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GLOBAL EXISTENCE OF SOLUTIONS TO DEGENERATE WAVE EQUATIONS WITH DISSIPATIVE TERMS

Mohammed Aassila

In this paper we prove the global existence and study the asymptotic behaviour of solutions to a degenerate wave equation with a nonlinear dissipative term.

1. Introduction

Nonlinear vibrations of an elastic string are written in the form of partial integro-differential equations

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left(p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} + f \tag{1.1}$$

for $0 < x < L$, $t \geq 0$, where u is the lateral deflection, x the space co-ordinate, t the time, E the Young's modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension, and f the external force. Kirchhoff **[10]** first introduced (1.1) in the study of the oscillations of strechted strings and plates, so that (1.1) is called the wave equation of Kirchhoff type after him. Moreover, we call (1.1) a degenerate equation when $p_0 = 0$ and a non-degenerate one when $p_0 > 0$. Concerning the solvability of (1.1), the analytic case is rather well known in general dimension, see for example **[3, 17, 15, 2, 6, 7, 5]** among others. On the other hand, in the case of Sobolev space we know only local solutions in time solvability, see for example **[1, 4, 8, 9, 11, 12, 18, 19, 20]**. So far, there has been no work to determine the global solutions in time existence in Sobolev spaces, because the problem is given by an interior initial boundary value problem for a hyperbolic equation, the solutions of which have a non-decay property. As well known now, deriving solutions global in time solvability deeply depends on the decay structure of the solutions to the corresponding linearised problem of (1.1). Therefore, we are led naturally to the equations of Kirchhoff type with a dissipative term which guarantees the decay properties of the solutions to the linearised problem. To be precise, in this paper we are interested in the following problem

$$\text{(P)}\qquad \begin{cases} u'' - \|\nabla u\|_2^2 \Delta u + g(u') = 0 \quad \text{in} \quad \Omega \times [0, +\infty), \\ u = 0 \quad \text{on} \quad \Gamma \times [0, +\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega, \end{cases}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and $g:\mathbb{R}\to\mathbb{R}$ is a non-decreasing continuous function such that

$$g(0)=0, \tag{1.2}$$

$$g'(x)\geq\tau>0, \tag{1.3}$$

$$\left|g(x)\right|\leq c_1|x|^q, \tag{1.4}$$

c_1 and τ are two positive constants and $q\geq 1$ is such that $(n-2)q\leq n+2$.

Concerning global existence of solutions to quasilinear wave equations, the degenerate case is more difficult to handle than the non-degenerate case, but when the equation includes some dissipative terms u', $-\Delta u'$, $\Delta^2 u$, et cetera, we may expect some decay properties of solutions under suitable assumptions and these are useful for analyses of solutions global in time solvability. When the damping term is linear, that is, $g(x)=\delta x$, problem (P) was investigated by Nishihara-Yamada [**16**] and Mizumachi [**13**]. In this paper, we prove an existence and uniqueness theorem and study the asymptotic behaviour for solutions to (P) under the hypotheses (1.2)-(1.4).

Throughout this paper the functions are all real valued and the notations are as usual, in particular we shall denote by $\|\cdot\|_p$ $(p\geq 1)$ the usual L^p-norm. Our main results are

THEOREM 1.1. (Existence and uniqueness.) *Suppose (1.2)-(1.4) hold and* $(u_0,u_1)\in\left(H_0^1\cap H^2\right)\times\left(H_0^1\cap L^{2q}\right)$ *with* $u_0(x)\not\equiv 0$ *for* $x\in\Omega$. *Then there exists a positive number* ε *depending on* τ, $\|\nabla u_0\|_2$ *and* $\|u_1\|_2$ *such that if:*

$$\frac{\|\nabla u_1\|_2^2}{\|\nabla u_0\|_2^2}+\|\Delta u_0\|_2^2\leq\varepsilon \tag{1.5}$$

then (P) admits a unique weak solution u *which satisfies* $\|\nabla u(t)\|>0$ *for all* $t\in[0,+\infty)$.

THEOREM 1.2. (Energy decay.)

In addition to (1.2)-(1.5), assume that

$$\left|g(x)\right|\leq c_2|x|\quad \text{if}\quad |x|\leq 1. \tag{1.6}$$

Then the total energy

$$E(t)=\left\|u'(t)\right\|_2^2+\frac{1}{2}\left\|\nabla u(t)\right\|_2^4$$

satisfies

$$E(t)\leq\frac{c_3E(0)}{(1+t)^2},\quad \text{for all}\quad t\geq 0,$$

where c_2 *and* c_3 *are positive constants.*

The contents of this paper are as follows. In Section 2, the existence and uniqueness of a solution are proved (Theorem 1.1). In Section 3, asymptotic behaviour is established (Theorem 1.2).

2. Existence and uniqueness

It is well known that the operator $-\Delta$ with Dirichlet condition has an infinite sequence of eigenvalues (λ_j^2) such that

$$0 < \lambda_1^2 \le \lambda_2^2 \le \cdots \le \lambda_j^2 \le \cdots \to +\infty \quad \text{as} \quad j \to \infty,$$

and that there exists a complete orthonormal system (w_j) in $L^2(\Omega)$, each w_j being an eigenvector corresponding to λ_j^2. Therefore, each $u \in L^2(\Omega)$ has a Fourier expansion in $L^2(\Omega)$:

$$u = \sum_{j=1}^{\infty} u_j w_j \quad \text{with} \quad \|u\|_2 = \left(\sum_{j=1}^{\infty} u_j^2\right)^{1/2}.$$

We apply the Faedo-Galerkin procedure. For each $m \ge 1$, we take an approximating solution $u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j$ as a solution of the initial value problem for the following system of ordinary differential equations:

$$\left(u_m''(t) - \left(\frac{1}{m} + \left\|\nabla u_m(t)\right\|_2^2\right)\Delta u_m(t) + g\left(u_m'(t)\right), w\right) = 0 \quad \forall w \in V_m \tag{2.1}$$

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w_j) w_j, \; u_{0m} \to u_0 \quad \text{in} \quad H_0^1 \cap H^2 \tag{2.2}$$

$$u_m'(0) = u_{1m} = \sum_{j=1}^{m} (u_1, w_j) w_j, \; u_{1m} \to u_1 \quad \text{in} \quad H_0^1 \cap L^{2q}, \tag{2.3}$$

where V_m is an m-dimensional vector space spanned by $\{w_1, \cdots, w_m\}$. By virtue of the theory of ordinary differential equations $u_m(t)$ can be defined on some interval $[0, t_m)$. In the next step, we obtain a priori estimates for the solution $u_m(t)$, so that it can be extended outside $[0, t_m)$, to obtain one solution defined for all $t > 0$.

(i) A priori estimate 1: Taking $w = 2u_m'(t)$ in (2.1), we have

$$\frac{d}{dt}\left(\left\|u_m'(t)\right\|_2^2 + \frac{1}{m}\left\|\nabla u_m(t)\right\|_2^2 + \frac{1}{2}\left\|\nabla u_m(t)\right\|_2^4\right) + 2\int_\Omega g\left(u_m'(t)\right)u_m'(t)\,dx = 0.$$

Integrating in $[0, t]$, $t < t_m$, we obtain

$$\left\|u_m'(t)\right\|_2^2 + \frac{1}{2}\left\|\nabla u_m(t)\right\|_2^4 + 2\int_0^t \int_\Omega g\left(u_m'(s)\right)u_m'(s)\,ds dx \le \|u_1\|_2^2 + \|\nabla u_0\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^4. \tag{2.4}$$

Hence,

$$\left\|u_m'(t)\right\|_2, \left\|\nabla u_m(t)\right\|_2 \le c. \tag{2.5}$$

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Here and after we denote by c various positive constants independent of m and t. From (2.4) we conclude that $u_m(t)$ can be extended to $[0,T)$ for any $0 < T < +\infty$. Furthermore, we conclude from (2.4) and (1.4) that

$$u'_m g(u'_m) \quad \text{is bounded in} \quad L^1(\Omega \times [0,T]), \tag{2.6}$$

$$g(u'_m) \quad \text{is bounded in} \quad L^{\frac{q+1}{q}}(\Omega \times [0,T]). \tag{2.7}$$

(ii) A PRIORI ESTIMATE 2: Let us define

$$F_m(t) := \frac{\left\|\nabla u'_m(t)\right\|_2^2}{m^{-1} + \left\|\nabla u_m(t)\right\|_2^2} + \left\|\Delta u_m(t)\right\|_2^2 := f_m(t) + \left\|\Delta u_m(t)\right\|_2^2.$$

A simple computation shows that

$$\begin{aligned} F'_m(t) &= \frac{\left(u''_m - (m^{-1} + \|\nabla u_m\|_2^2)\Delta u_m, -2\Delta u'_m\right)}{m^{-1} + \|\nabla u_m\|_2^2} - \frac{2\alpha(\nabla u_m, \nabla u'_m)\|\nabla u'_m\|_2^2}{(m^{-1} + \|\nabla u_m\|_2^2)^2} \\ &= \frac{2\left(-g(u'_m), -\Delta u'_m\right)}{m^{-1} + \|\nabla u_m\|_2^2} - \frac{2\alpha(\nabla u_m, \nabla u'_m)\|\nabla u'_m\|_2^2}{(m^{-1} + \|\nabla u_m\|_2^2)^2} \\ &\le -2\tau f_m(t) + c f_m(t)^{3/2}. \end{aligned}$$

Since

$$F'_m(t) + \left(2\tau - c f_m(t)^{1/2}\right) f_m(t) \le 0, \tag{2.8}$$

it is easy to see that $F_m(t) \le F_m(0)$ for $0 \le t \le t^*$ if $f_m(t) \le (\tau/c)^2$ for $0 \le t \le t^*$.

Assume $F(0) \le (\tau/c)^2/2$. Since

$$F_m(0) \to F(0) = \frac{\|\nabla u_1\|_2^2}{\|\nabla u_0\|_2^2} + \|\Delta u_0\|_2^2 \quad \text{as} \quad m \to +\infty,$$

it follows that $F_m(0) \le (\tau/c)^2$ for sufficiently large m, and therefore $f_m(t) \le F_m(t) \le (\tau/c)^2$. Thus, taking $\varepsilon = (\tau/c)^2/2$ in (1.5) we may get $t^* = \infty$. Integrating (2.8) over $[0,t)$, we obtain

$$F_m(t) + \tau \int_0^t f_m(x)\,dx \le \left(\frac{\tau}{c}\right)^2,$$

which implies

$$\left\|\Delta u_m(t)\right\|_2 \le c, \tag{2.9}$$

$$\frac{\left\|\nabla u'_m(t)\right\|_2}{\left(m^{-1} + \left\|\nabla u_m(t)\right\|_2^2\right)^{1/2}} \le c, \tag{2.10}$$

$$\int_0^t \frac{\left\|\nabla u'_m(s)\right\|_2^2}{m^{-1} + \left\|\nabla u_m(s)\right\|_2^2}\,ds \le c. \tag{2.11}$$

(iii) A PRIORI ESTIMATE 3: Taking $w = u''_m(t)$ in (2.1) and choosing $t = 0$, we obtain that

$$\begin{aligned}\|u''_m(0)\|_2 &\le \Big(\frac{1}{m} + \|\nabla u_{0m}\|_2^2\Big)\|\Delta u_{0m}\|_2 + \|g(u_{1m})\|_2 \\ &\le \big(1 + \|\nabla u_{0m}\|_2^2\big)\|\Delta u_{0m}\|_2 + \|g(u_{1m})\|_2.\end{aligned}$$

Since $g(u_{1m})$ is bounded in $L^2(\Omega)$ by (1.4), hence $u''_m(0)$ is bounded in $L^2(\Omega)$. Next, by differentiation of (2.1) and multiplication with $2u''_m(t)$ we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|u''{}_m(t)\|_2^2 + \Big(\frac{1}{m} + \|\nabla u_m(t)\|_2^2\Big)\|\nabla u'_m(t)\|_2^2\Big) + 2\int_\Omega u''^2_m(t) g'(u'_m(t))\,dx \\ &\quad = 2\big(\nabla u_m(t), \nabla u'_m(t)\big)\|\nabla u'_m(t)\|_2^2 + 4(\nabla u_m, \nabla u'_m) \times \int_\Omega u''_m \Delta u_m\,dx \\ &\quad \le 2\|\nabla u_m\|_2\|\nabla u'_m\|_2^3 + 4\|\nabla u_m\|_2\|\nabla u'_m\|_2\|u''_m\|_2\|\Delta u_m\|_2 \\ &\quad \le 2\|\nabla u_m\|_2\|\nabla u'_m\|_2^3 + 16\|\nabla u_m\|_2^2\|\nabla u'_m\|_2^2\|\Delta u_m\|_2^2 + \|u''_m\|_2^2,\end{aligned}$$

and then

$$\begin{aligned}&\frac{d}{dt}\Big(\|u''_m(t)\|_2^2 + \frac{1}{m}\|\nabla u_m(t)\|_2^2 + \|\nabla u_m(t)\|_2^2\|\nabla u'_m(t)\|_2^2\Big) \\ &\qquad\qquad + 2\int_\Omega u''^2_m g'(u'_m)\,dx \le g_m(t) + \|u''_m(t)\|_2^2\end{aligned}$$

where

$$g_m(t) = 2\|\nabla u_m\|_2\|\nabla u'_m\|_2^3 + \|\nabla u_m\|_2^2\|\nabla u'_m\|_2^2\|\Delta u_m\|_2^2.$$

Whence

$$\begin{aligned}&\|u''_m(t)\|_2^2 + \frac{1}{m}\|\nabla u_m(t)\|_2^2 + \|\nabla u_m(t)\|_2^2\|\nabla u'_m(t)\|_2^2 \\ &\quad \le e^T\Big(\|u''_m(0)\|_2^2 + \frac{1}{m}\|\nabla u_m(0)\|_2^2 + \|\nabla u_m(0)\|_2^2\|\nabla u'_m(0)\|_2^2\Big) + e^T\int_0^T g_m(s)\,ds\end{aligned}$$

for all $t \in \mathbb{R}_+$, and we deduce that

(2.13) $$u''_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;L^2).$$

(iv) PASSAGE TO THE LIMIT: By applying the Dunford-Pettis theorem and the Riesz lemma, we conclude from (2.5)-(2.7), (2.9)-(2.12) and (2.13), replacing the sequence u_m with a subsequence if needed, that

(2.14) $$u_m \rightharpoonup u \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1 \cap H^2)$$

(2.15) $$u'_m \rightharpoonup u' \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1)$$

(2.16) $$u''_m \rightharpoonup u'' \quad \text{weak-star in} \quad L^\infty(0,T;L^2)$$

(2.17) $$u'_m \to u' \quad \text{almost everywhere in} \quad \Omega \times [0,T]$$

(2.18) $$g(u'_m) \rightharpoonup \chi \quad \text{weak-star in} \quad L^{(q+1)/q}(\Omega \times (0,T))$$

(2.19) $$\|\nabla u_m\|_2^2 \Delta u_m \rightharpoonup \psi \quad \text{weak-star in} \quad L^\infty(0,T;L^2)$$

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for suitable functions $u \in L^\infty(0,T;H_0^1 \cap H^2)$, $\psi \in L^\infty(0,T;L^2)$ and $\chi \in L^{(q+1)/q}(\Omega \times (0,T))$. We have to show that u is a solution of (P). From (2.14)-(2.16) we deduce that

$$\int_\Omega u_m(0) w_j \, dx \to \int_\Omega u(0) w_j \, dx \quad \text{and} \quad \int_\Omega u'_m(0) w_j \, dx \to \int_\Omega u'(0) w_j \, dx$$

for any fixed $j \geq 1$. From (2.2)-(2.3) we deduce that $u(0) = u_0$ and $u'(0) = u_1$.

Now let us prove that $\psi = \|\nabla u\|_2^2 \Delta u$, that is,

$$\|\nabla u_m\|_2^2 \Delta u_m \rightharpoonup \|\nabla u\|_2^2 \Delta u \quad \text{weak-star in} \quad L^\infty(0,+\infty;L^2).$$

For $v \in L^2(0,T;L^2)$, we have

$$\text{(2.20)} \quad \int_0^T (\psi - \|\nabla u\|_2^2 \Delta u, v)\, dt = \int_0^T (\psi - \|\nabla u_m\|_2^2 \Delta u_m, v)\, dt \\ + \int_0^T \|\nabla u\|_2^2 (\Delta u_m - \Delta u, v)\, d + \int_0^T (\|\nabla u_m\|_2^2 - \|\nabla u\|_2^2)(\Delta u_m, v)\, dt.$$

The first and second term in (2.20) tend to zero as $m \to +\infty$, and for the third one we have

$$\int_0^T (\|\nabla u_m\|_2^2 - \|\nabla u\|_2^2)(\Delta u_m, v)\, dt \\ \leq c \int_0^T (\|\nabla u_m - \nabla u\|_2)(\|\nabla u_m\|_2 + \|\nabla u\|_2)\|\Delta u_m\|_2 \|v\|_2 \, dt.$$

As (u_m) is bounded in $L^\infty(0,T;H_0^1(\Omega))$ and the injection $H_0^1 \hookrightarrow L^2$ is compact, we have $u_m \to u$ strongly in $L^2(0,T;L^2)$, and hence

$$\int_0^T \int_\Omega (u''_m - \|\nabla u_m\|_2^2 \Delta u_m) v \, dx dt \to \int_0^T \int_\Omega (u'' - \|\nabla u\|_2^2 \Delta u) v \, dx dt$$

as $m \to +\infty$ for each fixed $v \in L^{q+1}(0,T;H_0^1)$.

It remains to show that

$$\text{(2.21)} \qquad \int_0^T \int_\Omega v g(u'_m)\, dx dt \to \int_0^T \int_\Omega v g(u')\, dx dt \quad \text{as} \quad m \to +\infty.$$

It follows from (2.6) and Fatou's lemma that $u' g(u') \in L^1(\Omega \times (0,T))$. This yields $g(u') \in L^1(\Omega \times (0,T))$. On the other hand, $g(u'_m) \to g(u')$ almost everywhere in $\Omega \times [0,T]$.

Let $E \subset \Omega \times [0,T]$ and set

$$E_1 := \left\{(x,t) \in E\,;\, g\big(u_m'(x,t)\big) \le \frac{1}{\sqrt{|E|}}\right\}, \quad E_2 := E - E_1,$$

where $|E|$ is the measure of E.

If $M(r) := \inf\left\{|x|\,;\, x \in \mathbb{R} \text{ and } |g(x)| \ge r\right\}$, then we have

$$\int_E |g(u_m')|\,dxdt \le \sqrt{|E|} + \left(M\Big(\frac{1}{\sqrt{|E|}}\Big)\right)^{-1} \int_{E_2} |u_m' g(u_m')|\,dxdt.$$

Applying (2.6), we deduce that $\sup_m \int_E |g(u_m')|\,dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g(u_m') \to g(u')$ in $L^1\big(\Omega \times (0,T)\big)$, hence

$$g(u_m') \rightharpoonup g(u') \quad \text{weak-star in} \quad L^{(q+1)/q}\big(\Omega \times [0,T]\big)$$

and this implies (2.2). Hence

$$\int_0^T \int_\Omega \big(u'' - \|\nabla u\|_2^2 \Delta u + g(u')\big) v\,dxdt = 0, \quad \forall v \in L^{q+1}\big(0,T; H_0^1\big).$$

(v) $\|\nabla u(t)\|_2 > 0$ for $0 \le t < +\infty$: We need the following lemma

Lemma. *If $v : [-T,T] \to H_0^1 \cap H^2$ is a weak solution of*

$$\begin{cases} v''(t) - \|\nabla v(t)\|_2^2 \Delta v(t) + g(v'(t)) = 0 & -T \le t \le T, \\ v(0) = 0, \quad v'(0) = 0, \end{cases}$$

then $v(t) = 0$ for $t \in [-T,T]$.

Proof: Multiplying with $2v'(t)$, we obtain

$$\frac{d}{dt}\left[\|v'(t)\|_2^2 + \frac{1}{2}\|\nabla v(t)\|_2^4\right] + 2\int_\Omega g(v')v'\,dx = 0,$$

and the integration of the above identity with (1.3) gives

$$\|v'(t)\|_2^2 + \frac{1}{2}\|\nabla v(t)\|_2^4 \le 2|\tau| \int_0^{|t|} \|v'(s)\|_2^2\,ds.$$

Gronwall's inequality assures $v'(t) = 0$ and $v(t) = 0$ for $t \in [-T,T]$. ∎

We now turn to the proof of $\|\nabla u(t)\| > 0$, $\forall t \ge 0$:

Let T be a point such that $\nabla u(T) = 0$. Since the a priori estimates imply that $\|\nabla u'(t)\|_2 / \|\nabla u(t)\|_2$ is bounded, then $\nabla u'(T)$ must be zero. Hence, the above lemma implies that $u(t) = 0$ $(0 \le t \le T)$, which contradicts $u_0 \ne 0$. Thus we obtain $\|\nabla u(t)\|_2 > 0$ for all $t > 0$.

(vi) Uniqueness: The uniqueness is a consequence of the monotonicity of g and Gronwall's lemma. We shall omit the proof since it can be obtained in a standard way.

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3. Energy estimate

For the proof of the energy decay, we need the following lemma

Lemma. (Nakao [**14**].) *Let $\phi(t)$ be a bounded and nonnegative function on $[0,\infty)$ satisfying*

$$\sup_{t\le s\le t+1}\phi^{1+r}(s)\le k_0\{\phi(t)-\phi(t+1)\}$$

for $r>0$ and $k_0>0$. Then

$$\phi(t)\le \frac{c}{(1+t)^{1/r}}\quad \text{for}\quad t\ge 0,$$

with some constant $c=c\big(r,k_0,\phi(0)\big)$.

We shall follow the method developed in [**14**], so we give here only the main steps of the proof.

Taking the scalar product of the first equation of (P) with $2u'$, we have

$$E'(t)+2\int_\Omega u'g(u')\,dx=0. \tag{3.1}$$

Integrating (3.1) over $[t,t+1]$, we have

$$2\int_t^{t+1}\int_\Omega g(u')u'\,dxds=E(t)-E(t+1)\ \big(:=D(t)^2\big). \tag{3.2}$$

Then there exist $t_1\in\big[t,t+(1/4)\big]$, $t_2\in\big[t+(3/4),t+1\big]$ such that

$$\int_\Omega g\big(u'(t_i)\big)u'(t_i)\,dx\le 4D(t)^2\quad\text{for}\quad i=1,2. \tag{3.3}$$

Taking the scalar product of the first equation in (P) with $2u$ and integrating it over $[t_1,t_2]$, we have from (3.2)-(3.3) that

$$\int_{t_1}^{t_2}\|\nabla u(s)\|_2^4\,ds\le c\Big\{D(t)^2+D(t)\sup_{t\le s\le t+1}\big(\nabla u(s)\big)_2\Big\}\ \big(:=A(t)^2\big). \tag{3.4}$$

Using the Poincaré inequality, we obtain from (3.1), (3.2), (3.4) that

$$E(t_2)\le\frac{1}{t_2-t_1}\int_{t_1}^{t_2}E(s)\,ds\le cA(t)^2$$

and hence

$$\begin{aligned}\sup_{t\le s\le t+1}E(s)^{3/2}&\le E(t_2)+2\int_t^{t+1}\int_\Omega u'g(u')\,dxds\\ &\le cA(t)^2\le c\Big\{D(t)^2+D(t)\sup_{t\le s\le t+1}E(s)^{1/4}\Big\}.\end{aligned}$$

Using Young's inequality, we arrive at

$$\sup_{t\leq s\leq t+1} E(s)^{3/2} \leq cD(t)^2 = c\big(E(t) - E(t+1)\big).$$

Hence, Nakao's lemma gives

$$E(t) \leq \frac{cE(0)}{(1+t)^2}, \quad t \geq 0.$$

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Centre de Recherches Mathematiques
Universite de Montreal
C.P. 6128-A
Montreal (QC) H3C 3J7
Canada
e-mail: aassila@crm.umontreal.ca

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