c)$ is a leaf of the invariant foliation;*
- *$h_c : \theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ is a semi-conjugacy between the projected dynamics $g_c : \theta \mapsto \pi_1 \circ f(\theta, c + (\partial u/\partial \theta)(\theta, c))$ and a rotation R of $\mathbb{T}$, that is, $h_c \circ g_c = R \circ h_c$.*

This allows us to give an example of a foliation of the annulus into smooth graphs that cannot be invariant by a C^0-integrable symplectic twist diffeomorphism. But we will see in §7 that it can be invariant by an exact symplectic twist homeomorphism that is a C^1-diffeomorphism.

COROLLARY 1.1. *Let $\varepsilon : \mathbb{R} \to \mathbb{R}$ be a non-C^1 function that is k-Lipschitz for some $k < 1$. Then the function*

$$(\theta, c) \mapsto u(\theta, c) = \frac{\varepsilon(c)}{2\pi} \sin(2\pi\theta)$$

is the generating function of a foliation of $\mathbb{A}$ into smooth graphs of $\theta \in \mathbb{T} \mapsto c + \varepsilon(c)\cos(2\pi\theta)$ that is not invariant by any C^0-integrable symplectic twist (C^1) diffeomorphism.

The striking fact is the regularity in c. Indeed, if we have a C^k foliation in graphs for some $k \geq 1$, we can only claim that u and $\partial u/\partial\theta$ are C^k. So in the C^0 case, even the derivability with respect to c, which is a result of the invariance by a symplectic *twist* diffeomorphism, is surprising. Also, the fact that the semi-conjugacy h_c continuously depends on c even at a c where the rotation number is rational is very surprising. At an irrational rotation number, this is an easy consequence of the uniqueness of the invariant measure supported on the corresponding leaf, but what happens for a rational rotation number is more subtle.

Another result for C^0-integrable twist diffeomorphisms is that the invariant foliation is not only C^0 but also $\frac{1}{2}$-Hölder. It has been well known since Birkhoff that it is locally uniformly Lipschitz in the variable θ, and we prove here some regularity with respect to c.

THEOREM 1.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a C^1 symplectic twist diffeomorphism that is C^0 integrable with generating function u of its invariant foliation. Then on every compact subset of $\mathbb{A}$, the foliation $(\theta, c) \mapsto \eta_c(\theta) = c + (\partial u/\partial\theta)(\theta, c)$ is uniformly $\frac{1}{2}$-Hölder in the variable c.*

In the C^0-integrable case, the dynamics restricted to a leaf with a rational rotation number is completely periodic.

It is an open question whether there can be a Denjoy counter-example when restricted to a leaf with an irrational rotation number.

With the notation of Theorem 1.1, let us observe that when $f : \mathbb{A} \to \mathbb{A}$ is C^0 integrable, there exists a dense G_δ subset $\mathcal{G}$ of $\mathbb{R}$ such that for every $c \in \mathcal{G}$, the dynamics restricted to the graph of η_c is minimal. Indeed, the set $\mathcal{R}$ of recurrent points is a G_δ set with full Lebesgue measure, hence $\mathcal{R}$ is dense. Hence there exists a dense G_δ subset G_1 of $\mathbb{R}$ such that for every $c \in G_1$, the set $\{\theta \in \mathbb{T}, (\theta, \eta_c(\theta)) \text{ is recurrent}\}$ is a dense G_δ subset of $\mathbb{T}$. Hence, for $c \in G_1$, the dynamics restricted to the graph of η_c cannot be Denjoy. If we remove from G_1 the countable set of cs that correspond to a rational rotation number, we obtain a dense G_δ subset of $\mathbb{R}$ such that the dynamics $f_{|\mathrm{Graph}(c+(\partial u/\partial\theta)(\cdot,c))}$ is minimal.

We will give some conditions that imply that the dynamics restricted to a leaf cannot be Denjoy.

Before this, we need to explain the notion of straightenable foliation.

Notation. We will work in some open subsets $\mathcal{U}, \mathcal{V}$ of either $\mathbb{A}$ or $\mathbb{R}^2$, on which we have global symplectic coordinates that we denote by (θ, r) or (θ, c). Moreover, we will assume that:

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- $\mathcal{V} = \{(\theta, r); \theta \in (\alpha, \beta)$ and $a(\theta) < r < b(\theta)\}$ or $\mathcal{V} = \mathbb{A}$ where a, b are continuous functions defined on $[\alpha, \beta]$ and $0 \in (\alpha, \beta)$;
- $\mathcal{U} = \{(x, c); c \in (c_-, c_+)$ and $d(c) < x < e(c)\}$ or $\mathcal{U} = \mathbb{A}$ where d, e are continuous functions defined on $[c_-, c_+]$.

With this notation, $\partial^+\mathcal{U} = [d(c_+), e(c_+)] \times \{c_+\}$ (respectively, $\partial^-\mathcal{U} = [d(c_-), e(c_-)] \times \{c_-\}$) is the upper (respectively, *lower*) boundary of $\mathcal{U}$ and $\partial^+\mathcal{V} = \{(t, b(t)); t \in [\alpha, \beta]\}$ (respectively, $\partial^-\mathcal{V} = \{(t, a(t)); t \in [\alpha, \beta]\}$) is the *upper* (respectively, *lower*) boundary of $\mathcal{V}$.

Whenever $\mathcal{U}$ appears, the hypotheses will have as a consequence that $\mathcal{U}$ (respectively, $\overline{\mathcal{U}}$) will be homeomorphic to $\mathcal{V}$ (respectively, $\overline{\mathcal{V}}$) and foliated by horizontal lines. In particular, $\mathcal{U}$ will always be homeomorphic to a disc or to an annulus. Firstly, we introduce the notion of exact symplectic homeomorphism, which is a particular case of the notion of symplectic homeomorphism that is due to Oh and Müller [18]. Their notion coincides in this two-dimensional setting with that of orientation- and Lebesgue measure-preserving homeomorphisms.

Definition. An *exact symplectic homeomorphism* from $\mathbb{A}$ onto $\mathbb{A}$ is a homeomorphism that is the limit for the C^0 compact-open topology of a sequence of exact symplectic diffeomorphisms. (We recall that a diffeomorphism $f : \mathbb{A} \to \mathbb{A}$ is exact symplectic if the 1-form $f^*(r d\theta) - r d\theta$ is exact.)

A homeomorphism $\phi : \mathcal{U} \to \mathcal{V}$ is exact symplectic if there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of exact symplectic diffeomorphisms $\phi_n : \mathcal{U} \hookrightarrow \mathcal{V}$ such that the sequence (ϕ_n) converges to ϕ for the C^0 compact-open topology.

Remark.

- In $\mathbb{R}^2$, every 1-form is exact and then the notions of symplectic homeomorphism and exact symplectic homeomorphism coincide.
- Let us recall that a symplectic diffeomorphism f of $\mathbb{A}$ that is isotopic to the identity is exact symplectic if and only if for every essential (an essential curve is a simple closed curve that is not homotopic to a point) curve γ of the annulus, the algebraic area between γ and $f(\gamma)$ is zero.

A remarkable tool can be associated to the exact symplectic homeomorphisms that map the standard horizontal foliation onto a foliation that is transverse to the vertical one. This is called a generating function.

THEOREM 1.3. (And definition) *Recall that $\mathcal{U}$, $\mathcal{V}$ are open subsets of either $\mathbb{A}$ or $\mathbb{R}^2$. Moreover, $\mathcal{V} = \{(\theta, r); \theta \in (\alpha, \beta)$ and $a(\theta) < r < b(\theta)\}$ or $\mathcal{V} = \mathbb{A}$ where a, b are continuous functions and $0 \in (\alpha, \beta)$ and $\mathcal{U} = \{(x, c); c \in (c_-, c_+)$ and $d(c) < x < e(c)\}$ or $\mathcal{U} = \mathbb{A}$ where d, e are continuous functions. We use the notation*

$$\mathcal{W} = \{(\theta, c);\ \text{there exists } (x, c) \in \mathcal{U} \text{ and there exists } (\theta, r) \in \mathcal{V}\} = I \times J.$$

Then $\mathcal{W} = (\alpha, \beta) \times (c_-, c_+)$ or $\mathcal{W} = \mathbb{A}$.

Let $\Phi : \mathcal{U} \to \mathcal{V}$ be an exact symplectic homeomorphism that maps the standard horizontal foliation onto a foliation $\mathcal{F}$ that is transverse to the vertical one and that preserves the orientation of the leaves (here we mean the orientation projected on the horizontal foliation). Assume that Φ extends to a homeomorphism $\overline{\Phi} : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ and that $\overline{\Phi}(\partial^{\pm}\mathcal{U}) = \partial^{\pm}\mathcal{V}$ (this implies that the endpoints of the leaves of $\mathcal{F}$ are not in $\partial^{\pm}\mathcal{V}$ and then that all the leaves of $\mathcal{F}$ are graphs above (α, β)).

Then there exists a C^1 function $u : \mathcal{W} \to \mathbb{R}$ such that

$$\Phi(x, c) = (\theta, r) \Longleftrightarrow x = \theta + \frac{\partial u}{\partial c}(\theta, c) \quad \text{and} \quad r = c + \frac{\partial u}{\partial \theta}(\theta, c).$$

In particular, when defined, every map $\theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ is injective and every set $\{(\theta, c + (\partial u/\partial\theta)(\theta, c)); \theta \in I\}$ is a leaf of the image of the standard horizontal foliation (where $I = \mathbb{T}$ or $I = (\alpha, \beta)$).

The function u is called a generating function for Φ.

Conversely, we consider a C^0-foliation of $\mathcal{V}$ into graphs $\eta_c : \mathbb{T} \to \mathbb{R}$ or $\eta_c : (\alpha, \beta) \to \mathbb{R}$ for $c \in J$ and assume that there exists a C^1 map $u : \mathcal{W} \to \mathbb{R}$ such that:

- *$u(0, c) = 0$ for all $c \in J$;*
- *the graph of $\theta \mapsto c + (\partial u/\partial\theta)(\theta, c)$ defines a leaf of the original foliation;*
- *for all $c \in J$, the map $\theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ is increasing.*

Then

$$\Phi(x, c) = (\theta, r) \Longleftrightarrow x = \theta + \frac{\partial u}{\partial c}(\theta, c) \quad \text{and} \quad r = c + \frac{\partial u}{\partial \theta}(\theta, c)$$

defines an exact symplectic homeomorphism from $\mathcal{U}$ onto $\mathcal{V}$ that maps the standard horizontal foliation onto the original foliation.

COROLLARY 1.2. *The foliation given in Corollary 1.1 cannot be straightened via an exact symplectic homeomorphism that preserves the horizontal orientation of the leaves.*

With some extra hypothesis and work we also obtain the following result.

COROLLARY 1.3. *Assuming that the function ε is not derivable on a dense set, the foliation given in Corollary 1.1 cannot be locally straightened via an exact symplectic homeomorphism that preserves the horizontal orientation of the leaves.*

Remarks.

(1) The formulas of Theorem 1.3 can be also written as

$$\Phi\left(\theta + \frac{\partial u}{\partial c}(\theta, c), c\right) = \left(\theta, c + \frac{\partial u}{\partial \theta}(\theta, c)\right).$$

(2) Observe that Theorem 1.1 gives us a C^1 function u, but not the injectivity of $\theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$. This is why *a priori* the maps h_c are not conjugacies, but only semi-conjugacies and in this case the restricted dynamics may be Denjoy.

We will now give a condition that implies that a foliation is straightenable by an exact symplectic homeomorphism.

Definition.

- A foliation by graphs $a \mapsto \eta_a$ is a *Lipschitz foliation* if $(\theta, a) \mapsto (\theta, \eta_a(\theta))$ is a homeomorphism that is locally bi-Lipschitz.
- If f has an invariant Lipschitz foliation, f is *Lipschitz integrable*.

The following proposition is a consequence of Theorem 1.3 and results of Minguzzi on the mixed derivative [**16**].

PROPOSITION 1.1. *Let $u : \mathbb{A} \to \mathbb{R}$ be the generating function of a continuous foliation of $\mathbb{A}$ into graphs. We assume that u is C^1. Then the two following assertions are equivalent.*

(1) *The foliation is Lipschitz.*

(2) *We have:*
   - *$\partial u/\partial\theta$ is locally Lipschitz continuous;*
   - *$\partial u/\partial c$ is uniformly Lipschitz continuous in the variable θ on any compact set of cs;*
   - *for every compact subset $\mathcal{K} \subset \mathbb{A}$, there exist two constants $k_+ > k_- > -1$ such that $k_+ \geq \partial^2 u/\partial\theta\partial c \geq k_-$ Lebesgue almost everywhere in $\mathcal{K}$.*

In this case, u is the generating function of an exact symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the standard foliation onto the invariant one.

Observe that Corollary 1.2 gives an example of a Lipschitz foliation by smooth curves that is not straightenable via a symplectic homeomorphism. Hence the hypothesis that u is C^1 is crucial in Proposition 1.1.

Definition.

- A map $a \mapsto \eta_a$ defines a C^k *foliation* if $(\theta, a) \mapsto (\theta, \eta_a(\theta))$ is a C^k diffeomorphism. If f has an invariant C^k foliation, f is C^k *integrable*.
- Following [**11**], a map $a \mapsto \eta_a$ defines a C^k *lamination* if $(\theta, a) \mapsto (\theta, \eta_a(\theta))$ is a homeomorphism, every η_a is C^k and the map $a \mapsto \eta_a$ is continuous when $C^k(\mathbb{T}, \mathbb{R})$ is endowed with the C^k topology.

COROLLARY 1.4. *Let $k \geq 1$ and $r \mapsto f_r$ be a C^k-foliation in graphs. Then there exists a C^{k-1} exact symplectic diffeomorphism (a C^0 diffeomorphism is a homeomorphism) Φ : $(\theta, r) \mapsto (h(\theta, r), \eta(h(\theta, r), r))$ such that for each $r \in \mathbb{R}$, the set $\{(\theta, \eta(\theta, r)), \theta \in \mathbb{T}\}$ is a leaf of the foliation.*

When a symplectic homeomorphism preserves a symplectic foliation that is symplectically straightenable, the dynamics is very simple. Let us introduce the following notion before explaining this point.

Definition. If $f : \mathbb{A} \to \mathbb{A}$ is a symplectic homeomorphism, we call C^0 *Arnol'd–Liouville coordinates* a symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ such that the standard foliation by graphs $\mathbb{T} \times \{c\}$ is invariant by $\Phi^{-1} \circ f \circ \Phi$ and

$$\Phi^{-1} \circ f \circ \Phi(x, c) = (x + \rho(c), c)$$

for some (continuous) function $\rho : \mathbb{R} \to \mathbb{R}$.

PROPOSITION 1.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a symplectic homeomorphism that preserves (each leaf of) a C^0-foliation $\mathcal{F}$. If the foliation is symplectically straightenable (by $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the standard foliation $\mathcal{F}_0$ to $\mathcal{F} = \Phi(\mathcal{F}_0)$), then the homeomorphism Φ provides C^0 Arnol'd–Liouville coordinates.*

COROLLARY 1.5. *A symplectic twist diffeomorphism $f : \mathbb{A} \to \mathbb{A}$ is C^0-integrable with the dynamics on each leaf conjugate to a rotation if and only if the invariant foliation is exact symplectically homeomorphic to the standard foliation. In this case, f admits global C^0 Arnol'd–Liouville coordinates.*

In the case where the invariant foliation by a symplectic twist diffeomorphism is Lipschitz, we are in the case of Proposition 1.2 and so for every leaf, the restricted dynamics is not Denjoy.

COROLLARY 1.6. *Let $f : \mathbb{A} \to \mathbb{A}$ be a symplectic twist diffeomorphism that is Lipschitz integrable. We denote by u the generating function of its invariant foliation. Then u is the generating function of an exact symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the standard foliation onto the invariant one such that*

$$\text{for all } (x, c) \in \mathbb{A}, \quad \Phi^{-1} \circ f \circ \Phi(x, c) = (x + \rho(c), c),$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is an increasing bi-Lipschitz homeomorphism.

Moreover, the invariant foliation is a C^1 lamination and Φ admits a partial derivative with respect to θ. The projected dynamics g_c restricted to every leaf is C^1 conjugate to a rotation via the C^1 diffeomorphism $h_c = Id_{\mathbb{T}} + (\partial u/\partial c)(\cdot, c) : \mathbb{T} \to \mathbb{T}$ such that $h_c \circ g_c = R \circ h_c$.

Corollary 1.6 provides some C^0 Arnol'd–Liouville coordinates. A similar statement for Tonelli Hamiltonians is proved in [3], without the fact that the conjugation is C^1.

1.2. *Some notation and definitions.* We will use the following notation.

Notation.

- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle and $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is the annulus; $\pi : \mathbb{R} \to \mathbb{T}$ is the usual projection.
- The universal covering of the annulus is denoted by $p : \mathbb{R}^2 \to \mathbb{A}$.
- The corresponding projections are $\pi_1 : (\theta, r) \in \mathbb{A} \mapsto \theta \in \mathbb{T}$ and $\pi_2 : (\theta, r) \in \mathbb{A} \mapsto r \in \mathbb{R}$; we denote also the corresponding projections of the universal covering by π_1, $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$.
- The Liouville 1-form is defined on $\mathbb{A}$ as being $\lambda = \pi_2 d\pi_1 = r d\theta$; then $\mathbb{A}$ is endowed with the symplectic form $\omega = -d\lambda$.

Let us recall the definition of a symplectic twist diffeomorphism.

Definition. A *symplectic twist diffeomorphism* $f : \mathbb{A} \to \mathbb{A}$ is a C^1 diffeomorphism such that:

- f is isotopic to the identity;
- f is symplectic, that is, $f^*\omega = \omega$;

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- f has the *twist property*, that is, $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f;
- for any $\tilde{\theta} \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde{\theta}, r) \in \mathbb{R}$ is an increasing C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

1.3. *Content of the different sections.* The main tools that we will use are tools of ergodic theory and symplectic (continuous or differentiable) dynamics, in particular symplectic twist maps and Green bundles. The rest of the paper is organized as follows.

- Section 2 contains the proof of Theorem 1.1. After recalling some generalities about symplectic twist diffeomorphisms, we consider the case of rational curves by using some ergodic theory. Then we prove the regularity of u also by using ergodic theory.
- Section 3 contains the proof of Theorem 1.2.
- Section 4 contains the proof of Theorem 1.3.
- Section 5 contains the proofs of Proposition 1.1 and Corollary 1.4.
- Section 6 contains the proofs of Proposition 1.2, Corollaries 1.5 and 1.6.
- Section 7 introduces a strange foliation, provides the proofs of Corollaries 1.1 and 1.2 and an example of an exact symplectic twist map that leaves the strange foliation invariant.
- Appendix A contains an example of a foliation by graphs that is the inverse image of the standard foliation by a symplectic map but not by a symplectic homeomorphism. Appendix B recalls some results about Green bundles.

2. *Proof of Theorem 1.1*

We assume that $f : \mathbb{A} \to \mathbb{A}$ is a C^k symplectic twist diffeomorphism (with $k \geq 1$) that has a continuous invariant foliation by continuous graphs with generating function denoted by u. We write $\eta_c = c + (\partial u/\partial\theta)(\cdot, c)$ and we recall that Birkhoff's theorem (see [**4**, **7**, **10**]) implies that all the η_c are Lipschitz.

Notation. For every $c \in \mathbb{R}$, we will denote by $g_c : \mathbb{T} \to \mathbb{T}$ the restricted-projected dynamics to the graph of η_c, that is,

$$g_c(\theta) = \pi_1 \circ f(\theta, \eta_c(\theta)).$$

2.1. *Some generalities.*

Notation.

- In $\mathbb{R}^2$ we denote by $B(x, r)$ the open disc for the usual Euclidean distance with centre x and radius r.
- We denote by $R_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation $R_\alpha(\theta) = \theta + \alpha$.
- If E is a finite set, then $\sharp(E)$ is the number of elements it contains.
- We denote by $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ the integer part.

We fix a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f. We denote by $\tilde{\eta}_c : \mathbb{R} \to \mathbb{R}$ a lift of η_c. We denote by ρ the function that maps $c \in \mathbb{R}$ to the rotation number $\rho(c) \in \mathbb{R}$ of the restriction of F to the graph of $\tilde{\eta}_c$.

The map ρ is then an increasing homeomorphism.

When the foliation is bi-Lipschitz, we will prove that ρ is a bi-Lipschitz homeomorphism (see Proposition 6.1). We recall a well-known result concerning the link between invariant measures and semi-conjugacies for orientation-preserving homeomorphisms of $\mathbb{T}$.

PROPOSITION 2.1. *Let $g : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism. Assume that μ is a non-atomic Borel invariant probability measure by g. Then, if the rotation number ρ of g is irrational or g is C^0 conjugate to a rotation, the map $h : \mathbb{T} \to \mathbb{T}$ defined by $h(\theta) = \int_0^\theta d\mu$ is a semi-conjugacy between g and the rotation of angle ρ, that is,*

$$h(g(\theta)) = h(\theta) + \rho.$$

Proof. Let $\tilde{\mu}$ be the pullback measure of μ to $\mathbb{R}$ and let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be a lift of g to $\mathbb{R}$. Then we have, for every $\Theta \in [0, 1]$ lift of $\theta \in \mathbb{T}$,

$$\tilde{\mu}([0, \Theta]) = \tilde{\mu}([\tilde{g}(0), \tilde{g}(\Theta)]) = \tilde{\mu}_c([\lfloor \tilde{g}(0) \rfloor, \tilde{g}(\Theta)]) - \tilde{\mu}([\lfloor \tilde{g}(0) \rfloor, \tilde{g}(0)]),$$

where $\lfloor \tilde{g}(0) \rfloor$ is the integer part of $\tilde{g}(0)$. This implies

$$h(\theta) = h(g(\theta)) - \tilde{\mu}([0, g(0)]) = h(g(\theta)) - \hat{r}.$$

(Recall that if $f : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism then either $\rho(f)$ is irrational, f is semi-conjugate (by h) to the rotation $R_{\rho(f)}$ and the only invariant measure is the pullback of the Lebesgue measure by h; or $\rho(f)$ is rational and the ergodic invariant measures are supported on periodic orbits. When $\rho(f)$ is irrational or when f is C^0 conjugate to a rotation and $\rho(f) \in [0, 1[$, then for any invariant measure μ and $x \in \mathbb{T}$, $\mu([x, f(x)[) = \rho(f)$.) Moreover, as we assumed that μ is non-atomic, h is continuous. □

Remarks.

(1) Conversely, if h is a (non-decreasing) semi-conjugacy such that $h \circ g = h + \rho$, then $\mu([0, \theta]) = h(\theta) - h(0)$ defines a g-invariant Borel probability measure.
(2) When ρ is irrational, it is well known that the Borel invariant probability measure μ is unique and that the semi-conjugacy h is unique up to a constant.

Notation. When $\rho(c)$ is irrational, we will denote by h_c the semi-conjugacy such that $h_c(0) = 0$.

Before entering the core of the proof, let us mention a useful fact about iterates of C^0-integrable symplectic twist diffeomorphisms.

PROPOSITION 2.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a C^0-integrable C^1 symplectic twist diffeomorphism. Then so is f^n for all $n > 0$.*

This is specific to the integrable case: in general, an iterated twist diffeomorphism is not a twist diffeomorphism, as can be seen in the neighborhood of an elliptic fixed point.

Proof. We argue by induction on $n > 0$. The initialization being trivial, let us assume the result true for some $n > 0$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. For any $c \in \mathbb{R}$, using the

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notation given at the beginning of §2, we have

$$\text{for all } \theta \in \mathbb{T}, \quad \text{for all } m > 0, \quad f^m(\theta, \eta_c(\theta)) = (g_c^m(\theta), \eta_c \circ g_c^m(\theta)).$$

Observe that if f^m satisfies the twist condition and $c_1 < c_2$ are two real numbers, denoting by $\sim$ the lifts of the considered functions, then we have

$$\tilde{g}_{c_1}^m(t) = \pi_1 \circ F^m(t, \eta_{c_1}(t)) < \pi_1 \circ F^m(t, \eta_{c_2}(t)) = \tilde{g}_{c_2}^m(t)$$

and $\lim_{t \to \pm\infty} \tilde{g}_{c_1}(t) = \pm\infty$.

Let $c_1 < c_2$ and $t \in \mathbb{R}$. We obtain that

$$\begin{aligned}\pi_1(F^{n+1}(t, \eta_{c_2}(t))) &- \pi_1(F^{n+1}(t, \eta_{c_1}(t))) \\ &= \tilde{g}_{c_2} \circ \tilde{g}_{c_2}^n(t) - \tilde{g}_{c_1} \circ \tilde{g}_{c_1}^n(t) > \tilde{g}_{c_2} \circ \tilde{g}_{c_1}^n(t) - \tilde{g}_{c_1} \circ \tilde{g}_{c_1}^n(t),\end{aligned}$$

where we have used the induction hypothesis, $\tilde{g}_{c_2}^n(t) > \tilde{g}_{c_1}^n(t)$, and the fact that $\tilde{g}_{c_2}$ is increasing. It follows that $c \mapsto \pi_1(F^{n+1}(t, \eta_c(t)))$ is an increasing diffeomorphism onto its image. Observe also that this inequality implies that

$$\lim_{c_2 \to +\infty} \pi_1(F^{n+1}(t, \eta_{c_2}(t))) = +\infty$$

because $\lim_{c_2 \to +\infty} \tilde{g}_{c_2}(s) = +\infty$. Moreover,

$$\begin{aligned}\pi_1(F^{n+1}(t, \eta_{c_2}(t))) &- \pi_1(F^{n+1}(t, \eta_{c_1}(t))) \\ &= \tilde{g}_{c_2} \circ \tilde{g}_{c_2}^n(t) - \tilde{g}_{c_1} \circ \tilde{g}_{c_1}^n(t) \le \tilde{g}_{c_2} \circ \tilde{g}_{c_2}^n(t) - \tilde{g}_{c_1} \circ \tilde{g}_{c_2}^n(t)\end{aligned}$$

implies that $\lim_{c_1 \to -\infty} \pi_1(F^{n+1}(t, \eta_{c_1}(t))) = -\infty$ because $\lim_{c_1 \to -\infty} \tilde{g}_{c_1}(s) = -\infty$. So finally $c \mapsto \pi_1(F^{n+1}(t, \eta_c(t)))$ is an increasing diffeomorphism onto $\mathbb{R}$. □

2.2. *Differentiability and conjugacy along rational curves.* It is proved in [1] that for every $r = p/q \in \mathbb{Q}$, $\eta_c = \eta_{\rho^{-1}(r)}$ is C^k and the restriction of f to the graph Γ_c of η_c is completely periodic: $f^q_{|\Gamma_c} = \mathrm{Id}_{\Gamma_c}$. Moreover, along these particular curves, the two Green bundles (see Appendix B for definition and results) are equal:

$$G_-(\theta, \eta_c(\theta)) = G_+(\theta, \eta_c(\theta)).$$

THEOREM 2.1.

- *Along every leaf Γ_c such that $\rho(c) \in \mathbb{Q}$, the derivative $\partial\eta_c(\theta)/\partial c = 1 + (\partial^2 u/\partial c \partial\theta)$ exists, is positive and C^{k-1} depends on θ.*
- *For any c such that $\rho(c)$ is rational, the Borel probability measure μ_c on $\mathbb{T}$ of density $\partial\eta_c/\partial c$ is invariant by g_c and, for $\theta \in [0, 1]$, the equality*

$$h_c(\theta) = \mu_c([0, \theta]) = \int_0^\theta \frac{\partial \eta_c}{\partial c}(t)\, dt$$

defines a conjugacy between g_c and the rotation with angle $\rho(c)$.
- *Then the map $c \in \mathbb{R} \mapsto \mu_c$ is continuous for the weak* topology on measures and $c \in \mathbb{R} \mapsto h_c$ is continuous for the uniform C^0 topology. Thus $(\theta, c) \mapsto h_c(\theta)$ is continuous.*

Remarks.

(1) Observe that because $c \mapsto \eta_c$ is increasing, we know that for Lebesgue almost every $(\theta, c) \in \mathbb{A}$, the derivative $\partial\eta_c(\theta)/\partial c$ exists (see [**12**]). But our theorem says something different.

(2) Because of the continuous dependence on θ along the rational curve, we obtain that $\partial\eta_c(\theta)/\partial c$ restricted to every rational curve is bounded (this is clear when we assume that the foliation is Lipschitz but not if the foliation is just continuous).

Proof of the first point of Theorem 2.1. We fix $A \in \mathbb{R}$ such that $\rho(A) = p/q \in \mathbb{Q}$. Replacing f by f^q, we can assume that $\rho(A) \in \mathbb{Z}$. Observe that because of the C^0-integrability of f, f^q is also a C^k symplectic twist diffeomorphism that is C^0-integrable with the same invariant foliation (Proposition 2.2).

We define $G_A : \mathbb{A} \to \mathbb{A}$ by

$$G_A(\theta, r) = (\theta, r + \eta_A(\theta)). \tag{1}$$

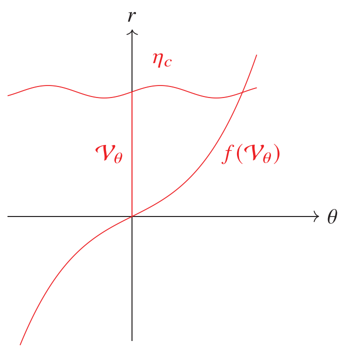
Then $G_A^{-1} \circ f^q \circ G_A$ is also a C^0-integrable C^k symplectic twist diffeomorphism and $\mathbb{T} \times \{0\}$ is filled with fixed points.

We finally have to prove our theorem in this case and we use the notation f instead of $G_A^{-1} \circ f^q \circ G_A$. We can assume that $\rho(A) = 0$ instead of $\rho(A) \in \mathbb{Z}$.

If $\varepsilon > 0$ is arbitrarily small, because of the semi-continuity of the two Green bundles $G_- = \mathbb{R}(1, s_-)$ and $G_+ = \mathbb{R}(1, s_+)$, we have that for any point $x = (\theta, r)$ sufficiently close to $\mathbb{T} \times \{0\}$: $\max\{|s_-(x)|, |s_+(x)|\} < \varepsilon$ is small.

Now we fix c small and consider for every $\theta \in \mathbb{T}$ the small triangular domain $\mathcal{T}(\theta)$ that is delimited by the following three curves:

- the graph of η_c;
- the vertical $\mathcal{V}_\theta = \{\theta\} \times \mathbb{R}$;
- the image $f(\mathcal{V}_\theta)$ of the vertical at θ.



To be more precise, $\mathcal{T}(\theta)$ is 'semi-open' in the following sense; it contains its entire boundary except for the image $f(\mathcal{V}_\theta)$ of the vertical at θ.

We assume that $c > 0$. The case $c < 0$ is similar.

As the slope of η_c is almost 0 (because between the slope of the two Green bundles; see Proposition B.1) and the slope of the side of the triangle that is in $f(\mathcal{V}_\theta)$ is almost $1/s(\theta)$

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where $s(\theta) > 0$ is the torsion that is defined by

$$Df(\theta, 0) = \begin{pmatrix} 1 & s(\theta) \\ 0 & 1 \end{pmatrix}, \tag{2}$$

the area of this triangle is

$$\lambda(\mathcal{T}(\theta)) = \tfrac{1}{2}(\eta_c(\theta))^2(s(\theta) + \varepsilon(\theta, c)), \tag{3}$$

where

$$\text{uniformly for } \theta \in \mathbb{T}, \quad \lim_{c \to 0} \varepsilon(\theta, c) = 0.$$

Let λ be the Lebesgue measure restricted to the invariant sub-annulus

$$\mathcal{A}_c = \bigcup_{\theta \in \mathbb{T}} \{\theta\} \times [0, \eta_c(\theta)].$$

Being symplectic, f preserves λ. Moreover, every ergodic measure μ for f with support in $\mathcal{A}_c$ is supported on one curve Γ_A with $A \in [0, c]$. But $f_{|\Gamma_A}$ is semi-conjugate to a rotation with an angle $\rho(A)$ that is in $[0, 1[$. Hence every interval in Γ_A that is between some $(\theta, \eta_A(\theta))$ and $f(\theta, \eta_A(\theta))$ has the same μ-measure, which is just given by the rotation number $\rho(A) \in [0, 1]$ on the graph of η_A. This implies that $\theta \mapsto \mu(\mathcal{T}(\theta))$ is constant. Hence for every $\theta, \theta' \in \mathbb{T}$ and for every ergodic measure μ with support in $\mathcal{A}_c$, we have $\mu(\mathcal{T}(\theta)) = \mu(\mathcal{T}(\theta'))$.

We now use the ergodic decomposition of invariant measures (see, for example, [14]) applied to the Lebesgue measure restricted to $\mathcal{A}_c$. Let δ_x be the notation for the Dirac measure at x. For every $a \in \mathbb{A}$, we denote $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n-1} \delta_{f^n(a)}$ by λ_a (such a limit exists and is ergodic as all Birkhoff averages converge to an ergodic measure for orientation-preserving circle homeomorphisms). Then we have $\lambda = \int_{\mathcal{A}_c} \lambda_a d\lambda(a)$. Denoting the rotation number of a point $a \in \mathbb{A}$ by $\mathcal{R}(a)$, we deduce that,

$$\text{for all } \theta, \theta' \in \mathbb{T}, \quad \lambda(\mathcal{T}(\theta)) = \lambda(\mathcal{T}(\theta')) = \int_{\mathcal{A}_c} \mathcal{R}(a)\, d\lambda(a). \tag{4}$$

This last inequality is again a consequence of the fact that for all $a \in \mathcal{A}_c$, $\lambda_a(\mathcal{T}(\theta)) = \mathcal{R}(a)$ as explained above.

We deduce from equation (3) that

$$\text{uniformly for } \theta, \theta' \in \mathbb{T}, \quad \lim_{c \to 0} \frac{\eta_c(\theta')}{\eta_c(\theta)} = \sqrt{\frac{s(\theta)}{s(\theta')}}.$$

Integrating with respect to θ' and recalling that $\int_{\mathbb{T}} \eta_c = c$, we deduce that uniformly in θ, we have

$$\lim_{c \to 0} \frac{c}{\eta_c(\theta)} = \sqrt{s(\theta)} \int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}.$$

This implies that

$$\frac{\partial \eta_c(\theta)}{\partial c}_{|c=0} = \left(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}} \right)^{-1} \frac{1}{\sqrt{s(\theta)}}, \tag{5}$$

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and even

$$\eta_c(\theta) = c\left(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\right)^{-1}\left(\frac{1}{\sqrt{s(\theta)}} + \varepsilon(\theta, c)\right), \tag{6}$$

where

$$\text{uniformly for } \theta \in \mathbb{T}, \quad \lim_{c\to 0} \varepsilon(\theta, c) = 0.$$

Observe that $\partial\eta_c/\partial c = (\int_{\mathbb{T}}(dt/\sqrt{s(t)}))^{-1}(1/\sqrt{s(\cdot)})$ is a C^{k-1} function of θ. This proves the first point of Theorem 2.1. □

Proof of the second point of Theorem 2.1. We deduce from the first point that for any c such that $\rho(c)$ is rational, the function $\partial\eta_c/\partial c$ is continuous and positive. Moreover, its integral on $\mathbb{T}$ is 1. Hence $\partial\eta_c/\partial c$ is the density of a Borel probability measure that is equivalent to Lebesgue. We now introduce some notation.

Notation. If $c < c'$, we denote by $\Lambda_{c,c'}$ the normalized Lebesgue measure between the graph of η_c and the graph of $\eta_{c'}$, that is, the Lebesgue measure divided by the area between the graph of η_c and the graph of $\eta_{c'}$.

Then f preserves $\Lambda_{c,c'}$. Observe that for any measurable $I \subset \mathbb{T}$, we have

$$\Lambda_{c,c'}(\{(\theta, r); \theta \in I,\ r \in [\eta_c(\theta), \eta_{c'}(\theta)]\}) = \frac{1}{c - c'}\int_I (\eta_c(\theta) - \eta_{c'}(\theta))\, d\theta. \tag{7}$$

LEMMA 2.1. *If $\rho(c)$ is rational, then $\lim_{c'\to c} \Lambda_{c,c'}$ is a measure supported on the graph of η_c whose projected measure μ_c has density $\partial\eta_c/\partial c$ with respect to the Lebesgue measure of $\mathbb{T}$. Hence if $h_c(\theta) = \int_0^\theta (\partial\eta_c/\partial c)(t)\, dt$, we have*

$$h_c \circ \pi_1 \circ f(\theta, \eta_c(\theta)) = h_c(\theta) + \rho(c).$$

Proof. Using equation (6), we can take the limit in equation (7), or more precisely for any $\psi \in C^0(\mathbb{A}, \mathbb{R})$ in

$$\int \psi(\theta, r)\, d\Lambda_{c,c'}(\theta, r) = \int_{\mathbb{T}} \frac{1}{c - c'}\left(\int_{\eta_{c'}(\theta)}^{\eta_c(\theta)} \psi(\theta, r)\, dr\right) d\theta,$$

and obtain that the limit is an invariant measure supported in the graph of η_c whose projected measure μ_c has a density with respect to Lebesgue (hence has no atoms) that is equal to $\partial\eta_c/\partial c$. Moreover, by [**1**] (this fact was recalled at the beginning of §2.2), g_c is C^0 conjugate to a rotation. We then use Proposition 2.1 to conclude that h_c is the desired conjugacy. □

□

Proof of the third point of Theorem 2.1. We notice that when $\rho(c)$ is irrational, there is only one invariant Borel probability measure that is supported on the graph of η_c. This implies the continuity of the map $c \mapsto \mu_c$ at such a c. Let us look at what happens when $\rho(c)$ is rational.

PROPOSITION 2.3. *For every $c_0 \in \mathbb{R}$ such that $\rho(c_0)$ is rational, for every $\theta \in [0, 1]$, we have*

$$\lim_{c\to c_0} \mu_c([0,\theta]) = \mu_{c_0}([0,\theta])$$

and the limit is uniform in θ.

Together with the continuity of h_{c_0}, this implies the continuity of $(\theta, c) \mapsto h_c(\theta)$ at (θ, c_0).

Proof. In this proof, we will use different functions $\varepsilon_i(\tau, c)$ and, uniformly in τ, all these functions will satisfy

$$\lim_{c\to 0} \varepsilon_i(\tau, c) = 0.$$

As in the proof of the first point of Theorem 2.1, we can assume that $\eta_{c_0} = 0$ (and then $c_0 = 0$) and $\rho(0) = 0$.

We fix $\varepsilon > 0$. Because of the continuity of ρ, we can choose α such that if $|c| < \alpha$, then $|\rho(c)| < \varepsilon$.

Let us introduce the notation $N_c = \lfloor 1/\rho(c)\rfloor$ for $c \neq 0$. Let us assume that $c > 0$ and $\theta \in (0, 1]$. We also denote by $\tilde{g}_c : \mathbb{R} \to \mathbb{R}$ the lift of g_c such that $\tilde{g}_c(0) \in [0, 1)$ and by $M_c(\theta)$,

$$M_c(\theta) = \sharp\{j \in \mathbb{N};\ \tilde{g}_c^j(0) \in [0,\theta]\}.$$

Hence, $M_c(\theta)$ is the number of points of the orbit of 0 under $\tilde{g}_c$ that belong to $[0, \theta]$. Observe that $M_c(\theta)$ is non-decreasing with respect to θ.

As $\eta_c > 0$, any primitive $\mathcal{N}_c$ of η_c is increasing, hence $M_c(\theta)$ is also the number of $\tilde{g}^k(0)$ such that $\mathcal{N}_c(\tilde{g}^k(0))$ belongs to $[\mathcal{N}_c(0), \mathcal{N}_c(\theta)]$, that is,

$$\begin{aligned} M_c(\theta) &= \sharp\left\{ j \in \mathbb{N};\ \int_0^{\tilde{g}_c^j(0)} \eta_c(t)\,dt \le \int_0^{\theta} \eta_c(t)\,dt \right\} \\ &= \sup\left\{ j \in \mathbb{N};\ \int_0^{\tilde{g}_c^j(0)} \eta_c(t)\,dt \le \int_0^{\theta} \eta_c(t)\,dt \right\}. \end{aligned} \tag{8}$$

Note that $M_c(1) = N_c$ because g_c has rotation number $\rho(c)$ and that we have, for all $\theta \in (0, 1]$, $M_c(\theta) \le N_c$ as M_c is non-decreasing. We also have

$$\mu_c([0,\theta]) = \sum_{j=0}^{M_c(\theta)-1} \mu_c([\tilde{g}_c^j(0), \tilde{g}_c^{j+1}(0)[) + \mu_c([\tilde{g}^{M_c(\theta)}(0), \theta])$$

and thus $\mu_c([0, \theta]) = M_c(\theta)\rho(c) + \Delta\rho(c)$ with $\Delta \in [0, 1]$ because $[\tilde{g}^{M_c(\theta)}(0), \theta] \subset [\tilde{g}^{M_c(\theta)}(0), \tilde{g}^{M_c(\theta)+1}(0)]$.

Hence

$$\mu_c([0,\theta]) \in [M_c(\theta)\rho(c), M_c(\theta)\rho(c) + \rho(c)] \subset \left[\frac{M_c(\theta)}{N_c+1}, \frac{M_c(\theta)+1}{N_c}\right]. \tag{9}$$

Hence to estimate the measure $\mu_c([0,\theta])$ we need a good estimate of the number of j such that $\tilde{g}_c^j(0)$ belongs to $[0,\theta]$. We have proved in equation (6) that

$$\eta_c(\tau) = \bigg(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\bigg)^{-1} \frac{c(1+\varepsilon_0(\tau,c))}{\sqrt{s(\tau)}}. \tag{10}$$

We deduce from equation (2) that $\tilde{g}_c(\tau) = \tau + (s(\tau)+\varepsilon_1(\tau,c))\eta_c(\tau)$ where, uniformly in τ, we have $\lim_{c\to 0}\varepsilon_1(\tau,c) = 0$ and then, by equation (10),

$$\int_\tau^{\tilde{g}_c(\tau)} \eta_c(t)\,dt = \eta_c(\tau)^2(s(\tau)+\varepsilon_2(\tau,c)) = \frac{c^2(1+\varepsilon_3(\tau,c))}{\big(\int_{\mathbb{T}}(dt/\sqrt{s(t)})\big)^2}. \tag{11}$$

This says that the area that is delimited by the zero section, the graph of η_c and the vertical lines between $\mathcal{V}_\tau$ and $\mathcal{V}_{\tilde{g}_c(\tau)}$ is almost constant (that is, does not depend much on τ).

We deduce from equation (8) that

$$\int_0^{\tilde{g}_c^{M_c(\theta)}(0)} \eta_c(t)\,dt \le \int_0^\theta \eta_c(t)\,dt < \int_0^{\tilde{g}_c^{M_c(\theta)+1}(0)} \eta_c(t)\,dt.$$

Hence

$$\sum_{j=0}^{M_c(\theta)-1} \int_{\tilde{g}^j(0)}^{\tilde{g}^{j+1}(0)} \eta_c(t)\,dt \le \int_0^\theta \eta_c(t)\,dt \le \sum_{j=0}^{M_c(\theta)} \int_{\tilde{g}^j(0)}^{\tilde{g}^{j+1}(0)} \eta_c(t)\,dt.$$

Using equation (11), we deduce that

$$M_c(\theta)\frac{c^2(1+\varepsilon_4(\theta,c))}{\big(\int_{\mathbb{T}}(dt/\sqrt{s(t)})\big)^2} \le \frac{c(1+\varepsilon_5(\theta,c))}{\int_{\mathbb{T}}(dt/\sqrt{s(t)})}\int_0^\theta \frac{dt}{\sqrt{s(t)}} < (M_c(\theta)+1)\frac{c^2(1+\varepsilon_6(\theta,c))}{\big(\int_{\mathbb{T}}(dt/\sqrt{s(t)})\big)^2},$$

and then

$$M_c(\theta) = \bigg\lfloor \frac{1}{c}\bigg(\bigg(\int_{\mathbb{T}} \frac{du}{\sqrt{s(u)}}\bigg)\bigg(\int_0^\theta \frac{dt}{\sqrt{s(t)}}\bigg) + \varepsilon_7(\theta,c)\bigg)\bigg\rfloor. \tag{12}$$

This implies that

$$N_c = M_c(1) = \bigg\lfloor \frac{1}{c}\bigg(\bigg(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\bigg)^2 + \varepsilon_8(1,c)\bigg)\bigg\rfloor, \tag{13}$$

and by equations (9), (12) and (13),

$$\mu_c([0,\theta]) = \frac{M_c(\theta)}{N_c} + \varepsilon_9(\theta,c) = \frac{\int_0^\theta (dt/\sqrt{s(t)})}{\int_{\mathbb{T}}(dt/\sqrt{s(t)})} + \varepsilon_{10}(\theta,c) = \mu_0([0,\theta]) + \varepsilon_{11}(\theta,c). \tag{14}$$

As none of the measures μ_c has atoms, this implies that $c \mapsto \mu_c$ and all the maps $c \mapsto \mu_c([0,\theta]) = h_c(\theta)$ are continuous. As every map h_c is non-decreasing in the variable θ, we deduce from the Dini–Polyá theorem [**19**, Exercise 13.b, p. 167] that $c \mapsto h_c$ is continuous for the C^0 uniform topology. □

□

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Remark. If $\rho(A) = p/q$, then we proved that

$$(\partial\eta_c(\theta)/\partial c)_{|c=A} = \bigg(\int_{\mathbb{T}} \frac{dt}{\sqrt{s_q(t, \eta_A(t))}}\bigg)^{-1}\bigg(\frac{1}{\sqrt{s_q(\theta, \eta_A(\theta))}}\bigg),$$

where

$$Df^q(x) = \begin{pmatrix} a_q(x) & s_q(x) \\ c_q(x) & d_q(x) \end{pmatrix}.$$

Indeed, the term $s_q(t, \eta_c(t))$ does not change when we conjugate by the map G_A where $G_A(\theta, r) = (\theta, r + \eta_A(\theta))$ as we did in §2.2.

This gives for the conjugacy

$$h_A(\theta) = \mu_A([0, \theta]) = \bigg(\int_{\mathbb{T}} \frac{dt}{\sqrt{s_q(t, \eta_A(t))}}\bigg)^{-1} \int_0^\theta \frac{1}{\sqrt{s_q(t, \eta_A(t))}}\, dt.$$

Observe that this C^k depends on θ.

Observe too that equation (6) can be rewritten as

$$\eta_c(\theta) = \eta_A(\theta) + (c - A)\bigg[\bigg(\int_{\mathbb{T}} \frac{dt}{\sqrt{s_q(t, \eta_A(t))}}\bigg)^{-1} \frac{1}{\sqrt{s_q(\theta, \eta_A(\theta))}} + \varepsilon(\theta, c)\bigg], \tag{15}$$

where

$$\text{uniformly for } \theta \in \mathbb{T}, \quad \lim_{c\to A} \varepsilon(\theta, c) = 0.$$

Observe that the formula does not give any continuous dependence of h_c or $\partial\eta_c/\partial c$ in the c variable, because q can become very large when c changes.

2.3. *Generating function and regularity.* To finish the proof of Theorem 1.1, we have to prove that u admits a derivative with respect to c everywhere and that

$$\text{for all } \theta \in \mathbb{T}, \text{ for all } c \in \mathbb{R}, \quad h_c(\theta) = \theta + \frac{\partial u}{\partial c}(\theta, c).$$

Because we proved that $(\theta, c) \mapsto h_c(\theta)$ is continuous, we will deduce that u is C^1.

Observe that for every θ, the map $c \mapsto u(\theta, c) + c\theta$ is increasing because every $c \mapsto \eta_c(\theta)$ is increasing.

Let us restate Theorem 1.1 with the new notation.

THEOREM 2.2. *The map u is C^1. Moreover, in this case, we have the following statements.*

- *The graph of $c + (\partial u/\partial\theta)(\cdot, c)$ is a leaf of the invariant foliation.*
- *$\theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ is the semi-conjugacy h_c between g_c and $R_{\rho(c)}$ given in Theorem 2.1. We have $h_c \circ g_c = h_c + \rho(c)$.*

Proof. The first point is a consequence of the definition of u.

Then $u(\cdot, c)$ and $\partial u/\partial\theta = \eta_c - c$ continuously depend on (θ, c).

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Observe that with the notation (7), we have

$$\Lambda_{c,c'}(\{(\theta, r);\ \theta \in [\theta_1, \theta_2], r \in [\eta_c(\theta), \eta_{c'}(\theta)]\})$$
$$= \frac{1}{c'-c}((u(\theta_2, c') - u(\theta_1, c')) - (u(\theta_2, c) - u(\theta_1, c))) + (\theta_2 - \theta_1).$$

Moreover, if $\rho(c_0) \in \mathbb{Q}$, we deduce from Lemma 2.1 that $u(\cdot, c)$ admits a derivative with respect to c at c_0,

$$\frac{\partial u}{\partial c}(\theta, c_0) = \lim_{c\to c_0} \frac{1}{c-c_0}((u(\theta, c) - u(0, c)) - (u(\theta, c_0) - u(0, c_0))),$$

that is given by

$$\frac{\partial u}{\partial c}(\theta, c_0) = \mu_{c_0}([0, \theta]) - \theta = h_{c_0}(\theta) - \theta$$

and this derivative continuously depends on θ.

Assume now that $\rho(c_0)$ is irrational and let c tend to c_0. Every limit point of Λ_{c,c_0} when c tends to c_0 is a Borel probability measure that is invariant by f and supported on the graph of η_{c_0}. As there exists only one such measure, whose projection was denoted by μ_{c_0}, we deduce that

$$\pi_{1*}(\lim_{c\to c_0} \Lambda_{c,c_0}) = \mu_{c_0}.$$

As μ_{c_0} has no atom, we have, for all $\theta_0 \in [0, 1)$,

$$\begin{aligned} h_{c_0}(\theta_0) = {} & \mu_{c_0}([0, \theta_0]) = \lim_{c\to c_0} \Lambda_{c_0,c}(\{(\theta, r); \theta \in [0, \theta_0], r \in [\eta_{c_0}(\theta), \eta_c(\theta)]\}) \\ = {} & \lim_{c\to c_0} \frac{1}{c-c_0}((u(\theta_0, c) - u(0, c)) - (u(\theta_0, c_0) - u(0, c_0))) + \theta_0 \\ = {} & \frac{\partial u}{\partial c}(\theta_0, c_0) + \theta_0. \end{aligned}$$

Let us explain the above equality. If $\varepsilon > 0$, as μ_{c_0} has no atoms, there exist two continuous functions $\chi^{\pm}$ such that $\chi^- \le \chi_{[0,\theta]} \le \chi^+$ and $\int_{\mathbb{T}}(\chi^+ - \chi^-)\, d\mu_{c_0} < \varepsilon$. Setting $\nu_c = \pi_{1*}(\Lambda_{c,c_0})$, we have $\int_{\mathbb{T}} \chi^-\, d\nu_c \le \nu_c([0, \theta]) \le \int_{\mathbb{T}} \chi^+\, d\nu_c$. As $\int_{\mathbb{T}} \chi^{\pm}\, d\nu_c \to \int_{\mathbb{T}} \chi^{\pm}\, d\mu_{c_0}$ when $c \to c_0$, we conclude that $|\nu_c([0, \theta]) - \mu_{c_0}([0, \theta])| < 2\varepsilon$ for c close to c_0.

Hence u admits a derivative with respect to c and

$$h_{c_0}(\theta) = \mu_{c_0}([0, \theta]) = \theta + \frac{\partial u}{\partial c}(\theta, c_0).$$

Because of Theorem 2.1, $(\theta, c) \mapsto (\partial u/\partial c)(\theta, c) = h_c(\theta) - \theta$ is continuous. As the two partial derivatives $\partial u/\partial\theta$ and $\partial u/\partial c$ are continuous in (θ, c), we conclude that u is C^1. This finishes the proof of Theorem 1.1. □

3. *Proof of Theorem 1.2*

We assume that $f : \mathbb{A} \to \mathbb{A}$ is a C^0 integrable symplectic twist diffeomorphism with generating function u for its invariant foliation and use the notation $\eta_c(\theta) = c + (\partial u/\partial\theta)(\theta, c)$. We also denote the projected dynamics on the graph of η_c by $\tilde{g}_c(\theta) = \pi_1 \circ F(\theta, \eta_c(\theta))$

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where we fix a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f. We work on the compact set

$$K = \{(\theta, \eta_c(\theta)); \theta \in \mathbb{T}, c \in [c_1, c_2]\}.$$

Replacing F by $F + (0, p)$ for some integer $p \in \mathbb{N}$, we can assume that the rotation number $\mathcal{R}(x)$ of every $x \in K$ is positive. We denote $U = \{(\theta, \eta_c(\theta)); \theta \in \mathbb{T}, c \in (c_1, c_2)\}$. Being C^1, u is C-Lipschitz on K for some constant $C > 0$. Using the notation of Appendix B, we recall that

$$\text{for all } x \in \mathbb{A}, \text{ for all } n \in \mathbb{N}, \quad s_{-n}(x) < s_{-(n+1)} < s_-(x) \le s_+(x) < s_{n+1}(x) < s_n(x) \tag{16}$$

and that all the maps s_k are continuous. Hence there exist $b > a > 0$ and $r \in (0, 1)$ such that

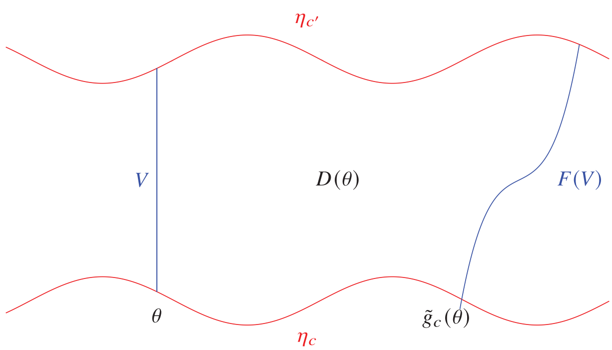
$$\text{for all } x, y \in K, \quad d(x, y) < r \Rightarrow 0 < a \le s_1(x) - s_2(y) < s_1(x) - s_{-1}(y) \le b. \tag{17}$$

We deduce that,

$$\text{for all } x, y \in K, \quad d(x, y) < r \Rightarrow 0 < a \le s_1(x) - s_+(y) < s_1(x) - s_-(y) \le b.$$

Working in $\mathbb{R}^2$, we consider for any $c, c' \in (c_1, c_2)$ such that $c < c'$ the domain $D(\theta)$ whose boundary is the union of:

- the small piece V of the vertical $\{\theta\} \times \mathbb{R}$ that is between η_c and $\eta_{c'}$;
- the arc $F(V)$; and
- pieces of η_c and $\eta_{c'}$ that are between V and $F(V)$.



Using the same method as in §2.2 for the rational curve case (that is, the decomposition of Lebesgue measure into ergodic measures), we see that the area of $D(\theta)$ does not depend on θ.

We then cut $D(\theta)$ into three subsets.

- If $p = \lfloor \tilde{g}_c(\theta) - \theta \rfloor$, $D_1(\theta)$ is the domain that is between V, $V + p$ and the graphs of η_c and $\eta_{c'}$. Observe that $p = \lfloor \mathcal{R}(\theta, \eta_c(\theta)) \rfloor = \lfloor \rho(c) \rfloor$ does not depend on θ (recall that when restricted to the graph of η_c, either f is periodic and all points have the same period, or f has no periodic orbit).
- $D_2(\theta)$ is the domain between $V + p$, the vertical V^* at $F(\theta, \eta_c(\theta))$ and the graphs of η_c and $\eta_{c'}$.
- $D_3(\theta)$ is the triangular domain between V^*, $F(V)$ and $\eta_{c'}$.

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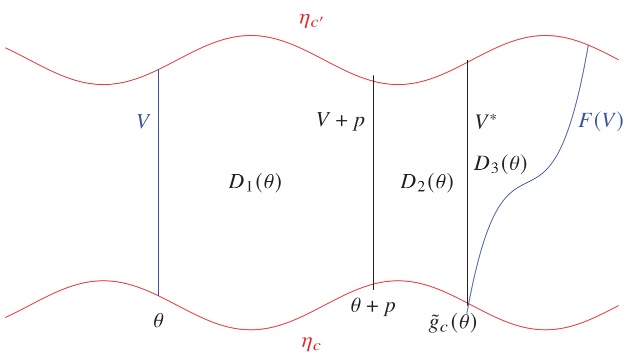

Then the area of $D_1(\theta)$ is $p(c'-c)$ and does not depend on θ.

The area of $D_2(\theta)$ is

$$\int_{\theta+p}^{\tilde{g}_c(\theta)} \left(\left(c' + \frac{\partial u}{\partial \theta}(t, c')\right) - \left(c + \frac{\partial u}{\partial \theta}(t, c)\right)\right) dt,$$

a positive number equal to

$$(c'-c)(\tilde{g}_c(\theta) - \theta - p) + u(\tilde{g}_c(\theta), c') - u(\tilde{g}_c(\theta), c) + u(\theta+p, c) - u(\theta+p, c').$$

We recall that u is C Lipschitz on K and that $\tilde{g}_c(\theta) - \theta - p \in (0, 1)$. We deduce that the area of $D_2(\theta)$ belongs to

$$(0, (2C+1)(c'-c)).$$

We want now to estimate the area of the triangle $D_3(\theta)$. This triangle has three (curved) sides:

- the vertical side V^* whose length is equal to $\eta_{c'}(\tilde{g}_c(\theta)) - \eta_c(\tilde{g}_c(\theta))$;
- the side $F(V)$ with slope at $x \in F(V)$ equal to $s_1(x)$;
- the side H supported in $\eta_{c'}$ that is Lipschitz with tangent cone at x that is contained in $[s_-(x), s_+(x)] \subset [s_{-1}(x), s_2(x)]$.

Because of the continuity of the foliation, there exists $\nu > 0$ such that if $c_1 \le c \le c' \le c_2$ and $c'-c < \nu$, then the length of every piece of vertical V between η_c and $\eta_{c'}$ is less than r (recall that r was chosen to satisfy Formula (17)) and the same is true for $F(V)$ because of the uniform continuity of F on the strip between η_{c_1} and η_{c_2}. Without loss of generality, we assume furthermore that $\nu < 1$. Using the fact that the tangent cone to the graph of $\eta_{c'}$ is between the two Green bundles (see Appendix B) and equation (16), we deduce that if $c'-c \in [0, \nu)$, then $D_3(\theta)$

- is contained in a true triangle with vertical side equal to V^*, upper side with slope equal to $\max_{x\in H} s_2(x)$ and slope of lower side equal to $\min_{x\in F(V)} s_1(x)$, and
- contains a true triangle with vertical side equal to V^*, upper side with slope equal to $\min_{x\in H} s_{-1}(x)$ and slope of lower side equal to $\max_{x\in F(V)} s_1(x)$;

Observe that when the triangle is a true triangle, its horizontal height has length $\delta/S-T$ where δ is the length of the vertical side, T is the slope of the side coming from the upper point of the vertical side and S is the slope of the side coming from the lower point of the vertical side. The area is then $\delta^2/2(S-T)$

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These remarks and equation (17) imply that the area of $D_3(\theta)$ belongs to the interval

$$\left[\frac{(\eta_{c'}(\tilde g_c(\theta))-\eta_c(\tilde g_c(\theta)))^2}{2b}, \frac{(\eta_{c'}(\tilde g_c(\theta))-\eta_c(\tilde g_c(\theta)))^2}{2a}\right].$$

Finally, the sum $A(\theta)$ of the area of $D_2(\theta)$ and $D_3(\theta)$ does not depend on θ and, recalling we are assuming $c'-c<\nu<1$:

- at a point (which always exists because $\int_{\mathbb{T}}(\eta_c-\eta_{c'})=c-c'$) such that $\eta_{c'}(\tilde g_c(\theta))-\eta_c(\tilde g_c(\theta))=c'-c$, we have

$$\begin{aligned} A(\theta) &\in \left[\frac{(c'-c)^2}{2b}, \frac{(c'-c)^2}{2a}+(2C+1)(c'-c)\right] \\ &\subset \left[\frac{(c'-c)^2}{2b}, \left((2C+1)+\frac{1}{2a}\right)(c'-c)\right]; \end{aligned}$$

- at every point, we have

$$A(\theta)\ge \text{area}(D_3(\theta))\ge \frac{(\eta_{c'}(\tilde g_c(\theta))-\eta_c(\tilde g_c(\theta)))^2}{2b};$$

This implies that for $c_1\le c\le c'\le c_2$ such that $c'-c<\nu$, we have

$$\text{for all } \theta\in\mathbb{R},\quad \frac{(\eta_{c'}(\theta)-\eta_c(\theta))^2}{2b}\le\left((2C+1)+\frac{1}{2a}\right)(c'-c),$$

so

$$\text{for all } \theta\in\mathbb{R},\quad \eta_{c'}(\theta)-\eta_c(\theta)\le\sqrt{2b\left((2C+1)+\frac{1}{2a}\right)(c'-c)}.$$

Hence we obtain on the compact K a uniform local constant of Hölder, and this implies that η_c is uniformly $\frac{1}{2}$-Hölder in the variable c on K.

4. *Proof of Theorem 1.3*

We recall that $\mathcal{U}$, $\mathcal{V}$ are open subsets of either $\mathbb{A}$ or $\mathbb{R}^2$ and that:

- $\mathcal{V}=\{(\theta,r);\theta\in(\alpha,\beta)\text{ and }a(\theta)<r<b(\theta)\}$ or $\mathcal{V}=\mathbb{A}$ where a,b are continuous functions defined on $[\alpha,\beta]$ and that $0\in(\alpha,\beta)$;
- $\mathcal{U}=\{(x,c);c\in(c_-,c_+)\text{ and }d(c)<x<e(c)\}$ or $\mathcal{U}=\mathbb{A}$ where d,e are continuous functions defined on $[c_-,c_+]$.

With this notation, $\partial^+\mathcal{U}=[d(c_+),e(c_+)]\times\{c_+\}$ (respectively, $\partial^-\mathcal{U}=[d(c_-),e(c_-)]\times\{c_-\}$) is the upper (respectively, lower) boundary of $\mathcal{U}$ and $\partial^+\mathcal{V}=\{(t,b(t));t\in[\alpha,\beta]\}$ (respectively, $\partial^-\mathcal{V}=\{(t,a(t));t\in[\alpha,\beta]\}$) is the upper (respectively, lower) boundary of $\mathcal{V}$.

We use the notation

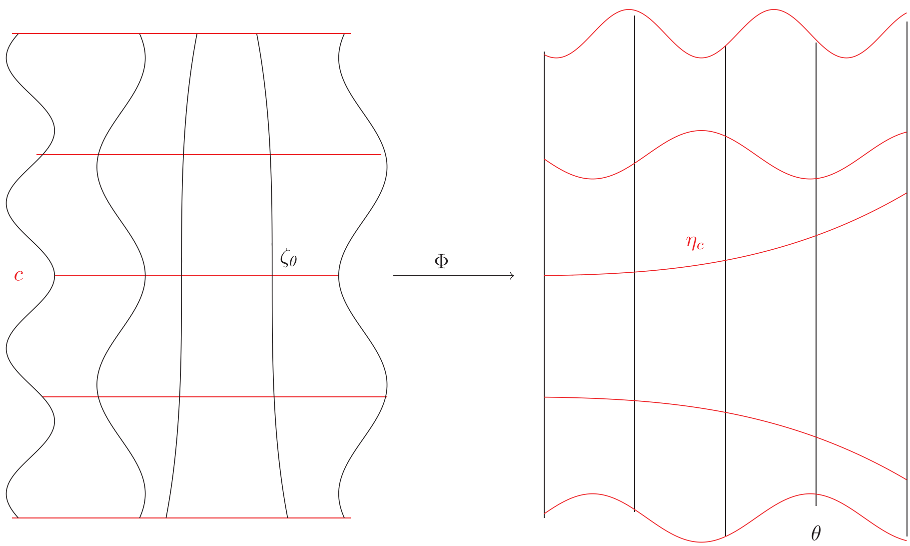
$$\mathcal{W}=\{(\theta,c);\ \text{there exists } (x,c)\in\mathcal{U} \text{ and there exists } (\theta,r)\in\mathcal{V}\}=I\times J.$$

Then $\mathcal{W}=(\alpha,\beta)\times(c_-,c_+)$ or $\mathcal{W}=\mathbb{A}$.

Let us consider a C^0-foliation $\mathcal{F}$ of $\mathcal{V} = \{(\theta, r); \theta \in (\alpha, \beta) \text{ and } \eta_{c_-}(\theta) < r < \eta_{c_+}(\theta)\}$ or $\mathcal{V} = \mathbb{A}$ into graphs: $(\theta, c) \in \mathcal{W} \mapsto (\theta, \eta_c(\theta))$ (when $\mathcal{V} = \mathbb{A}$, $\mathcal{W} = \mathbb{A}$ and in the other cases $\mathcal{W} = (\alpha, \beta) \times (c_-, c_+)$). When $\mathcal{V} = \mathbb{A}$ we insist that $\int_{\mathbb{T}} \eta_c = c$ for all $c \in \mathbb{R}$. Then there exists a continuous function $u : \mathcal{W} \to \mathbb{R}$ that admits a continuous derivative with respect to θ such that $\eta_c(\theta) = c + (\partial u/\partial\theta)(\theta, c)$ and $u(0, c) = 0$, a function that we called generating function of the foliation when $\mathcal{V} = \mathbb{A}$.

4.1. *Proof of the first implication.* Let $\Phi : \mathcal{U} \to \mathcal{V}$ be an exact symplectic homeomorphism that maps the standard horizontal foliation onto a foliation $\mathcal{F}$ that is transverse to the vertical one and that preserves the orientation of the leaves. (When $\mathcal{V} = \mathbb{A}$, the exactness condition implies that Φ maps the circle $\mathbb{T} \times \{c\}$ to the graph of η_c. In the other cases, as the domains are simply connected, the exactness condition is empty.) Assume that Φ extends to a homeomorphism $\overline{\Phi} : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ and that $\overline{\Phi}(\partial^{\pm}\mathcal{U}) = \partial^{\pm}\mathcal{V}$. Then the endpoints of the leaves of $\mathcal{F}$ are not in $\partial^{\pm}\mathcal{V}$. As the foliation $\mathcal{F}$ is transverse to the 'vertical' foliation $\mathcal{G}_0$ into $\pi_1^{-1}(\{\theta\})$ for $\theta \in \pi_1(\mathcal{V})$, all the leaves of $\mathcal{F}$ are graphs above (α, β).

Another result of the transversality of $\mathcal{F}$ and the vertical foliation $\mathcal{G}_0$ is that the foliation $\mathcal{G} = \Phi^{-1}(\mathcal{G}_0)$ is a foliation of $\mathcal{U}$ that is transverse to the standard ('horizontal') foliation $\mathcal{F}_0 = \Phi^{-1}(\mathcal{F})$. This means precisely that the foliation $\mathcal{G}$ is a foliation by graphs of maps $\zeta_\theta : I = \pi_2(\mathcal{U}) \to \pi_1(\mathcal{U})$. Hence there exists a continuous function $v : \mathcal{W} \to \mathbb{R}$ that admits a continuous derivative with respect to r such that the foliation $\mathcal{G}$ is the foliation by graphs $\Phi^{-1}(\pi_1^{-1}(\{\theta\}))$ of $\zeta_\theta : r \mapsto \theta + (\partial v/\partial r)(\theta, r)$. Observe that by definition of ζ_θ, we have $\Phi(\zeta_\theta(c), c) = (\theta, \eta_c(\theta))$. As a result, every map $\theta \mapsto \zeta_\theta(c)$ is a homeomorphism onto its image.



We now use the preservation of the area. We fix $\theta_1 < \theta_2$ in $\pi_1(\mathcal{V})$ and $c_1 < c_2$ in $\pi_2(\mathcal{U})$ such that the domain D_1 delimited by the horizontals $\pi_2^{-1}(\{c_1\})$, $\pi_2^{-1}(\{c_2\})$, the graph of $c \in [c_1, c_2] \mapsto \zeta_{\theta_1}(c)$ and the graph of $c \in [c_1, c_2] \mapsto \zeta_{\theta_2}(c)$ is contained in $\mathcal{U}$. Because

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Φ is a symplectic homeomorphism, Φ preserves the area and so D_1 and $\Phi(D_1)$ have the same area. Observe that $\Phi(D_1)$ is the domain delimited by the graphs of η_{c_1}, η_{c_2} and the verticals $\pi_1^{-1}(\{\theta_1\})$ and $\pi_1^{-1}(\{\theta_2\})$. This can be written

$$\int_{c_1}^{c_2}\bigg(\bigg(\theta_2+\frac{\partial v}{\partial c}(\theta_2,c)\bigg)-\bigg(\theta_1+\frac{\partial v}{\partial c}(\theta_1,c)\bigg)\bigg)\,dc$$
$$=\int_{\theta_1}^{\theta_2}\bigg(\bigg(c_2+\frac{\partial u}{\partial \theta}(\theta,c_2)\bigg)-\bigg(c_1+\frac{\partial u}{\partial \theta}(\theta,c_1)\bigg)\bigg)\,d\theta.$$

It follows that

$$u(\theta_2,c_2)-u(\theta_1,c_2)-u(\theta_2,c_1)+u(\theta_1,c_1)$$
$$=v(\theta_2,c_2)-v(\theta_2,c_1)-v(\theta_1,c_2)+v(\theta_1,c_1).$$

Evaluating for $\theta_1=0$ we find

$$u(\theta_2,c_2)-u(\theta_2,c_1)=v(\theta_2,c_2)-v(\theta_2,c_1)-v(0,c_2)+v(0,c_1).$$

Finally, as v admits a continuous partial derivative with respect to c, we conclude that $(\partial u/\partial c)(\theta,c)=(\partial v/\partial c)(\theta,c)-(\partial v/\partial c)(0,c)$ exists and is continuous. Hence u is C^1. Moreover, every map $\theta\mapsto\theta+(\partial u/\partial c)(\theta,c)=\zeta_c(\theta)-(\partial v/\partial c)(0,c)$ is a homeomorphism onto its image and we have established the first implication.

4.2. *Proof of the second implication.* We assume that there exists a C^1 map $u:\mathcal{W}\to\mathbb{R}$ such that:

- $u(0,c)=0$ for all $c\in I$ where $I=(c_-,c_+)$ or $I=\mathbb{R}$;
- $\eta_c(\theta)=c+(\partial u/\partial\theta)(\theta,c)$ for all $(\theta,c)\in\mathcal{W}$;
- for all $c\in I$, the map $h_c:\theta\mapsto\theta+(\partial u/\partial c)(\theta,c)$ is increasing.

Then we can define a unique homeomorphism Φ by

$$\Phi\bigg(\theta+\frac{\partial u}{\partial c}(\theta,c),c\bigg)=\bigg(\theta,c+\frac{\partial u}{\partial\theta}(\theta,c)\bigg).$$

We first prove the result when $\mathcal{U}=\mathbb{A}$. Let $v:\mathbb{R}^2\to\mathbb{R}_+$ be the C^∞ function with support in $B(0,1)$ defined by $v(\theta,c)=a\exp((1-\|(\theta,c)\|)^{-2})$ for $(\theta,c)\in B(0,1)$ and where a is such that $\int v=1$. We denote by v_ε the function $v_\varepsilon(x)=(1/\varepsilon^2)v(x/\varepsilon)$.

For every $\varepsilon>0$, we define

$$U_\varepsilon(\theta,c)=(u*v_\varepsilon)(\theta,c), \tag{18}$$

where we recall the convolution formula

$$u*v(x)=\int u(x-y)v(y)\,dy.$$

Then when ε tends to 0, the functions U_ε tend to U in the C^1 compact-open topology. Moreover, U_ε is 1-periodic in θ and smooth. Observe that for every θ, the function $c\mapsto c+(\partial u/\partial\theta)(\theta,c)$ is increasing. We deduce that the convolution $c\mapsto c+(\partial U_\varepsilon/\partial\theta)(\theta,c)$ is a C^∞ diffeomorphism as it is a mean of C^∞ diffeomorphisms thanks to Lemma 4.1.

Finally, the maps $F_\varepsilon : (\theta, c) \mapsto (\theta, c + (\partial U_\varepsilon/\partial\theta)(\theta, c))$ define C^∞ foliations that converge to the initial foliation $F_0 : (\theta, c) \mapsto (\theta, c + (\partial u/\partial\theta)(\theta, c))$ for the C^0 compact-open topology when ε tends to 0.

Observe that the $h_c : \theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ are assumed to be increasing. We deduce that the maps $G_\varepsilon : (\theta, c) \mapsto (\theta + (\partial U_\varepsilon/\partial c)(\theta, c), c)$ are C^∞ diffeomorphisms of $\mathbb{A}$ that converge for the C^0 compact-open topology to $G_0 : (\theta, c) \mapsto (\theta + (\partial u/\partial c)(\theta, c), c)$.

Finally, the maps $\mathcal{H}_\varepsilon = F_\varepsilon \circ G_\varepsilon^{-1}$ are C^∞ diffeomorphisms of $\mathbb{A}$ that converge for the C^0 compact-open topology to $F_0 \circ G_0^{-1} = \Phi$. Moreover, because they are defined via generating functions U_ε in the classical sense, we know that the diffeomorphisms $\mathcal{H}_\varepsilon$ are all exact symplectic.

This means precisely that Φ is an exact symplectic homeomorphism.

We now deal with the local case when $\mathcal{U} \neq \mathbb{A}$. The computation from the first implication (with $u = v$) shows that Φ is area preserving. We conclude with the next proposition.

PROPOSITION 4.1. *Let $\psi : \mathcal{U} \to \mathcal{V}$ be a homeomorphism that preserves the Lebesgue measure λ. Then there exists a sequence $\psi_n : \mathcal{U} \to \mathcal{V}$ of area-preserving diffeomorphisms that converges to ψ for the compact-open topology.*

Proof. As $\mathcal{U}$ and $\mathcal{V}$ are bounded and simply connected they are diffeomorphic. Let $\phi_0 : \mathcal{V} \to \mathcal{U}$ be a diffeomorphism. As $\phi_{0*}\lambda(\mathcal{U}) = \lambda(\mathcal{V}) = \lambda(\mathcal{U})$ (because ψ preserves λ) we can apply [9, Theorem 1] to obtain a diffeomorphism $\phi_1 : \mathcal{U} \to \mathcal{U}$ such that $\phi_{1*}(\phi_{0*}\lambda) = \lambda$.

Let $S \subset \mathbb{R}^3$ be a two-dimensional Euclidean sphere that has same total Lebesgue area as $\mathcal{U}$. We denote by λ_S the Lebesgue area on S. Let $N \in S$ be a point. As $S \setminus \{N\}$ is diffeomorphic to $\mathcal{U}$, arguing as before, and again using [9] yields a diffeomorphism $\phi_2 : \mathcal{U} \to S \setminus \{N\}$ such that $\phi_{2*}\lambda = \lambda_S$.

Set $\Psi = \phi_2 \circ \phi_1 \circ \phi_0 \circ \psi \circ \phi_2^{-1}$, a homeomorphism of $S \setminus \{N\}$ that preserves λ_S. It extends to a homeomorphism $\widetilde{\Psi}$ of S by setting $\widetilde{\Psi}(N) = N$ that still preserves λ_S.

We may now apply a classical dynamical systems result to approximate $\widetilde{\Psi}$ by area-preserving diffeomorphisms $(\widetilde{\Psi}_n)_{n>0}$ of S ([17]; see also [13] for a simpler proof adapted from an idea of Sikorav). Up to composing by rotations of S (that preserve λ_S) we may assume furthermore that $\widetilde{\Psi}_n(N) = N$ for all $n > 0$. Therefore by restriction, they give rise to $(\Psi_n)_{n>0}$ that are diffeomorphisms of $S \setminus \{N\}$ that uniformly converge to Ψ.

Finally, setting $\psi_n = \phi_0^{-1} \circ \phi_1^{-1} \circ \phi_2^{-1}\Psi_n \circ \phi_2$ yields a family $(\psi_n)_{n>0}$ of area-preserving diffeomorphisms that converge to ψ for the compact–open topology. □

LEMMA 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, non-trivial, smooth, compactly supported in $[-M, M]$ and even function such that $f' \leq 0$ on $[0, +\infty)$. Then:*

- *if $g : \mathbb{R} \to \mathbb{R}$ is an increasing function, $f * g$ is an increasing C^∞ diffeomorphism onto its image; if, moreover, g is a homeomorphism of $\mathbb{R}$, then $f * g$ is a C^∞ diffeomorphism of $\mathbb{R}$;*
- *if $g : [a - M, b + M] \to \mathbb{R}$ is an increasing function, we can define $f * g : [a, b] \to \mathbb{R}$ that is an increasing C^∞ diffeomorphism onto its image.*

Proof. We only prove the first point, the second one being very similar. As f is smooth and compactly supported, $f * g$ is well defined and smooth. As f is even, f' is odd. Just notice that

$$(f * g)'(x) = \int_{\mathbb{R}} f'(y)g(x-y)\,dy = \int_0^{+\infty} f'(y)(g(x-y) - g(x+y))\,dy.$$

Then $f * g$ is an increasing diffeomorphism onto its image as $g(x-y) - g(x+y) < 0$ and $f'(y) \le 0$ and f' is not the zero function.

Let us now assume that $\lim_{x\to+\infty} g(x) = +\infty$. We will prove that $\lim_{x\to+\infty} f * g(x) = +\infty$. Let $M > 0$ such that the support of f is included in $[-M, M]$. Let $A > 0$ and $B \in \mathbb{R}$ such that $A < g(x)$ as soon as $x > B$. Then if $x > M + B$,

$$f * g(x) = \int_{-M}^{M} f(y)g(x-y)\,dy > A \int_{\mathbb{R}} f(y)\,dy.$$

This proves our result as $\int_{\mathbb{R}} f(y)\,dy > 0$. □

5. *Proof of Proposition 1.1 and Corollary 1.4*

5.1. *Proof of Proposition 1.1: bi-Lipschitz foliations with C^1 generating functions are straightenable.*

Let $u : \mathbb{A} \to \mathbb{R}$ be the C^1 generating function of a continuous foliation of $\mathbb{A}$ into graphs. We recall the following result due to Minguzzi [16].

THEOREM. (Minguzzi) *Let Ω be an open subset of $\mathbb{R}^2$ and let $f \in C^1(\Omega, \mathbb{R})$. Then the following conditions are equivalent:*

(1) *for every x, the partial derivative $(\partial f/\partial x)(x, \cdot)$ is locally Lipschitz, locally uniformly with respect to x;*

(2) *for every y, the partial derivative $(\partial f/\partial y)(\cdot, y)$ is locally Lipschitz, locally uniformly with respect to y.*

If they hold true, then on a subset $E \subset \Omega$ with full Lebesgue measure in Ω, $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ exist and are equal.

5.1.1. *Proof of the first implication.*

We assume that the invariant foliation is K-Lipschitz on a compact $\mathcal{K} = \{(\theta, \eta_c(\theta)); \theta \in \mathbb{T}, c \in [a, b]\}$, which means

$$\text{for all } \theta \in \mathbb{T}, \text{ for all } c_1, c_2 \in [a, b], \quad \frac{|c_1 - c_2|}{K} \le |\eta_{c_1}(\theta) - \eta_{c_2}(\theta)| \le K|c_1 - c_2|. \tag{19}$$

As $\eta_c(\theta) = c + (\partial u/\partial\theta)(\theta, c)$, this means that $(\partial u/\partial\theta)(\theta, \cdot)$ is locally Lipschitz, locally uniformly with respect to θ. Hence, by Minguzzi's theorem, for every $c \in (a, b)$, $(\partial u/\partial c)(\cdot, c)$ is locally Lipschitz, locally uniformly with respect to c, and at almost all $(\theta, c) \in \mathbb{T} \times (a, b)$ we have that $\partial^2 u/\partial c\partial\theta$ and $\partial^2 u/\partial\theta\partial c$ exist and are equal and uniformly bounded.

Hence $h_c = Id_{\mathbb{T}} + (\partial u/\partial c)(\cdot, c)$ is locally uniformly Lipschitz and because of Inequality (19), we have Lebesgue almost everywhere

$$\frac{\partial^2 u}{\partial\theta\partial c}(\theta, c) = \frac{\partial^2 u}{\partial c\partial\theta}(\theta, c) = \frac{\partial\eta}{\partial c}(\theta_0, c_0) - 1 \in \left[-1 + \frac{1}{K}, -1 + K\right] = [k_-, k_+]. \tag{20}$$

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This implies that $h_c(\theta) = \theta + (\partial u/\partial c)(\theta, c)$ defines a $(1/K, K)$-bi-Lipschitz homeomorphism of $\mathbb{T}$ for almost every $c \in [a, b]$ and then for all $c \in [a, b]$ by continuity. By Theorem 1.3, we deduce that u is the generating function of an exact symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the invariant foliation onto the standard one.

5.1.2. *Proof of the second implication.* We assume that the map u is C^1 with $\partial u/\partial\theta$ locally Lipschitz continuous and $\partial u/\partial c$ uniformly Lipschitz in the variable θ on any compact set of cs, and there exist two constants $k_+ > k_- > -1$ such that $(\partial^2 u/\partial\theta\partial c)(\theta, c) \in [k_-, k_+]$ almost everywhere. Because $\partial u/\partial c$ is uniformly Lipschitz in the variable θ on any compact set of cs, we can apply the Minguzzi theorem and equation (20) which implies that the foliation is bi-Lipschitz.

5.2. *Proof of Corollary 1.4.* Let $k \geq 1$ and $r \mapsto f_r$ be a C^k-foliation in graphs and let u be its generating function. As $k \geq 1$, the foliation is Lipschitz when restricted to every compact set. Hence we can use Proposition 1.1. In this case, u is the generating function of an exact symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the standard foliation onto the invariant one and we have

$$\Phi\left(\theta + \frac{\partial u}{\partial c}(\theta, c), c\right) = \left(\theta, c + \frac{\partial u}{\partial\theta}(\theta, c)\right).$$

Moreover, u is C^k, hence $F_0(\theta, c) = (\theta, c + (\partial u/\partial\theta)(\theta, c))$ defines a C^{k-1} homeomorphism that is locally bi-Lipschitz, hence a C^{k-1} diffeomorphism.

Also $h_c(\theta) = \theta + (\partial u/\partial c)(\theta, c)$ is C^{k-1} in (θ, c). Observe that every h_c is a bi-Lipschitz homeomorphism that is C^{k-1}, hence $G_0 : (\theta, c) \mapsto (h_c(\theta), c)$ is also a C^{k-1} diffeomorphism and then $\Phi = F_0 \circ G_0^{-1}$ is a C^{k-1} symplectic diffeomorphism (where a C^0-diffeomorphism is a homeomorphism).

6. *More results on symplectic homeomorphisms that are C^0-integrable*

6.1. *Proof of Proposition 1.2.* Let $f : \mathbb{A} \to \mathbb{A}$ be an exact symplectic homeomorphism. We assume that f has an invariant foliation $\mathcal{F}$ into C^0 graphs that is symplectically homeomorphic (by $\Phi^{-1} : \mathbb{A} \to \mathbb{A}$) to the standard foliation $\mathcal{F}_0 = \Phi^{-1}(\mathcal{F})$. Then the standard foliation is invariant by the exact symplectic homeomorphism $g = \Phi^{-1} \circ f \circ \Phi$. Hence we have

$$g(\theta, r) = (g_1(\theta, r), r).$$

As g is area preserving, for every $\theta \in [0, 1]$ and every $r_1 < r_2$, the area of $[0, \theta] \times [r_1, r_2]$ is equal to the area of $g([0, \theta] \times [r_1, r_2])$, that is,

$$\theta(r_2 - r_1) = \int_{r_1}^{r_2} (g_1(\theta, r) - g_1(0, r))\, dr.$$

Dividing by $r_2 - r_1$ and taking the limit when r_2 tends to r_1, we obtain

$$g_1(\theta, r_1) = \theta + g(0, r_1).$$

This proves the proposition for $\rho = g_1(0, \cdot)$.

6.2. *Proof of Corollary 1.5.* The 'if' part is obvious by Proposition 1.2.

Let us prove the 'only if' part, that is, we assume f is C^0-integrable with the dynamics on each leaf conjugate to a rotation. We denote by $u : \mathbb{A} \to \mathbb{R}$ the map given by Theorem 1.1 and which enjoys the properties of Theorem 2.1. Hence $h_c : \theta \mapsto \theta + (\partial u/\partial c)(\theta, c)$ is a semi-conjugation between the projected dynamics $g_c : \theta \mapsto \pi_1 \circ f(\theta, c + (\partial u/\partial\theta)(\theta, c))$ and the rotation $R_{\rho(c)}$ of $\mathbb{T}$ and is even a conjugation when $\rho(c)$ is rational.

If $\rho(c)$ is irrational, it follows from the hypothesis that g_c is conjugate to a rotation. As the dynamics is minimal, there is up to constants a unique (semi)-conjugacy and then h_c is a true conjugation. We then conclude by using Theorem 1.3.

6.3. *Proof of Corollary 1.6.*

6.3.1. *Arnol'd–Liouville coordinates for f.* Let $f : \mathbb{A} \to \mathbb{A}$ be a symplectic twist diffeomorphism that is Lipschitz integrable with generating function u of its invariant foliation.

By Theorem 1.1 and Proposition 1.1, u is the generating function of an exact symplectic homeomorphism $\Phi : \mathbb{A} \to \mathbb{A}$ that maps the standard foliation onto the invariant one, and for every compact subset $\mathcal{K} \subset \mathbb{A}$ there exist two constants $k_+ > k_- > -1$ such that $\partial^2 u/\partial\theta\partial c \in [k_-, k_+]$ Lebesgue almost everywhere in $\mathcal{K}$.

By Proposition 1.2, we have

$$\text{for all } (x, c) \in \mathbb{A}, \quad \Phi^{-1} \circ f \circ \Phi(x, c) = (x + \rho(c), c),$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, because of the twist condition, ρ is an increasing homeomorphism of $\mathbb{R}$.

6.3.2. *Proof that $\rho : \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz homeomorphism.*

PROPOSITION 6.1. *Assume that a C^1 symplectic twist diffeomorphism $f : \mathbb{A} \to \mathbb{A}$ has an invariant locally Lipschitz continuous foliation by graphs $c \in \mathbb{R} \mapsto \eta_c \in C^0(\mathbb{T}, \mathbb{R})$. Then the map $\rho : c \in \mathbb{R} \mapsto \rho(c)$ is a locally bi-Lipschitz homeomorphism.*

We will use the following lemma.

LEMMA 6.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be lifts of homeomorphisms of $\mathbb{T}$ that preserve orientation (implying $f(\cdot + 1) = f(\cdot) + 1$ and $g(\cdot + 1) = g(\cdot) + 1$). Assume that:*

- *either f or g is conjugate to a translation $t_\alpha : x \mapsto x + \alpha$ by a homeomorphism h that is a lift of a homeomorphism of $\mathbb{T}$ that preserves orientation;*
- *h and h^{-1} are K-Lipschitz.*

Then:

(1) *if there exists $d > 0$ such that $f < g + d$, then $\rho(f) \leq \rho(g) + Kd$;*
(2) *if there exists $d > 0$ such that $f + d < g$ then $\rho(f) + d/K \leq \rho(g)$.*

Proof. Let us say that $h \circ g \circ h^{-1} = t_\alpha$, hence $\rho(g) = \alpha$ (the proof when f is conjugate to a translation is the same).

(1) By hypothesis, $f \circ h^{-1} < g \circ h^{-1} + d$. Using that h is increasing and K-Lipschitz, it follows that for all $x \in \mathbb{R}$,

$$h \circ f \circ h^{-1}(x) < h(g \circ h^{-1}(x) + d) \leq h \circ g \circ h^{-1}(x) + Kd = x + \alpha + Kd.$$

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Finally, as $\rho(f)=\rho(h\circ f\circ h^{-1})$, we conclude that

$$\rho(f)\leq\alpha+Kd=\rho(g)+Kd.$$

(2) By hypothesis, $f\circ h^{-1}+d<g\circ h^{-1}$. Using that h is increasing, it follows that

$$\text{for all } x\in\mathbb{R},\quad h(f\circ h^{-1}(x)+d)<h\circ g\circ h^{-1}(x)=x+\alpha.$$

Because h^{-1} is K-Lipschitz and increasing, observe that

$$\begin{aligned} d&=h^{-1}(h(f\circ h^{-1}(x)+d))-h^{-1}(h\circ f\circ h^{-1}(x))\\ &\leq K(h(f\circ h^{-1}(x)+d)-h\circ f\circ h^{-1}(x)).\end{aligned}$$

Then

$$h\circ f\circ h^{-1}(x)\leq h(f\circ h^{-1}(x)+d)-\frac{d}{K}<x+\alpha-\frac{d}{K}.$$

Hence $\rho(f)+d/K\leq\rho(g)$. □

Proof of Proposition 6.1. The proof is now a direct application of the previous lemma. Indeed, we have seen that when the foliation is K-Lipschitz, if c varies in a compact set $\mathcal{K}$, the dynamics g_c are all conjugate to rotations. We have, moreover, proven there exists a constant $\widetilde{K}$ such that the conjugating functions h_c may be chosen equi-bi-Lipschitz (for $c\in\mathcal{K}$) of constant $\widetilde{K}$.

We denote the minimum and maximum torsions on $\mathcal{K}$ by

$$b_{\min}=\min_{x\in\mathcal{K}}\frac{\partial f_1}{\partial\theta}(x)\quad\text{and}\quad b_{\max}=\max_{x\in\mathcal{K}}\frac{\partial f_1}{\partial\theta}(x).$$

For $c_1<c_2$ in $[a,b]$, we have

$$\tilde{g}_{c_2}(\theta)-\tilde{g}_{c_1}(\theta)=F_1(\theta,\eta_{c_2}(\theta))-F_1(\theta,\eta_{c_1}(\theta))$$

and so

$$\tilde{g}_{c_2}(\theta)-\tilde{g}_{c_1}(\theta)\in[b_{\min}(\eta_{c_2}(\theta)-\eta_{c_1}(\theta)),b_{\max}(\eta_{c_2}(\theta)-\eta_{c_1}(\theta))]$$

and

$$\tilde{g}_{c_2}(\theta)-\tilde{g}_{c_1}(\theta)\in\bigg[\frac{b_{\min}}{K}(c_2-c_1),K.b_{\max}(c_2-c_1)\bigg].$$

We deduce from Lemma 6.1 that

$$K.\tilde{K}.b_{\max}(c_2-c_1)\geq\rho(g_{c_2})-\rho(g_{c_1})\geq\frac{b_{\min}}{K.\tilde{K}}(c_2-c_1).$$ □

6.3.3. *Proof of the C^1 regularity.* Here we prove that $\Phi:\mathbb{A}\to\mathbb{A}$ is C^1 in the θ variable, that the invariant foliation is a C^1 lamination and that the dynamics restricted to every leaf is C^1 conjugate to a rotation.

Let us fix c. Then $h_c=Id_{\mathbb{T}}+(\partial u/\partial c)(\cdot,c)$ is a bi-Lipschitz homeomorphism of $\mathbb{T}$ by Proposition 1.1. Then [**1**, Corollary 4] tells us that η_c is in fact C^1 (and the two Green bundles coincide along its graphs) and that h_c is a C^1 diffeomorphism.

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Hence all the points of $\mathbb{A}$ are recurrent. Moreover, as the two Green bundles are equal everywhere, they are continuous. Because they coincide with the tangent space to the foliation, the foliation is a C^1 lamination. This is equivalent to the continuity (in the two variables) of $\partial^2 u/\partial\theta^2$.

As $\Phi(\Theta, c) = (h_c^{-1}(\Theta), \eta_c \circ h_c^{-1}(\Theta))$, we deduce that Φ is C^1 in the θ-direction.

Remark. We do not know if $\partial^2 u/\partial\theta\partial c$ is continuous, and then if $c \mapsto h_c$ is continuous for the C^1 topology.

7. *A strange foliation*

We consider the foliation of $\mathbb{A}$ by the graphs of $\eta_c(\theta) = c + \varepsilon(c)\cos(2\pi\theta)$ where ε is a contraction (k-Lipschitz with $k < 1$) that is not everywhere differentiable. It is a bi-Lipschitz foliation with smooth leaves. Indeed, for all $c < c'$ and θ,

$$(1-k)(c'-c) \leq |c + \varepsilon(c)\cos(2\pi\theta) - c' - \varepsilon(c')\cos(2\pi\theta)| \leq (1+k)(c'-c).$$

Observe that the generating function of this foliation is given by

$$u(\theta, c) = \frac{\varepsilon(c)}{2\pi}\sin(2\pi\theta).$$

7.1. *Proof of Corollaries 1.1–1.3.*

As u is not C^1, we deduce from Theorems 1.1 and 1.3 that this foliation cannot be globally straightenable by a symplectic homeomorphism and also that it cannot be invariant by a symplectic twist diffeomorphism.

Proof of Corollary 1.3. We now work in $\mathbb{R}^2$ and assume by contradiction that there are two open sets $\mathcal{U}$ and $\mathcal{V}$ as in Theorem 1.3, and a symplectic homeomorphism $\Phi : \mathcal{U} \to \mathcal{V}$ that maps the horizontal lines of $\mathcal{U}$ to the leaves of the foliation in $\mathcal{V}$ preserving the orientation of the leaves. Let us denote $(c_-, c_+) = \pi_2(\mathcal{U})$ and $J \subset \mathbb{R}$ such that $\mathcal{V} = \{(\theta, \eta_c(\theta)),\ \theta \in (\alpha, \beta),\ c \in J\}$. By hypothesis, there exists an increasing homeomorphism $\mathfrak{h} : (c_-, c_+) \to J$ such that if $c \in (c_-, c_+)$, the horizontal line $\mathbb{R} \times \{c\} \cap \mathcal{U}$ is mapped to the leaf of the foliation $\{(\theta, \eta_{\mathfrak{h}(c)}(\theta)),\ \theta \in (\alpha, \beta)\}$ by Φ.

Returning to the generating function, if $c \in (c_-, c_+)$ we have $u(0, c) = 0$ and $c + (\partial u/\partial\theta)(c, \theta) = \eta_{\mathfrak{h}(c)}(\theta)$ from which we deduce that

$$\text{for all } (\theta, c) \in \mathcal{W}, \quad u(\theta, c) = \theta(\mathfrak{h}(c) - c) + \frac{\varepsilon \circ \mathfrak{h}(c)}{2\pi}\sin(2\pi\theta).$$

Now let $\theta_1, \theta_2 \in (\alpha, \beta)$ such that the matrix

$$A = \begin{pmatrix} \theta_1 & \sin(2\pi\theta_1)/2\pi \\ \theta_2 & \sin(2\pi\theta_2)/2\pi \end{pmatrix}$$

is invertible. It follows that

$$\text{for all } c \in (c_-, c_+), \quad \begin{pmatrix} \mathfrak{h}(c) \\ \varepsilon \circ \mathfrak{h}(c) \end{pmatrix} = A^{-1} \begin{pmatrix} u(\theta_1, c) + \theta_1 c \\ u(\theta_2, c) + \theta_2 c \end{pmatrix}.$$

As by Theorem 1.3 the function u is C^1, we deduce that $\mathfrak{h}$ and $\varepsilon \circ \mathfrak{h}$ are C^1 functions. It follows that $\mathfrak{h}'$ must vanish at all points c such that ε is not differentiable at $h(c)$.

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As such points are dense, we conclude that $\mathfrak{h}$ is constant which contradicts it being a homeomorphism onto its image. □

7.2. *An exact symplectic twist map that leaves the strange foliation invariant.* Let us prove, however, that this foliation, for a simple choice of ε, can be invariant by a certain C^1 exact symplectic (weakly) twist map as defined below.

Definition. An exact symplectic homeomorphism $f : \mathbb{A} \to \mathbb{A}$ has the *weak twist property* if when $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f, for any $\tilde{\theta} \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde{\theta}, r) \in \mathbb{R}$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Let us now assume that ε is a C^2 function away from $c = 0$ and that at 0 it has left and right derivatives up to order 2. For the sake of simplicity, let us assume also that $\varepsilon(0) = 0$ so that $\mathbb{T} \times \{0\}$ is a leaf of the foliation and that ε restricted to $[0, +\infty)$ (respectively, $(-\infty, 0]$) is the restriction of a C^2 periodic function.

The proof of Theorem 1.4 gives us two C^1 functions

$$\Phi^{\pm} : (\theta, r) \mapsto (h^{\pm}(\theta, r), \eta(h^{\pm}(\theta, r), r))$$

where Φ^+ is a C^1 exact symplectic diffeomorphism of $\mathbb{A}^+ = \mathbb{T} \times [0, +\infty)$ to itself (up to the boundary) and Φ_- is a C^1 exact symplectic diffeomorphism of $\mathbb{A}^- = \mathbb{T} \times (-\infty, 0]$ to itself (up to the boundary). Note that here Φ^+ and Φ^- do not coincide on $\mathbb{T} \times \{0\}$. explaining why the foliation is not straightenable.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be an increasing, C^1 homeomorphism such that $\rho(0) = \rho'(0) = 0$. We denote $f_\rho : (\theta, r) \mapsto (\theta + \rho(r), r)$. The function $f = \Phi^{\pm} \circ f_\rho \circ (\Phi^{\pm})^{-1}$ is well defined on $\mathbb{A}$; it is the identity on $\mathbb{T} \times \{0\}$. It is clearly an area-preserving homeomorphism that is C^1 away from $\mathbb{T} \times \{0\}$.

If $r > 0$ and $\theta \in \mathbb{T}$, let us set $(\Theta, R) = \Phi^+(\theta, r)$. Then one finds that

$$\begin{aligned} Df(\Theta, R) &= D\Phi^+(\theta + \rho(r), r) \cdot Df_\rho(\theta, r) \cdot D\Phi^+(\theta, r)^{-1} \\ &= D\Phi^+(\theta + \rho(r), r) \cdot \begin{pmatrix} 1 & \rho'(r) \\ 0 & 1 \end{pmatrix} \cdot D\Phi^+(\theta, r)^{-1}. \end{aligned}$$

It follows from the properties on Φ^+ and $\rho(0) = \rho'(0) = 0$ that as $R \to 0$, $Df(\Theta, R)$ uniformly converges to the identity. As the same holds for $R < 0$, we deduce that f is in fact C^1 with a differential on $\mathbb{T} \times \{0\}$ being identity.

It remains to choose ρ in such a way that the map obtained is a twist map. We construct it on $[0, +\infty)$. The twist condition we aim at is: for every $\Theta \in \mathbb{R}$, the map $r \mapsto h^+((h_r^+)^{-1}(\Theta) + \rho(r), r)$ is an increasing homeomorphism of $\mathbb{R}$ where $h_r^+ = h^+(\cdot, r)$.

After computation, if we denote $h^+(\theta_r, r) = \Theta$, the derivative of the above function is the following for $r > 0$ (the inequality is our goal):

$$\begin{aligned} &\frac{\partial h^+}{\partial r}(\theta_r + \rho(r), r) - \left(\frac{\partial h^+}{\partial \theta}(\theta_r, r)\right)^{-1} \frac{\partial h^+}{\partial \theta}(\theta_r + \rho(r), r) \frac{\partial h^+}{\partial r}(\theta_r, r) \\ &\quad + \frac{\partial h^+}{\partial \theta}(\theta_r + \rho(r), r)\rho'(r) > 0. \end{aligned}$$

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The first line above is smaller in absolute value than $M_1\rho(r)$ where (recall that by hypothesis, all the functions at play are continuous periodic, hence bounded)

$$M_1 = \left\|\frac{\partial^2 h^+}{\partial r \partial \theta}\right\|_\infty + \left\|\left(\frac{\partial h^+}{\partial \theta}\right)^{-1}\right\|_\infty . \left\|\frac{\partial^2 h^+}{\partial \theta^2}\right\|_\infty \left\|\frac{\partial h^+}{\partial r}\right\|_\infty .$$

On the other hand, the second line is greater than $M_2\rho'(r)$, where we set $M_2 = \min(\partial h^+/\partial\theta) > 0$. If $\rho(t) = t^2 e^{Mt}$ with $M = 2M_1/M_2$, then we have $\rho(0) = \rho'(0) = 0$ and $\rho'(t) > (M_1/M_2)\rho(t)$ which implies the twist condition.

Acknowledgements. The authors are grateful to Philippe Bolle for insightful discussions that helped clarify and simplify some proofs in this work. They thank Sobhan Seyfaddini for his insight on symplectic homeomorphisms. The authors thank the anonymous referees. Their careful reading helped to remove some unclear formulations and misleading notation. M. Zavidovique was financed by a bursary PEPS from CNRS.

A. *Appendix. Foliation by graphs that is the inverse image of the standard foliation by a symplectic map but not by a symplectic homeomorphism*

We will use two special functions:

- $\gamma : \mathbb{T} \to \mathbb{R}$, a C^∞ function such that $\gamma'_{[1/2-\varepsilon,1/2+\varepsilon]} = -1$ and $\gamma'_{\mathbb{T}\setminus[1/2-\varepsilon,1/2+\varepsilon]} > -1$;
- $\zeta : \mathbb{R} \to \mathbb{R}$, a C^∞ function that is increasing, such that $\zeta'(0) = 1$ and $\zeta'_{\mathbb{R}\setminus\{0\}} < 1$ with $\lim_{\pm\infty} \zeta' = \frac{1}{2}$.

The function $u(\theta, c) = \zeta(c)\gamma(\theta)$ defines the foliation in graphs of

$$\eta_c = c + \frac{\partial u}{\partial \theta} = c + \zeta(c)\gamma'.$$

The derivative with respect to c of $\eta_c(\theta)$ is then $(\partial\eta_c/\partial c)(\theta) = 1 + \zeta'(c)\gamma'(\theta)$ which is non-negative, vanishes only for $(\theta, c) \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \times \{0\}$ and is greater that $\frac{1}{3}$ close to $\pm\infty$. Hence every map $c \in \mathbb{R} \mapsto \eta_c(\theta) \in \mathbb{R}$ is a homeomorphism and we indeed have a C^0 foliation.

Let us introduce $h_c(\theta) = \theta + (\partial u/\partial c)(\theta) = \theta + \gamma(\theta)\zeta'(c)$. Its derivative is $1 + \zeta'(c)\gamma'(\theta)$ that is non negative and vanishes only if $(\theta, c) \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \times \{0\}$. Hence h_0 is not a homeomorphism but all the other h_c are homeomorphisms.

We deduce from Theorem 1.3 that this foliation is not symplectically homeomorphic to the standard one.

We will now prove that the map defined by $H(\theta, \eta_c(\theta)) = (h_c(\theta), c)$ is a symplectic map, the limit (for the C^0 topology) of a sequence of symplectic diffeomorphisms.

Let $\gamma_n : \mathbb{T} \to \mathbb{R}$ be a sequence of C^∞ maps that converges to γ in C^1 topology and satisfies $\gamma'_n > -1$. Let (ζ_n) be a sequence of C^∞ diffeomorphisms of $\mathbb{R}$ that C^1 converges to ζ and satisfies $\zeta'_n < 1$. We introduce $u_n(\theta, c) = \gamma_n(\theta)\zeta_n(c)$. Then $\eta_{c,n}(\theta) = c + \zeta_n(c)\gamma'_n(\theta)$ defines a smooth foliation, $h_{c,n}(\theta) = \theta + \gamma_n(\theta)\zeta'_n(c)$ is a smooth diffeomorphism of $\mathbb{T}$ and

$$K_n(\theta, c) = ((h_{c,n})^{-1}(\theta), \eta_{c,n}((h_{c,n})^{-1}(\theta)))$$

is a symplectic smooth diffeomorphism that maps the standard foliation to the foliations by the graphs of $(\eta_{c,n})_{c\in\mathbb{R}}$.

If $H_n = K_n^{-1}$, observe that $H_n = G_n \circ F_n^{-1}$ where:

- $F_n(\theta, c) = (\theta, c + (\partial u_n/\partial\theta)(\theta, c))$ converges uniformly to $F(\theta, c) = (\theta, c + (\partial u/\partial\theta)(\theta, c))$;
- $G_n(\theta, c) = (\theta + (\partial u_n/\partial c)(\theta, c), c)$ converges uniformly to $G(\theta, c) = (\theta + (\partial u/\partial c)(\theta, c), c)$.

Finally, $H_n = G_n \circ F_n^{-1}$ converges uniformly to $H = G \circ F^{-1}$.

B. *Appendix. Green bundles*

Here we recall the theory of Green bundles. More details and proofs can be found in [**1**, **2**]. We fix a lift F of a symplectic twist diffeomorphism f.

Notation.

- $V(x) = \{0\} \times \mathbb{R} \subset T_x\mathbb{R}^2$, and for $k \neq 0$ we set $G_k(x) = DF^k(F^{-k}x)V(f^{-k}x)$.
- The slope of G_k (when defined) is denoted by s_k:

$$G_k(x) = \{(\delta\theta, s_k(x)\delta\theta);\ \delta\theta \in \mathbb{R}\}.$$

- If γ is a real Lipschitz function defined on $\mathbb{T}$ or $\mathbb{R}$, then

$$\gamma'_+(x) = \limsup_{\substack{y,z\to x\\ y\neq z}} \frac{\gamma(y)-\gamma(z)}{y-z} \quad \text{and} \quad \gamma'_-(t) = \liminf_{\substack{y,z\to x\\ y\neq z}} \frac{\gamma(y)-\gamma(z)}{y-z}.$$

Then:

(1) if the orbit of $x \in \mathbb{R}^2$ is minimizing (see [**1**, **2**] for the definition; the only thing needed here is that an orbit that is contained in an invariant curve that is a graph is minimizing), we have

$$\text{for all } n \geq 1, \quad s_{-n}(x) < s_{-n-1}(x) < s_{n+1}(x) < s_n(x);$$

(2) in this case, the two *Green bundles* at x are $G_+(x), G_-(x) \subset T_x(\mathbb{R}^2)$ with slopes s_-, s_+ where $s_+(x) = \lim_{n\to+\infty} s_n(x)$ and $s_-(x) = \lim_{n\to+\infty} s_{-n}(x)$;

(3) the two Green bundles are invariant under Df: $Df(G_\pm) = G_\pm \circ f$;

(4) we have $s_+ \geq s_-$;

(5) the map s_- is lower semi-continuous and the map s_+ is upper semi-continuous;

(6) hence $\{G_- = G_+\}$ is a G_δ subset of the set of points whose orbit is minimizing (this last set is a closed set) and $s_- = s_+$ is continuous at every point of this set.

Let us focus on the case of an invariant curve that is the graph of $\gamma : \mathbb{T} \to \mathbb{R}$. Then we have the following propositions.

PROPOSITION B.1. *Assume that the graph of a function $\gamma \in C^0(\mathbb{T}, \mathbb{R})$ is invariant by F. Then the orbit of any point contained in the graph of γ is minimizing and we have*

$$\textit{for all } \theta \in \mathbb{T}, \quad s_-(\theta, \gamma(\theta)) \leq \gamma'_-(\theta) \leq \gamma'_+(\theta) \leq s_+(\theta, \gamma(\theta)).$$

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PROPOSITION B.2. (Dynamical criterion) *Assume that x has its orbit that is minimizing and that is contained in some strip $\mathbb{R} \times [-K, K]$ (for example, x is in some invariant graph) and that $v \in T_x\mathbb{R}^2\backslash\{0\}$. Then:*

- *if* $\liminf_{n\to+\infty}|D(\pi \circ F^n)(x)v| < +\infty$, *then* $v \in G_-(x)$*;*
- *if* $\liminf_{n\to+\infty}|D(\pi \circ F^{-n})(x)v| < +\infty$, *then* $v \in G_+(x)$.

In particular, if the dynamics restricted to some invariant graph is totally periodic, then along this graph we have $G_- = G_+$ and the graph is C^1. The C^1 property can also be proved by using the implicit functions theorem.

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