EXTREMA IN SPACE-TIME

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1. Introduction. Consider an astronomer and his observation field, i. e., the set of observable (light or radio) signal-emitting loci of the universe. Let the observation field be ordered by attaching a date to each observable locus indicating the time in the history of the universe that the signal was emitted from its source. Whereas both the astronomer and his observation field age with time, the observations of the astronomer may trace a sequence of loci whose time labels proceed *forward or backward in time* (cf. Appendix).

Consider now a finite set S of events in L^n, n-dimensional space–time (Riemannian n-space having the fundamental form

$$\Phi = (dx^1)^2 + \ldots + (dx^{n-1})^2 - (dt)^2, \qquad n \geqslant 2).$$

A rectilinear world-line segment with end points in S will be called a *rectilinear connection* in S, and a set of rectilinear connections that form a polygon with vertex set S a *polygonal connection* of S. The *clock time* of a polygonal connection is defined to be the sum of all the time separations of its rectilinear connections. (The *time separation* of a rectilinear connection with end points

$$E_u\colon (x_u^1, \ldots, x_u^{n-1}, t_u) \quad \text{and} \quad E_v\colon (x_v^1, \ldots, x_v^{n-1}, t_v)$$

is equal to

$$\sqrt{(t_v - t_u)^2 - \sum_{i=1}^{n-1} (x_v^i - x_u^i)^2}$$

and will be denoted by $s(E_u E_v)$.) A polygonal connection having either the least or the greatest clock time of all possible "circuit states," i.e., all possible polygonal connections of S, will be called *extreme* with respect to S. In this paper criteria are established, which, if satisfied by a set S of events in L^n, enable one to obtain extreme polygonal connections of S immediately.

2. Summary of results. A set S of events is called a *timelike distribution* if each event of S is in the interior of the time cone of every other event in S.

A timelike distribution S in L^n ($n \geqslant 2$) is said to satisfy the *Four Point Condition* if the events of S can be labelled $P_1, P_2, \ldots, P_k$ so as to satisfy the following condition:

$$\text{(2.1)} \quad \begin{cases} \textit{For all sets of integers } \{a, b, c, d\} \textit{ such that} \\ \qquad\qquad 1 \leqslant a < b < c < d \leqslant k \\ \textit{it follows that} \\ s(P_a P_b) + s(P_c P_d) \leqslant s(P_a P_c) + s(P_b P_d) \leqslant s(P_a P_d) + s(P_b P_c). \end{cases}$$

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BASIC LEMMA. *If a timelike distribution S of k events in L^n ($n \geqslant 2$) satisfies the Four Point Condition, then the polygonal connections*

$$[\ldots P_7 P_5 P_3 P_1 P_2 P_4 P_6 \ldots]$$

and

$$[\ldots P_{k-5} P_5 P_{k-3} P_3 P_{k-1} P_1 P_k P_2 P_{k-2} P_4 P_{k-4} P_6 \ldots]$$

have respectively the least and the greatest clock time of the set of all possible polygonal connections of S.

The proof of this lemma is identical with that of (**1**, Theorem III). The latter was proved in a Euclidean setting, but it is verified directly that this is not necessary.

In 1964, Lerman obtained the result (unpublished) that, if a set of k non-collinear points in the Euclidean plane satisfies the Four Point Condition, i.e., the hypothesis of (**1**, Theorem III), then $k \leqslant 8$. This is contrasted with the fact that in L^2, for any $k \geqslant 4$, it is possible to select k non-collinear events and label them so as to satisfy (2.1), e.g., any k events lying on a convex timelike arc (in L^2) can be so labelled (cf. Theorem 4 below). Higher-dimensional curves having this property are constructed in §10. A still wider class of event distributions in L^n ($n \geqslant 2$) satisfying the Four Point Condition is given in Theorem 2 below.

Our first theorem consists of a necessary and sufficient condition that a timelike distribution satisfy the Four Point Condition.

THEOREM 1. *Let*

$$S = \{E_1, E_2, \ldots, E_k\} \qquad (t_1 < t_2 < \ldots < t_k;\ k \geqslant 4)$$

denote a timelike distribution in L^n ($n \geqslant 2$). Then S satisfies the Four Point Condition if and only if at least one of the following four labellings of S

$$\begin{cases} \qquad \{P_1, P_2\} = \{E_1, E_2\}, \\ P_i = E_i \qquad (i = 3, 4, \ldots, k-2), \\ \textit{and} \qquad \{P_{k-1}, P_k\} = \{E_{k-1}, E_k\} \end{cases} \tag{2.2}$$

is a labelling for which (2.1) *is satisfied*; cf. §4.

Remark. Since there are only four labellings to check, Theorem 1 yields an effective computational method for determining whether or not a timelike distribution in L^n ($n \geqslant 2$) satisfies the Four Point Condition.

Let U, V, and W denote three events in L^n ($n \geqslant 2$), $F(W)$ the interior of the future time cone of W, and

$$R(U, V; W) = \{E \in F(W): s(UW) - s(VW) \leqslant s(UE) - s(VE)\}.$$

THEOREM 2. *Let $S = \{E_1, E_2, \ldots, E_k\}$ ($k \geqslant 4$) denote a set of events in L^n ($n \geqslant 2$) such that for $t = 2, 3,$*

$$E_t \in F(E_{t-1})$$

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and for $t = 4, 5, \ldots, k$,

$$E_t \in \cap R(E_i, E_j; E_{t-1}) \qquad (1 \leqslant i < j < t-1);$$

then S *is a timelike distribution satisfying the Four Point Condition*; cf. §5.

In the next theorem we define a class of event distributions which includes the class of event distributions given in Theorem 2; cf. (5.2). For each S in this class we are able to determine a minimal polygonal connection.

THEOREM 3. *Let* $S = \{E_1, E_2, \ldots, E_k\}$ $(k \geqslant 4)$ *denote a set of events in* L^n $(n \geqslant 2)$ *such that for* $t = 2, 3$,

$$E_t \in F(E_{t-1})$$

and for $t = 4, 5, \ldots, k$,

$$E_t \in \cap R(E_{i-3}, E_{i-2}; E_{i-1}) \qquad (i = 4, 5, \ldots, t);$$

then $[\ldots E_7 E_5 E_3 E_1 E_2 E_4 E_6 \ldots]$ *has the least clock time of the set of all possible polygonal connections of* S; cf. §6.

The following lemma and Theorem 4 below state some geometric properties of event distributions in L^2 that satisfy the Four Point Condition.

LEMMA 1. *If the events of a non-collinear timelike distribution of events in* L^2 *fall on the boundary* B *of their convex hull and are labelled so as to satisfy* (2.1), *then the labelling must be cyclic with respect to* B; cf. §7.

THEOREM 4. *Let* k *events* $(k \geqslant 4)$ *of a timelike distribution of events in* L^2 *fall on the boundary* B *of their convex hull. Let* P, Q, R, *and* S *denote the events with the minimal, next to minimal, next to maximal, and maximal t-co-ordinates respectively. Then, the* k *events satisfy the Four Point Condition if and only if* $B - \{P, S\}$ *consists of one component or one of its two components contains at most two events from among the* k *given events and if an event is in this component, then that event is either* Q *or* R; cf. §8.

A feature of L^n is the existence of rectilinear connections having time separation equal to zero (world-lines of photons). This gives rise to the consideration of a special class of minimal polygonal connections, namely those with clock time equal to zero. We call these *zero polygonal connections*. An event distribution S is said to be *zero separated* if each pair of events in S constitutes the end points of a polygonal path consisting of rectilinear connections of S each of which has time separation equal to zero.

Remark. The vertex set of a zero polygonal connection is zero separated, but not conversely; cf. (9.1).

Let $C(E)$ denote the *light cone centred at* E, i.e., the boundary of the time cone of the event E.

THEOREM 5. *Let* $S = \{E_1, E_2, \ldots, E_k\}$ $(k \geqslant 4)$ *denote a set of events in* L^n $(n \geqslant 2)$ *which is zero separated and such that for each* $i = 1, 2, \ldots, k$, *the set* $C(E_i) \cap S$ *contains exactly three events; then* S *is the vertex set of a unique zero polygonal connection*; cf. §9.

Remark. It is possible for a set $S = \{E_1, E_2, \ldots, E_k\}$ to have the property that for each $i = 1, 2, \ldots, k$, $C(E_i) \cap S$ contains exactly three events and yet not be the vertex set of a zero polygonal connection; cf. (9.2).

We conclude by formulating some open questions; cf. §11.

3. Some lemmas. Properties of sets in L^n $(n \geqslant 2)$ that satisfy the Four Point Condition.

(3.1) *If* E_a, E_b, E_x, *and* E_y *denote four events of a timelike distribution of events in* L^n $(n \geqslant 2)$ *such that* $t_a < t_b < t_x < t_y$, *then*

$$s(E_a E_b) + s(E_x E_y) < \begin{cases} s(E_a E_x) + s(E_b E_y), \\ s(E_a E_y) + s(E_b E_x). \end{cases}$$

Proof. By the triangle inequality in L^n, $s(E_a E_b) < s(E_a E_x)$ and $s(E_x E_y) < s(E_b E_y)$. Thus,

$$s(E_a E_b) + s(E_x E_y) < s(E_a E_x) + s(E_b E_y).$$

Since

$$s(E_a E_b) + s(E_x E_y) < s(E_a E_x) + s(E_x E_y)$$

and by the triangle inequality we have $s(E_a E_x) + s(E_x E_y) \leqslant s(E_a E_y)$, it follows that

$$s(E_a E_b) + s(E_x E_y) < s(E_a E_y) + s(E_b E_x).$$

(3.2) *Any set of four events of a timelike distribution in* L^n $(n \geqslant 2)$ *satisfies the Four Point Condition.*

Proof. Let the events be labelled as in (3.1). If

$$s(E_a E_x) + s(E_b E_y) \leqslant s(E_a E_y) + s(E_b E_x),$$

then if E_x and E_y are relabelled E_c and E_d respectively, the four events will satisfy (2.1). If

$$s(E_a E_x) + s(E_b E_y) > s(E_a E_y) + s(E_b E_x),$$

then relabel E_x and E_y, E_d and E_c respectively.

(3.3) *There exist timelike distributions* S *of five events in* L^2 *for which the events of* S *cannot be labelled so as to satisfy* (2.1).

Proof. Cf. §7; also see the six-point condition given at the end of §8.

(3.4) *Let* E_a, E_b, E_c, *and* E_d $(a < b < c < d)$ *denote four events of a timelike distribution of events in* L^n $(n \geqslant 2)$ *labelled so as to satisfy* (2.1). *If* t_a, t_b, t_c, *and*

t_d *denote the t-co-ordinates of these events, then the two intervals defined by* $\{t_a, t_b\}$ *and* $\{t_c, t_d\}$ *do not have any points in common.*

Proof. Let

$$t_\alpha = \min\{t_a, t_b, t_c, t_d\}, \qquad t_\beta = \min[\{t_a, t_b, t_c, t_d\} - \{t_\alpha\}],$$

and

$$\{t_x, t_y\} = \{t_a, t_b, t_c, t_d\} - \{t_\alpha, t_\beta\}.$$

Then by (3.1), $s(E_\alpha E_\beta) + s(E_x E_y)$ is strictly less than both

$$s(E_\alpha E_x) + s(E_\beta E_y) \quad \text{and} \quad s(E_\alpha E_y) + s(E_\beta E_x),$$

where E_α, E_β, E_x, and E_y are the events corresponding to t_α, t_β, t_x, and t_y respectively. Since under the present hypothesis, the events E_a, E_b, E_c, and E_d are labelled so as to satisfy (2.1), it is necessary that either $\{t_\alpha, t_\beta\} = \{t_a, t_b\}$ or $\{t_\alpha, t_\beta\} = \{t_c, t_d\}$. However, this condition is realized if and only if the two intervals defined by $\{t_a, t_b\}$ and $\{t_c, t_d\}$ do not have any points in common. Thus, (3.4) is proved.

(3.5) *If* $P_1, P_2, \ldots, P_k$ *is a timelike distribution of events in* L^n $(n \geqslant 2)$ *labelled so as to satisfy* (2.1), *then the relabelling* $Q_i = P_{k-i+1}$ $(i = 1, 2, \ldots . k)$ *of these events also satisfies* (2.1).

Proof. This assertion is proved by noting that, (i) if $1 \leqslant a < b < c < d \leqslant k$, then

$$1 \leqslant k - d + 1 < k - c + 1 < k - b + 1 < k - a + 1 \leqslant k,$$

and (ii) $s(P_{k-x+1} P_{k-y+1}) = s(Q_x Q_y)$.

4. Proof of Theorem 1. Let S satisfy the Four Point Condition and $P_1, P_2, \ldots, P_k$ be a labelling of the events of S for which (2.1) is satisfied.

We first show that $P_1 \in \{E_1, E_2, E_{k-1}, E_k\}$. If $P_1 \notin \{E_1, E_2, E_{k-1}, E_k\}$, then by (3.4) we have $P_2 \notin \{E_1, E_k\}$. Since $s(P_1 P_2) + s(E_1 E_k)$ is not less than both $s(P_1 E_1) + s(P_2 E_k)$ and $s(P_1 E_k) + s(P_2 E_1)$ (cf. (3.1)), no labelling of E_1 and E_k exists such that (2.1) is satisfied. Thus, $P_1 \in \{E_1, E_2, E_{k-1}, E_k\}$.

By (3.4), P_1 and P_2 must have t-co-ordinates that are not separated by any t-co-ordinate of any member of S. Thus, if $P_1 = E_1$ or $P_1 = E_k$, then $P_2 = E_2$ or $P_2 = E_{k-1}$ respectively. If $P_1 = E_2$ and $P_2 = E_3$, then $\{P_1, P_2, E_1, E_k\}$ would not satisfy (2.1), since $s(P_1 P_2) + s(E_1 E_k)$ is not minimal. Similarly, if $P_1 = E_{k-1}$ and $P_2 = E_{k-2}$, then $\{P_1, P_2, E_1, E_k\}$ would not satisfy (2.1). Thus, we have shown that $\{P_1, P_2\} = \{E_1, E_2\}$ or $\{P_1, P_2\} = \{E_{k-1}, E_k\}$.

Applying (3.4) again we see that, if $\{P_1, P_2\} = \{E_1, E_2\}$, then $P_i = E_i$ $(i = 3, 4, \ldots, k - 2)$ and since only P_{k-1} and P_k remain, we must have $\{P_{k-1}, P_k\} = \{E_{k-1}, E_k\}$. Similarly, if $\{P_1, P_2\} = \{E_{k-1}, E_k\}$, then $P_i = E_{k-i+1}$ $(i = 3, 4, \ldots, k - 2)$ and $\{P_{k-1}, P_k\} = \{E_1, E_2\}$. Thus, either we have (2.2) and we are finished, or we have the case of the preceding sentence. In the latter case, applying (3.5) we obtain a labelling $Q_1, Q_2, \ldots, Q_k$ of the events of S such that

$$\{Q_1, Q_2\} = \{E_1, E_2\},$$

$$Q_i = E_i \qquad (i = 3, 4, \ldots, k-2),$$

and

$$\{Q_{k-1}, Q_k\} = \{E_{k-1}, E_k\},$$

and for which (2.1) is satisfied. This completes the proof of the necessity part of the theorem.

Conversely, if one of the four labellings (2.2) of the events of S is a labelling for which (2.1) is satisfied, then by definition S satisfies the Four Point Condition. This completes the proof of Theorem 1.

5. Proof of Theorem 2. Let $1 \leqslant a < b < c < d \leqslant k$.

Since $E_t \in F(E_{t-1})$ for $t = 2, 3, \ldots, k$, the events $E_1, E_2, \ldots, E_k$ constitute a time-ordered, i.e. $t_1 < t_2 < \ldots < t_k$, timelike distribution of events in L^n. Thus, by (3.1),

$$s(E_a E_b) + s(E_c E_d) < s(E_a E_c) + s(E_b E_d). \tag{5.1}$$

We now show that

(5.2) *if* $1 \leqslant a < b < c < k$, *then* $R(E_a, E_b; E_{c+1}) \subset R(E_a, E_b; E_c)$.

This is proved by noting that, if $E \in R(E_a, E_b; E_{c+1})$, then

$$E \in F(E_{c+1}) \subset F(E_c)$$

and

$$s(E_a E_{c+1}) - s(E_b E_{c+1}) \leqslant s(E_a E) - s(E_b E).$$

Since, by hypothesis, $E_{c+1} \in R(E_a, E_b; E_c)$, we have

$$s(E_a E_c) - s(E_b E_c) \leqslant s(E_a E_{c+1}) - s(E_b E_{c+1}).$$

Therefore, $E \in R(E_a, E_b; E_c)$.

Now by hypothesis, $E_d \in R(E_a, E_b; E_{d-1})$ for $1 \leqslant a < b < d-1 < k$. By (5.2) we have

$$R(E_a, E_b; E_{d-1}) \subset R(E_a, E_b; E_c) \qquad \text{for} \quad c = b+1, b+2, \ldots, d-1.$$

Thus,

$$E_d \in R(E_a, E_b; E_c) \qquad (1 \leqslant a < b < c < d \leqslant k).$$

Therefore,

$$s(E_a E_c) - s(E_b E_c) \leqslant s(E_a E_d) - s(E_b E_d)$$

or

$$s(E_a E_c) + s(E_b E_d) \leqslant s(E_a E_d) + s(E_b E_c). \tag{5.3}$$

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The inequalities (5.1) and (5.3) taken together show that the events $E_1, E_2, \ldots, E_k$ satisfy (2.1). This completes the proof of Theorem 2.

6. Proof of Theorem 3. The proof of (**1**, Theorem I) does not depend on its Euclidean setting and it can be shown directly that it is valid for timelike distributions in L^n $(n \geqslant 2)$ and polygonal connections. We shall now show that any set of events satisfying the hypothesis of Theorem 3 of this paper also satisfies the system of inequalities (with d replaced by s) in the hypothesis of (**1**, Theorem I).

Since $E_t \in F(E_{t-1})$ for $t = 2, 3, \ldots, k$, the events $E_1, E_2, \ldots, E_k$ constitute a time-ordered timelike distribution of events in L^n. Thus, by (3.1),

$$s(E_1 E_2) + s(E_i E_j) < s(E_1 E_i) + s(E_2 E_j) \tag{6.1}$$

for all (i, j) $(i = 3, \ldots, k; j = 3, \ldots, k; i \neq j)$ and

$$s(E_h E_{h+2}) + s(E_i E_j) < s(E_h E_i) + s(E_{h+2} E_j) \tag{6.2}$$

where h assumes the values $1, 2, \ldots, k-3$ and for each h (6.2) holds for all pairs (i, j) of the set $\{h+3, h+4, \ldots, k\}$ $(i \neq j)$.

We now show that

$$\begin{cases} s(E_1 E_3) + s(E_i E_2) \leqslant s(E_1 E_i) + s(E_2 E_3) & (i = 4, 5, \ldots, k), \\ s(E_2 E_4) + s(E_i E_3) \leqslant s(E_2 E_i) + s(E_3 E_4) & (i = 5, 6, \ldots, k), \\ s(E_3 E_5) + \ldots, \\ \ldots\ldots \end{cases} \tag{6.3}$$

where the sequence of inequalities (6.3) is extended until the set of integers over which i ranges becomes void.

The inequalities (6.3) hold if

$$s(E_h E_{h+2}) - s(E_{h+1} E_{h+2}) \leqslant s(E_h E_i) - s(E_{h+1} E_i)$$

for $i = h+3, h+4, \ldots, k$ and $h = 1, 2, \ldots, k-3$. Now, if

$$E \in R(E_h, E_{h+1}; E_{h+2}),$$

then E satisfies

$$s(E_h E_{h+2}) - s(E_{h+1} E_{h+2}) \leqslant s(E_h E) - s(E_{h+1} E).$$

By hypothesis $\{E_{h+3}, E_{h+4}, \ldots, E_k\} \subset R(E_h, E_{h+1}; E_{h+2})$ for $h = 1, 2, \ldots, k-3$. Thus, (6.3) is true.

The inequalities (6.1), (6.2), and (6.3) taken together imply that the events $E_1, E_2, \ldots, E_k$ satisfy the system of inequalities in the statement of (**1**, Theorem I).

7. Proof of Lemma 1. Let $S = \{E_1, E_2, \ldots, E_k\}$ $(k \geqslant 4)$ denote a set of events satisfying the hypothesis of the lemma. Then, the events of S are cyclically labelled with respect to the boundary B of their convex hull if and only if B coincides with the polygonal connection $[E_1 E_2 \ldots E_k]$. If B does not coincide with $[E_1 E_2 \ldots E_k]$, then there exist four non-collinear events E_a, E_b, E_c, and E_d in S with $a < b < c < d$, for which the labelling is not cyclic with respect to B. We now show that this is impossible.

If $t_a = \min\{t_a, t_b, t_c, t_d\}$, then, by (3.4), $t_b = \min\{t_b, t_c, t_d\}$. From the triangle inequality in L^2 it follows directly that among the three pairs of non-adjacent rectilinear connections of the complete quadrilateral defined by a timelike distribution of four events that fall on the boundary of their convex hull, the diagonals have intermediate time separation. This fact together with the hypothesis that the events E_a, E_b, E_c, and E_d satisfy (2.1) implies that $E_a E_c$ and $E_b E_d$ are the diagonals. Therefore, a labelling that satisfies (2.1) is uniquely determined once the event with the minimal t-co-ordinate is labelled. Clearly, this labelling is cyclic with respect to the boundary of the convex hull of the given four events and consequently with respect to B.

By considering the cases where t_b, t_c, and t_d are respectively equal to $\min\{t_a, t_b, t_c, t_d\}$ and repeating the above argument, we arrive at the same conclusion in each case. This completes the proof of Lemma 1.

Proof of (3.3). Let four events E_a, E_b, E_c, and E_d $(t_a < t_b < t_c < t_d)$ constitute a set of events in L^2 that fall on a strictly convex world-line and let E_e denote any event on the rectilinear connection $E_b E_d$. We now show that no labelling of these five events satisfies (2.1).

By Lemma 1, if four non-collinear events of a timelike distribution of events in L^2 fall on the boundary of their convex hull, any labelling that satisfies (2.1) must be cyclic with respect to their convex hull. Consider now the sets:

$$\{E_a, E_b, E_c, E_d\}, \quad \{E_a, E_b, E_e, E_d\}, \quad \text{and } \{E_b, E_e, E_c, E_d\}.$$

Each of these sets has the property that its events fall on the boundary of their convex hull. Thus, if the five events are to be labelled so as to satisfy (2.1), the labelling must be cyclic with respect to each boundary of these convex hulls. However, it is directly verified that such a labelling cannot exist.

8. Proof of Theorem 4. (i) *Necessity*. Let the given k events satisfy the Four Point Condition. If $k = 4, 5$, or if all the given events other than P and S lie in the same component of $B - \{P, S\}$, then the assertion of the theorem is valid. Thus, we assume that $k \geqslant 6$ and $B - \{P, S\}$ consists of two components B_1 and B_2, each of which contains at least one event of the given set of events.

Let $P_1, P_2, \ldots, P_k$ denote a labelling of the given events $P, Q, E_3, \ldots, E_{k-2}, R, S$ $(t_p < t_q < t_3 < \ldots < t_{k-2} < t_r < t_s)$ such that (2.1) is satisfied and (by Theorem 1)

$$\{P_1, P_2\} = \{P, Q\},$$

$$P_i = E_i \qquad (i = 3, 4, \ldots, k-2),$$

and

$$\{P_{k-1}, P_k\} = \{R, S\}.$$

Assume that $P_3, P_4, \ldots, P_{k-2}$ is not entirely contained in the same component of $B - \{P, S\}$. Then there are two events P_b and P_c with

$$3 \leqslant b < c \leqslant k - 2$$

such that P_b and P_c lie in separate components of $B - \{P, S\}$. Then the events P, P_b, P_c, and S lie on the boundary of their convex hull, but are not cyclically labelled with respect to the boundary of the convex hull. This is a contradiction of Lemma 1. Thus, $\{P_3, P_4, \ldots, P_{k-2}\}$ is contained in the same component of $B - \{P, S\}$.

(ii) *Sufficiency.* Let

$$t_1 < t_2 < t_3 < t_4 < \ldots < t_{k-2} < t_{k-1} < t_k$$

denote the t-co-ordinates of the given k events

$$P, Q, E_3, E_4, \ldots, E_{k-2}, R, S. \tag{8.1}$$

If the events $Q, E_3, E_4, \ldots, E_{k-2}, R$ all lie in the same component of $B - \{P, S\}$, then let the events (8.1) be relabelled

$$E_1, E_2, E_3, E_4, \ldots, E_{k-2}, E_{k-1}, E_k$$

respectively. Now, if $1 \leqslant a < b < c < d \leqslant k$, we have by (3.1)

$$s(E_a E_b) + s(E_c E_d) < s(E_a E_c) + s(E_b E_d).$$

Since the events (8.1) lie on a convex world-line in L^2, it follows from the triangle inequality that

$$s(E_a E_c) + s(E_b E_d) \leqslant s(E_a E_d) + s(E_b E_c).$$

Thus, in the case just considered, the events (8.1) can be labelled so as to satisfy (2.1).

Let $B - \{P, S\} = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, where B_1 is the component containing the lesser number of events from (8.1). There are three cases to consider: (i) $Q \in B_1$ and $R \notin B_1$, (ii) $Q \notin B_1$ and $R \in B_1$, and (iii) $\{Q, R\} \subset B_1$. If the events (8.1) are relabelled in case (i)

$$E_2, E_1, E_3, E_4, \ldots, E_{k-2}, E_{k-1}, E_k,$$

in case (ii)

$$E_1, E_2, E_3, E_4, \ldots, E_{k-2}, E_k, E_{k-1},$$

and in case (iii)

$$E_2, E_1, E_3, E_4, \ldots, E_{k-2}, E_k, E_{k-1},$$

then it is verified directly that in each case the events (8.1) have been labelled so as to satisfy (2.1). This completes the proof of Theorem 4.

A six-point condition. The proof of the necessity part of Theorem 4 contains the following assertion:

Let S denote a set of events in L^n $(n \geqslant 2)$ such that S contains a timelike distribution

$$S' = \{E_1, E_2, \ldots, E_6\} \qquad (t_1 < t_2 < \ldots < t_6)$$

which lies in a plane and for which the sets $\{E_2, E_4\}$ and $\{E_3, E_5\}$ lie in opposite half-planes determined by the line through E_1 and E_6. Then S' cannot be labelled so as to satisfy (2.1) *and consequently S does not satisfy the Four Point Condition.*

9. Proof of Theorem 5. With respect to the hypothesis of Theorem 5 we note the following:
(9.1) $S = \{(0,0), (1,1), (0,2), (1,3)\}$ is zero separated but is not the vertex set of a zero polygonal connection.
(9.2) The events

$$E_1 : (0,0), \quad E_2 : (1,1), \quad E_3 : (0,2), \quad E_4 : (-1,1),$$
$$E_5 : (1,2), \quad E_6 : (2,3), \quad E_7 : (1,4), \quad E_8 : (0,3)$$

form a set S such that for each $i = 1, 2, \ldots, 8$, $C(E_i) \cap S$ contains exactly three events, but S is not the vertex set of a zero polygonal connection.
(9.3) $S = \{(0,0), (1,1), (2,2), (1,3), (-1,1)\}$ is the vertex set of exactly one zero polygonal connection and $C((0,0)) \cap S$ contains four events. Thus, the conditions given in Theorem 5 are sufficient but not necessary for the existence of a unique zero polygonal connection.

By a *light-line* we mean a rectilinear world-line having the property that any two events on this line have zero time separation.
(9.4) The hypothesis of Theorem 5 implies that no three events of S lie on the same light-line.

Proof of (9.4). Assume that P_1, P_2, and P_3 are three events of S that lie on the same light-line. Then, $\{P_1, P_2, P_3\}$ is contained in $C(P_j) \cap S$ $(j = 1, 2, 3)$. Let $E \in S - \{P_1, P_2, P_3\}$. Since S is zero separated, there is a polygonal path with end points P_1 and E consisting of rectilinear connections of S, each having time separation equal to zero. Let $P'E'$ be a rectilinear connection in this path such that

$$P' \in \{P_1, P_2, P_3\} \quad \text{and} \quad E' \in S - \{P_1, P_2, P_3\}.$$

Then, $\{P_1, P_2, P_3, E'\}$ is contained in $C(P') \cap S$. But, this contradicts the assumption that $C(P') \cap S$ contains exactly three events.

Before proving Theorem 5 we shall prove the following lemma and corollary.

LEMMA 2. *Let $S = \{E_1, E_2, \ldots, E_k\}$ $(k \geqslant 4)$ denote a set of events in L^n $(n \geqslant 2)$. Then*

$$\begin{aligned} &\{E_k, E_1, E_2\} \subset C(E_1) \quad \textit{and} \\ &\{E_{i-1}, E_i, E_{i+1}\} \subset C(E_i) \qquad (i = 2, 3, \ldots, k-1) \end{aligned} \tag{9.5}$$

if and only if $[E_1 E_2 \ldots E_k]$ is a zero polygonal connection of S.

Proof. (9.5) implies that $E_k E_1$ and $E_i E_{i+1}$ $(i = 1, 2, \ldots, k-1)$ each have time separation equal to zero. Hence, $[E_1 E_2 \ldots E_k]$ is a zero polygonal connection of S.

Conversely, if $[E_1 E_2 \ldots E_k]$ is a zero polygonal connection of S, then

$$\{E_k, E_1, E_2\} \subset C(E_1) \quad \text{and} \quad \{E_{i-1}, E_i, E_{i+1}\} \subset C(E_i)$$

$(i = 2, 3, \ldots, k-1)$.

COROLLARY. *If*

$$(9.6) \qquad \begin{aligned} \{E_k, E_1, E_2\} &= C(E_1) \cap S \quad \textit{and} \\ \{E_{i-1}, E_i, E_{i+1}\} &= C(E_i) \cap S \qquad (i = 2, 3, \ldots, k-1), \end{aligned}$$

then $[E_1 E_2 \ldots E_k]$ *is the unique zero polygonal connection of* S.

Proof. By Lemma 2, $h = [E_1 E_2 \ldots E_k]$ is a zero polygonal connection. Let $h' = [P_1 P_2 \ldots P_k]$ denote any zero polygonal connection of S. If $h \neq h'$, then there exists a vertex $E_j = P_m$ such that $E_{j-1} E_j E_{j+1} \neq P_{m-1} P_m P_{m+1}$ (subscripts reduced modulo k). Since each of the rectilinear connections $E_{j-1} E_j$, $E_j E_{j+1}$, $P_{m-1} P_m$, and $P_m P_{m+1}$ have time separation equal to zero, the set

$$U = \{E_{j-1}, E_j, E_{j+1}\} \cup \{P_{m-1}, P_{m+1}\}$$

is contained in $C(E_j) = C(P_m)$. Since U contains at least four events, this is a contradiction of (9.6). Hence h is unique.

Proof of Theorem 5. Let $P_1 = E_1$,

$$C(P_1) \cap S = \{P_k, P_1, P_2\}, \qquad C(P_m) \cap S = \{P_{m-1}, P_m, P_{m+1}\}$$
$$(2 \leqslant m \leqslant k-1).$$

We note first that $P_3 \neq P_k$; for $P_3 = P_k$ would imply that $[P_1 P_2 P_3]$ has clock time equal to zero. But this is impossible for any three non-collinear events. Now, let m be the least integer of $\{3, 4, \ldots, k-1\}$ such that $P_{m+1} = P_j$ with $P_j \in \{P_k, P_1, P_2, \ldots, P_{m-2}\}$. We shall show that $m = k-1$ and $P_j = P_k$. For if we assume that $P_{m+1} = P_j$ and $m < k-1$, then

$$S - \{P_j, P_{j+1}, \ldots, P_m\}$$

(subscripts reduced modulo k) is not empty and $[P_j P_{j+1} \ldots P_m]$ is a zero polygonal connection. Since S is zero separated, there exists a polygonal path $\mathfrak{P}$ consisting of rectilinear connections of S each having time separation equal to zero, such that $\mathfrak{P}$ has its initial point in $S - \{P_j, P_{j+1}, \ldots, P_m\}$ and its terminal point a vertex of $[P_j P_{j+1} \ldots P_m]$. Then, as in the proof of (9.4), it follows that there exists a vertex of $[P_j P_{j+1} \ldots P_m]$ whose light-cone intersected with S contains at least four events. Thus, if $P_{m+1} = P_j$ we must have $m = k-1$, i.e., when $m = k-1$, P_{m+1} is an event that has already been labelled. By the same method as above we see that this event must be P_k, i.e.,

$P_j \equiv P_k$. Consequently, the events $P_1, P_2, \ldots, P_k$ are distinct and satisfy the hypothesis (9.6). Therefore, $[P_1 P_2 \ldots P_k]$ is the unique zero polygonal connection of S.

10. A class of curves in L^n $(n \geqslant 2)$. Theorem 4 contains the result that, if k events fall on a convex timelike arc in L^2, then these events satisfy the Four Point Condition. The following generalizes this to a class of curves in L^n $(n \geqslant 2)$.

(10.1) *Let γ denote the curve in L^n $(n \geqslant 2)$ defined by*

$$\gamma(u) = \begin{cases} x^j = \gamma_j(u) & (j = 1, 2, \ldots, n-1), \\ t = \gamma_n(u) & d\gamma_n/du > 0. \end{cases}$$

Let $0 < u_1 < u_2 < \ldots < u_k$ denote k values of the parameter u, $E_i = \gamma(u_i)$ $(i = 1, 2, \ldots, k)$, and $s(\gamma(x)\gamma(y)) = h(x, y)$.

If $h(x, y)$ is a differentiable function defined on $0 < x < y$ such that

$$\frac{\partial h}{\partial x} \leqslant 0, \quad \frac{\partial h}{\partial y} \geqslant 0, \quad \textit{and} \ \frac{\partial^2 h}{\partial x \partial y} \leqslant 0,$$

then the events $E_1, E_2, \ldots, E_k$ satisfy (2.1).

Proof. Since $d\gamma_n/du > 0$, the events $E_i = \gamma(u_i)$ $(i = 1, 2, \ldots, k)$ are distinct and are ordered by their corresponding parameter values u_i. The inequalities $1 \leqslant a < b < c < d \leqslant k$ imply $u_a < u_b < u_c < u_d$. Then, a straightforward argument using the Law of the Mean yields

$$h(u_a, u_b) + h(u_c, u_d) \leqslant h(u_a, u_c) + h(u_b, u_d) \leqslant h(u_a, u_d) + h(u_b, u_c),$$

which when written in terms of the separation function s is precisely (2.1).

A realization of (10.1). *Let γ denote the curve in L^4 defined by* $\gamma_1(u) = \cos u$, $\gamma_2(u) = \sin u$, $\gamma_3(u) = \sinh u$, *and* $\gamma_4(u) = \cosh u$ $(u > 0)$. *Then γ satisfies the hypothesis of* (10.1).

Proof. $d\gamma_4/du = \sinh u > 0$ when $u > 0$,

$$s(\gamma(x)\gamma(y)) = h(x, y) = \sqrt{2[\cosh(y - x) - \cos(y - x)]},$$

which upon differentiation yields

$$\frac{\partial h}{\partial x} = -[\sinh(y - x) + \sin(y - x)]h(x, y)^{-1} < 0, \qquad \text{when } 0 < x < y,$$

$$\frac{\partial h}{\partial y} = [\sinh(y - x) + \sin(y - x)]h(x, y)^{-1} > 0, \qquad \text{when } 0 < x < y,$$

and

$$\frac{\partial^2 h}{\partial x \partial y} = -\left[h(x, y)[\cosh(y - x) + \cos(y - x)] + [\sinh(y - x) + \sin(y - x)]\frac{\partial h}{\partial x}\right]h(x, y)^{-2},$$

which is negative whenever $0 < x < y$.

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11. Open questions. We conclude by listing some open problems in the development of the preceding investigations.

1. What are extreme polygonal connections of a timelike distribution of k events in L^2 that fall on the boundary of their convex hull?

2. Describe the class of all timelike distributions of events in L^2 (then in L^n $(n \geqslant 2)$) that satisfy the Four Point Condition.

3. Establish criteria which if satisfied by a set of events offer a prespecified ordering for an extreme polygonal connection (as exemplified by the *convex case* in the Euclidean plane **(2**; **3**; **4)**, the Four Point Condition **(1**, Theorem III and §6; Basic Lemma §2 this paper**)**, the realization of Theorem I **(1**, §5**)**, and Theorems 3 and 5 of this paper).

Appendix. Record spaces. In this section we formalize more precisely (and more naïvely) the concept expressed in the first paragraph of §1. Let R denote a set, I a closed interval of real numbers $a \leqslant \tau \leqslant b$, and for each $\tau \in I$ let R_τ denote the set of ordered triplets (r, t_r, τ), where $r \in R$ and t_r is a real number dependent on r. We call the set $\mathfrak{R} = \cup R_\tau(\tau \in I)$ a *record space* and R_τ the *record at the time* τ. A function $\mathfrak{O}$ defined on a closed subset Dom $\mathfrak{O}$ of I such that $\mathfrak{O}(\tau) \in R_\tau$ we call an *observation function*. The set

$$T(\mathfrak{O}) = \{\mathfrak{O}(\tau) : \tau \in \text{Dom } \mathfrak{O}\}$$

will be called an *observation track* and is said to be *generated* as τ ranges increasingly over Dom $\mathfrak{O}$.

If (r, t_r, τ_0) is a point in R_{τ_0} and $\tau_0 < \tau_1$, then the point $(r, t_r, \tau_1) \in R_{\tau_1}$ is called the *projection* of (r, t_r, τ_0) onto R_{τ_1}. For each value $\tau_1 \in I$, let all the points $\mathfrak{O}(\tau)$ $(\tau \in \text{Dom } \mathfrak{O}, \tau \leqslant \tau_1)$ of an observation track be projected onto R_{τ_1}. The resultant point set is called the *cumulative track at the time* τ_1. The generation of the cumulative track as τ ranges increasingly over Dom $\mathfrak{O}$ induces a linear ordering on its points and hence a linear ordering on the *complete cumulative track* in R_d, where d is the greatest element in Dom $\mathfrak{O}$. (There are numerous examples of record spaces. A file of letters ordered by their dates corresponds to an R_τ at a given time τ and a set of letters selected from this file in linear order with respect to τ corresponds to an observation track. Similarly, a geologist analysing strata, a genealogist considering ancestral ordering, a palaeontologist studying evolutionary patterns, or a psychotherapist interpreting memory tracks can each be considered in a record space context.)

Now, let $\mathfrak{R}$ be the set of triplets $((x^1, \ldots, x^{n-1}, t), t, \tau)$ where $(x^1, \ldots, x^{n-1}, t)$ belongs to L^n $(n \geqslant 2)$, and $\tau \in I$, a closed interval. Then for each $\tau \in I$, R_τ can be thought of as L^n ordered by the t-co-ordinates of its elements. Let the n-flat $t = 0$ have the structure of n-dimensional space–time and be construed as the space of an observer's world-line. In what follows, the domain of the observation function $\mathfrak{O}$ will be equal to $I = \{\tau: a \leqslant \tau \leqslant b\}$. Consider now the following situation of "restricted observation." Let $P_1, P_2, \ldots, P_k$ be k

points that are fixed in the moving n-flat R_τ as τ ranges from a to b. We wish a cumulative track, as it is generated in R_τ, to pass through these k points. The ordering of these points in the track will depend on the discretion of the observer, i.e., on the observation function. This immediately gives rise to a set of possible "states" corresponding to the different ways in which cumulative tracks can be routed through these points. Questions may now be posed as to which state is "best" or "extreme" in accordance with some specified point of view. The results of this paper can thus be interpreted as extremum properties of complete cumulative tracks that are polygonal connections. Since we were concerned only with complete cumulative tracks we were able to restrict our attention to the n-flat R_b, or equivalently, to n-dimensional space–time L^n.

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