Clays and Clay Minerals, Vol. 56, No. 3, 389–395, 2008.

LOG SELFSIMILARITY OF CONTINUOUS SOIL PARTICLE-SIZE DISTRIBUTIONS ESTIMATED USING RANDOM MULTIPLICATIVE CASCADES

Miguel Ángel Martín* and Carlos García-Gutiérrez[1]

Department of Applied Mathematics to Agriculture Engineering, E.T.S.I Agrónomos,
Technical University of Madrid (UPM), 28040 Madrid, Spain

Abstract—Particle-size distribution (PSD) is a fundamental soil property usually reported as discrete clay, silt, and sand percentages. Models and methods to effectively generate a continuous PSD from such poor descriptions using another property would be extremely useful to predict and understand in fragmented distributions, which are ubiquitous in nature. Power laws for soil PSDs imply scale invariance (or selfsimilarity), a property which has proven useful in PSD description. This work is based on two novel ideas in modeling PSDs: (1) the concept of selfsimilarity in PSDs; and (2) mathematical tools to calculate fractal distributions for specific soil PSDs using few actual texture data. Based on these ideas, a random, multiplicative cascade model was developed that relies on a regularity of scale invariance called 'log-selfsimilarity.' The model allows the estimation of intermediate particle size values from common texture data. Using equivalent inputs, this new modeling approach was checked using soil data and shown to provide greatly improved results in comparison to the selfsimilar model for soil PSD data. The Kolmogorov-Smirnov D-statistic for the log-selfsimilar model was smaller than the selfsimilar model in 92.94% of cases. The average error was 0.74 times that of the selfsimilar model. The proposed method allows measurement of a heterogeneity index, *H*, defined using Hölder exponents, which facilitates quantitative characterization of soil textural classes. The average *H* value ranged from 0.381 for silt texture to 0.838 for sandy loam texture, with a variance of <0.034 for all textural classes. The index can also be used to distinguish textures within the same textural class. These results strongly suggest that the model and its parameters might be useful in estimating other soil physical properties and in developing new soil PSD pedotransfer functions. This modeling approach, along with its potential applications, might be extended to fine-grained mineral and material studies.

Key Words—Fractals, Fragmentation, Iterated Function Systems, Log-selfsimilarity, Particle-size Distribution, Random Cascades, Soil.

INTRODUCTION

The statistical description of PSD in soil is of scientific interest for a variety of reasons: *e.g.* mechanical, hydraulic, and agricultural soil properties are strongly influenced by soil texture. Hence, a statistical description of PSD might be useful to predict such properties, devise sustainable soil use, or implement effective conservation policies, among others. In a more theoretical framework, the study and characterization of size distributions which result from particle fragmentation processes still have unsolved challenges. For instance, Kolmogorov (1992) proved mathematically that a certain particle fragmentation process leads to a log-normal particle-size distribution. He wondered, however, what could be expected if a slightly different algorithm drove the fragmentation process and this remains an unanswered question today. In a more practical context, the different properties of mineral groups (*e.g.* cleavage, hardness) might result in different fragmentation processes.

* E-mail address of corresponding author:
miguelangel.martin@upm.es
DOI: 10.1346/CCMN.2008.0560308

Since the 1980s, 'fractal geometry' has been used to describe scenarios with scale-invariant features which are assumed to be the result of a more or less explicit iteration process acting along a range of scales. Vaguely, the term 'fractal' or 'selfsimilar' is applied to objects, distributions, or processes that have some scale-invariant features revealed by power laws. In such cases, the power exponents are usually called 'fractal dimensions.'

The connection between fractals and fragmentation was first established by B. Mandelbrot (1982) in his well known book *The Fractal Geometry of Nature*. Turcotte (1986) used Mandelbrot's seminal ideas to test 'fractal scaling' in soil and derived what is currently known as Turcotte's law, according to which, the number $N(R)$ of particles larger than a characteristic diameter R follows the scaling rule

$$N(R) \propto R^{-D} \quad (1)$$

where D is a number known as the 'scaling fractal dimension.' Also, a power scaling of the type

$$\mu(R) \propto R^{3-D} \quad (2)$$

for the distribution $\mu(R)$ of the mass of particles of size less than R can be derived from equation 1 or else a mass scaling

$$\mu(R) \propto R^d \tag{3}$$

where d is a positive exponent, may be tested directly (Tyler and Wheatcraft, 1992; Turcotte, 1992; Wu *et al.*, 1993). Since the development of equations 1 and 2, much work has been devoted to testing the fractal behavior of PSDs and various soil physical properties (see Anderson *et al.*, 1998 for a review), which strongly supports the view that fractality is a new regularity law that might be helpful for describing soil properties.

When a fractal behavior such as that expressed by equation 3 is shown for a mass-size distribution, μ, such power scaling indicates an underlying selfsimilar structure of the distribution (*i.e.* the PSD). Identical (or close) fractal dimensions (in the sense stated above), however, may correspond to soils of rather different textural composition (see Tyler and Wheatcraft, 1989; Martín *et al.*, 2001). Moreover, the mere knowledge of the fractal exponent does not allow one to estimate the mass of particles with size between prescribed limits a and b, which might be helpful for estimating other soil properties.

Once selfsimilarity in soil (conceived in a broad sense) becomes a sensible hypothesis for explaining soil PSDs, an interesting problem arises that involves the construction of models based on selfsimilarity with the intent to exploit limited information provided by common texture data in order to estimate PSDs beyond the scale afforded by available data. Martín and Taguas (1998) used "iterated function systems" (IFS) to estimate selfsimilar distributions to fit specific soil-texture data and characterized PSD heterogeneity *via* entropy-like parameters (Martin *et al.*, 2001). The ability to reproduce the entire PSD using IFS modeling was tested by Taguas *et al.* (1999).

Improving the ability to model PSDs based on the limited information provided by texture data is a crucial issue for ensuring accurate predictions and for understanding mass-size distribution features that result from various fragmentation processes. This paper revolves around two novel ideas for modeling PSDs. One relates to the way selfsimilarity in PSDs is conceived, and the other to the mathematical tools used to calculate a fractal distribution model for a specific soil PSD by using a small amount of texture data. This new modeling approach was tested using previously reported data.

MATERIAL AND METHODS

Multifractal distributions

An IFS is a set $\{\varphi_i, p_i, i = 1 \cdots m\}$, where $\varphi_i(x), i = 1 \ldots m$ are linear transformations and p_i, $i = 1 \cdots m$ are a series of positive numbers that together sum to one as indicated by equation 4.

$$\sum_{i=1}^{m} p_i = 1 \tag{4}$$

An IFS determines a 'selfsimilar mass distribution' that may be seen as the result of an iterative process guided by the linear transformations and probabilities (see Falconer 1997). One typical example is the binomial selfsimilar distribution determined by the IFS with two linear transformations φ_1 and φ_2 [$\varphi_1(x) = 0.5x$, $\varphi_2(x) = 0.5x + 0.5$], and two respective probabilities $p_1 = p$ and $p_2 = 1-p$. In this case, the support is the unit interval [0,1] and the binomial distribution is generated by the following infinite iterative process: in the first step, the unit mass is assumed to be uniformly distributed throughout the unit interval [0,1]; in the second step, a mass p_1 is assumed to be uniformly distributed on the left-half subinterval [0,0.5] and, similarly, a mass p_2 in the right-half subinterval [0.5,1]. These two steps are repeated in any of the 2^k subintervals appearing at step k and so on (see Eversetz and Mandelbrot, 1992).

Soil PSD has been modeled by means of selfsimilar mass distributions associated with certain IFSs that are assigned to any interval $I = [a,b]$ of the particle-size interval [0,2000 μm] for the mass $\mu(I)$ of soil particles that have sizes (equivalent diameters) between a and b (see Martín and Taguas, 1998).

Selfsimilar distributions are paradigmatic examples of so-called 'multifractal distributions', *i.e.* distributions for the local 'Hölder exponent' of the mass distribution μ at a point x are defined by the limit

$$\alpha(x) = \lim_{r \to 0} \frac{\log \mu(I_r(x))}{\log r}$$

and is not constant on the $I_r(x)$ support interval $[x-r, x+r]$.

The Hölder exponents for selfsimilar distributions typically span the entire interval between two extreme values α_{min} and α_{max} (see Eversetz and Mandelbrot, 1992, or Falconer, 1997, for further details). For the binomial distribution, $\alpha_{min} = -\log_2(p_2)$ and $\alpha_{max} = -\log_2(p_1)$, the exponent provides a measure of the mass concentration around the point: the greater $\alpha(x)$ is, the smaller will be the mass concentration and *vice versa*. For simulated selfsimilar (or experimental) distributions, however, the above theoretical approach is replaced with a 'coarse' version involving a scaling analysis of overall information quantities instead of the pointwise local Hölder exponents which lack practical sense in a natural setting. One common choice is to consider dyadic scaling down (Eversetz and Maldelbrot, 1992), *i.e.* successive partitions of I of size $L\ \varepsilon = L\ 2^{-k}$, where L is the length of I and $k = 1,2,3,\cdots$ At every size scale, ε, a number $N(\varepsilon) = 2^k$ of subintervals (cells) I_i, $I = 1 \ldots N(\varepsilon)$ are considered and their respective measures $\mu(I_i) = \mu_i(\varepsilon)$ are assumed to be provided by available

https://doi.org/10.1346/CCMN.2008.0560308 Published online by Cambridge University Press

data. Now, the ratio $\log(\mu_i(\varepsilon))/\log(\varepsilon)$ is called the coarse Hölder exponent of interval I_i, and the 'coarse Hölder spectrum' is defined *via* a parameter, q, such that

$$\alpha(q) \propto \frac{\sum_{i=1}^{N(\varepsilon)} \mu_i(q,\varepsilon) \log \mu_i(\varepsilon)}{\log \varepsilon} \tag{6}$$

where

$$\mu_i(q,\varepsilon) = \frac{\mu_i(\varepsilon)^q}{\sum_{i=1}^{N(\varepsilon)} \mu_i(\varepsilon)^q} \tag{7}$$

and where '$\propto$' means that a suitable linear fit occurs within a significant range of ε values (Eversetz and Mandelbrot, 1992; Chhabra and Jensen, 1989).

For exact selfsimilar distributions, $\propto$ in equation 6 may be replaced with the limit when $\varepsilon \to 0$. In such a case, the function $\alpha(q)$ for $-8 < q < +8$ parameterizes the interval $[\alpha_{\min}, \alpha_{\max}]$ of local Hölder exponents.

Conversely, an appropriate scale fitting of equation 6 over a significant range of scales may reveal that the distribution concerned possesses selfsimilar features.

THE LOG-SELFSIMILAR MODEL BASED ON RANDOM CASCADES

The selfsimilar mass distribution generated by the IFS proposed by Martín and Taguas (1998) appeared to be a sensible way to conceive selfsimilarity in soil PSDs. The IFS is an iterative process which spreads the relative mass proportions of elementary size classes over short subintervals that are smaller, linear copies of the three initial size intervals. Thus, it provides a useful method for testing soil selfsimilarity behavior and calculating selfsimilar PSDs from the limited information provided by standard texture data.

Closer examination of this model led us to rethink selfsimilarity in soil PSDs. Soil data are usually reported in terms of clay (<0.002 mm), silt (0.002–0.05 mm), and sand (0.05–2 mm). This assigns relatively similar importance to these textural separates. In contrast, the size intervals differ by several orders of magnitude (*i.e.* 0.002 mm, 0.048 mm, and 1.95 mm, respectively). Using these selfsimilar modeling approaches, the proportions and interval sizes would result in the accumulation of vast amounts of soil mass in very small linear copies of the size intervals. Specifically, much of the mass would be in linear copies (subfractions) of the clay interval. This might be unrealistic in pedological terms. In fact, the analysis by Montero and Martin (2003) using a Hölder spectrum, which was computed from laser-diffraction texture data, revealed an excellent scaling behavior when a log-rescaled interval was scaled down. Using the rescaled interval instead of the usual interval in scaling analyses is strongly supported by the nature of the data provided by texture-analysis instruments (Martin *et al.*, 2001; Montero and Martín, 2003).

The previous results suggest a different (selfsimilar) way of conceiving how particle-size, fine-scale structure is echoed by overall particle-size data based on particle-size analysis. The key idea is to view the PSD as the result of a fractal-like, mass-spreading iterative process occurring in the log-rescaled size interval at any scale. The new regularity may be termed 'log-selfsimilarity.' Just as normality is replaced with lognormality in order to explain fundamental aspects of natural order, log-selfsimilarity might help establish regularity laws for a wide range of scenarios where fractal features have been demonstrated.

The model described below produces log-selfsimilar distributions *via* random cascades instead of the self-similar distributions produced by IFS techniques which have a deterministic nature.

Roughly speaking, a cascade is a process which fragments a given set (the size interval in this case) into increasingly smaller pieces according to a certain rule and simultaneously divides the distribution of the set according to some (possibly random) mass fragmentation rule. The process defines the limits of a mass distribution that is multifractal.

In precise terms, let I be the size interval, I = [0,2000], and $I_1 = [0,\alpha]$, $I_2 = [\alpha,\beta]$, $I_3 = [\beta,2000]$, the subintervals of sizes corresponding to three size classes. Also, let $p_1 = \mu(I_1)$, $p_2 = \mu(I_2)$, and $p_3 = \mu(I_3)$ be the mass fractions for the intervals I_1, I_2, and I_3, respectively ($p_1 + p_2 + p_3 = 1$).

Consider the transformation given by $\Phi(x) = \ln(1+x)$ that transforms the texture interval $I = [0,2000]$ into $I^* = \Phi(I) = [\Phi(0),\Phi(2000)] = [0,7.601]$, where $7.601 = \ln(2001)$ and $\ln(1) = 0$.

For $I = 1,2,3$, let $I_i^* = \Phi(I_i)$.

Let φ_i be the linear transformations that transform I^* into I_i^* for $I = 1,2,3$; $I_i^* = \varphi_i(I_i)$. These three transformations are applied first to the interval I^* and then to any of the resulting intervals I_i^* ('subintervals') following the branching process *ad infinitum*.

One interval of the k^{th} stage of the multiplicative cascade that results from the iterative application of a certain sequence of linear transformations is denoted by I_{kj}^*. The mass, which is supposed to be uniformly spread in a 'son' (sub-subinterval) $I_{k+1,i}^* = \varphi_i(I_{kj}^*)$, is given by:

$$\mu(I_{k+1,I}^*) = \mu(I_{kj}^*)\ V$$

where V is a random variable that follows a normal distribution of mean p_{i} (see further details in the next section), and $\mu(I_{kj}^*)$ is the mass of the interval I_{kj}^*.

At the limit, the process defines a *statistically selfsimilar mass distribution* supported on I^* (see Falconer 1994, 1997).

Finally, let us define $I = \Phi^{-1}(I^*)$ and $\mu(J) = \mu(J^*)$, with $J \subset I$ and $J^* = \Phi(J)$, as a model for the PSD distribution.

https://doi.org/10.1346/CCMN.2008.0560308 Published online by Cambridge University Press

DATA

The data used to test the log-selfsimilarity modeling approach were soil data for the upper two horizons reported in *Soil Taxonomy* (1975). Soil data included the mass proportions, p_i, of particles in eight size classes (mm): clay (<0.002), silt (0.002−0.02) and (0.02−0.05), very fine sand (0.05−0.1), fine sand (0.1−0.25), medium sand (0.25−0.5), coarse sand (0.5−1), and very coarse sand (1−2).

For obvious reasons, selfsimilarity was not expected when one textural separate (sand, silt, or clay) was much greater than the others. Therefore, of the initial 170 soils, soils with clay or sand contents >85% and silt contents >90% were rejected. The elimination of these soil samples resulted in the use of 158 soils.

TESTING LOG-SELFSIMILARITY

To construct a multiplicative cascade from these data, let $[a,b]$ denote the sizes greater than or equal to a and less than or equal to b (a = size = b). These size classes define a set of seven particle-size division points (0.002, 0.02, 0.05, 0.1, 0.25, 0.5, 1 mm) and eight consecutive size intervals that correspond to the eight size classes in mm: $J_1 = [0,0.002]$, $J_2 = [0.002,0.02], \cdots, J_8 = [1,2]$.

Part of a soil particle-size analysis data set can be entered into the model to construct a log-selfsimilar cascade and, hence, calculate a fractal PSD associated with the soil data. Then, the part of the real soil data set not used to construct the cascade can be compared to the calculated fractal PSD. Similar calculations can be performed using the Taguas *et al.* (1999) method and can be compared to results calculated using the log-selfsimilar cascade model. Both fractal PSD calculation methods can use the same soil data for comparison.

Multiplicative cascades can be constructed from a number of input data values that can range from 2 to 8. Hence, the proposed method can potentially provide a large number of calculated fractal PSDs. In order to test log-selfsimilarity, only three input data values obtained from soil-texture data with three measured textural separates were used. One reason for this choice was that three is the number for most commonly available texture data (*i.e.* %clay, %silt, and %sand).

The different possible log-selfsimilar cascades were constructed:

$$\{\varphi_1,\varphi_2\varphi_3;p_1,p_2,p_3\}$$

by using the following procedure:

(1) Select two size-separation points α and β from the seven (*i.e.* 0.002 ⋯1) possible choices with α<β.

(2) Let p_1, p_2, and p_3 be the mass fractions of the three constructed intervals: $I_1 = [0,\alpha]$, $I_2 = [\alpha,\beta]$, and $I_3 = [\beta,2000]$.

(3) Let $I^* = \underline{\Phi}\ (I)$ and $I_i^* = \Phi(I_i)$, for $i = 1,2,3$. Also let p_1, p_2, and p_3 be the mass fractions of I_1^*, I_2^*, and I_3^*, respectively.

(4) Let φ_i be the linear transformation that maps the interval I^* into I_i^*.

(5) The random variable, V, follows a normal distribution of mean p_i. Different values of the variance (r_i max $\{|p_i-p_j|\}$, $r_i\ p_i \ldots r_i$ are the concentration ratios of the linear transformation φ_i) are used in the respective calculations. These choices are dictated by the potential dependence of such values on the particular scale (step) where they are considered.

The procedural steps just described permit construction of 21 basic, random log-selfsimilar cascades for each soil that were used in combination with variance choices to perform PSD calculations. Another 21 calculations were performed using the selfsimilar method of Taguas *et al.* (1999).

Heterogeneity parameters

Soil PSD, or texture, is known to be an important soil property that strongly influences other soil physical properties, such as the capacity to retain and transport water, and solutes. These properties can be obtained directly from detailed PSD data by using an appropriate mathematical-physical model (Arya and Paris, 1981; Haverkamp and Parlange, 1986) or by statistical regression using texture-related parameters (Martín *et al.*, 2005). The PSD and parameters derived from it are used in virtually every pedotransfer function to estimate properties that are difficult (or expensive) to measure directly. Pedotransfer functions use regression analysis and data mining techniques to approximate difficult-to-measure soil properties from readily available soil data, such as texture and pH. A more quantitative soil-texture classification might be most helpful for this purpose.

The US Department of Agriculture (USDA) textural triangle classification groups many different soil textures within the same textural class. Therefore, soils within the same textural class with different properties would not be differentiated by this classification. The proposed model provides heterogeneity parameters (Hölder exponents) that can be calculated using commonly available soil texture. This can lead to quantitative texture characterization that might finally be used to examine correlations between texture and other soil properties and facilitate development of different pedotransfer functions.

Two heterogeneity parameters from the Hölder spectrum are proposed: α(0) and α(1). These heterogeneity parameters may be computed from the available data (see Falconer, 1997) by means of the following formulae:

$$\alpha(0) = \frac{\sum_i r_i\ \log p_i}{\sum_i p_i \log r_i} \tag{8}$$

and

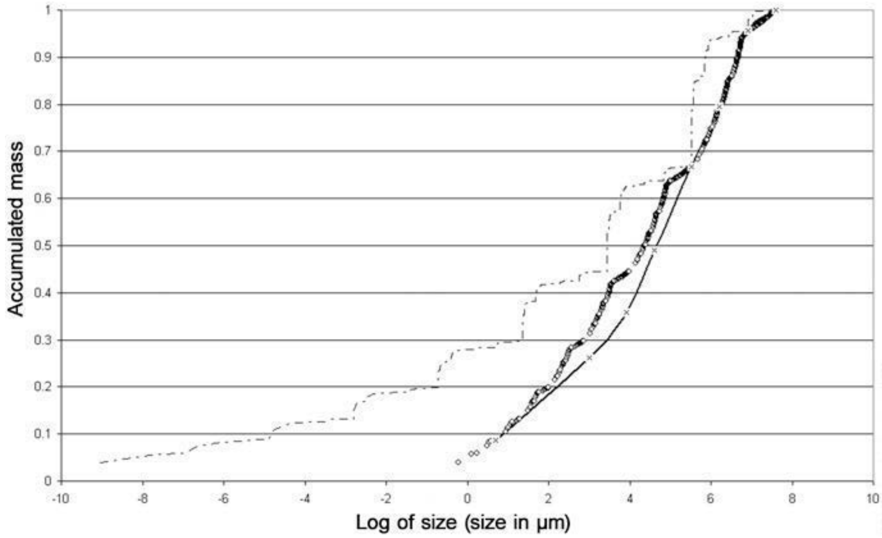

Figure 1. Real and calculated cumulative distributions of the PSD for a sandy loam soil (soil 526, 8.7% clay, 64.1% sand). Real data are represented by the solid line with crosses. Circles correspond to the log selfsimilar model simulation, and the dotted line to the selfsimilar model. Input data for the calculation were mass of intervals (mm) [0−0.250], [0.250−1], and [1−2]. The horizontal axis is log-rescaled in order to show more effectively the mass distributions for silt and clay.

$$\alpha(1) = \frac{\sum_i p_i \log p_i}{\sum_i p_i \log r_i} \qquad (9)$$

where p_i is the mass of particles within a certain range and r_i is the linear transformation ratio for the particle size range in the rescaled interval.

Using the model, the Hölder exponent, α(0), would approach the average value of coarse Hölder exponents. Alternatively, α(1) is the entropy dimension of the distribution and is consistent with the weighted average value of the coarse Hölder exponent. Thus, it accounts for the degree of heterogeneity of the distribution (Caniego *et al.*, 2001). The ratio $H = \alpha(1)/\alpha(0)$ ranges from 0 to 1 and can be used as a single heterogeneity index (Caniego *et al.*, 2001). As the H index value approaches one, the soil mass concentrations in the different size intervals become more and more similar.

All 170 soils were used for calculation of the heterogeneity parameters.

RESULTS AND DISCUSSION

Testing log-selfsimilarity

Soil PSDs calculated using the log-selfsimilar model and the selfsimilar models were compared using the errors for each calculation which were defined as:

$$\varepsilon = \frac{\sum |m_i - m_i'|}{2}$$

where m_i is the real mass fraction in size class I_i and m_i' is the mass fraction assigned to the size class by the calculation. This error term was used by Taguas *et al.* (1999) and a similar error term was employed by Vrscay (1991). The PSD calculations were performed for each soil using the 21 soil data sets as input. The minimum error value for each soil was used as the PSD calculation error.

Below, results obtained using the random cascade method are summarized and error values for validation of the log-selfsimilar model are presented.

(1) As the variance of the random variable in the log-selfsimilar random cascade model approaches zero, the resulting error also decreases. This somewhat surprising result, which was not expected by the authors before testing, strongly supports use of the log-selfsimilar hypothesis for calculating soil PSDs in strict terms (*i.e.* random factor not used through the scales).

Table 1. Mean and variance values of the proposed heterogeneity parameter $H = \alpha(1)/\alpha(0)$ for the different textural classes. The number of soils in each class is shown in parentheses.

	H	
Textural class	Average	Variance
Sand (6)	0.428	0.003
Loamy sand (5)	0.682	0.012
Sandy loam (50)	0.838	0.017
Silt loam (29)	0.550	0.034
Silt (3)	0.381	0.007
Loam (17)	0.801	0.020
Sandy clay loam (5)	0.818	0.009
Clay loam (13)	0.712	0.008
Silty clay loam (15)	0.548	0.025
Silty clay (7)	0.478	0.027
Clay (20)	0.420	0.020

Table 2. Soil particle-size analysis (%) of two sandy loams and two loams. Soil labels used by Taguas (1995) given in parentheses.

	Textural class	Soil A (498a) Sandy loam	Soil B (600) Sandy loam	Soil C (558a) Loam	Soil D (638a) Loam
Clay	[0–0.002]	17.3	5.4	25.6	9.3
Silt	[0.002–0.02]	11.4	9.6	26.7	20.9
	[0.02–0.05]	4.7	19.1	22.5	19.8
Sand	[0.05–0.1]	4.8	13.1	10.5	18.0
	[0.1–0.25]	25.9	23.0	5.2	11.7
	[0.25–0.5]	27.4	11.1	1.5	5.7
	[0.5–1]	8.3	12.6	3.2	8.0
	[1–2]	0.2	6.1	4.8	6.6

(2) The error was <15 for 126 (79.7% of all) soils with the log-selfsimilar model and 74 (46.8%) with the selfsimilar model.

(3) The average error of the log-selfsimilar cascade model was 10.90 and that of the selfsimilar model, 14.67.

(4) Errors in the log-selfsimilar model PSD calculations were smaller than the selfsimilar model calculation errors for 111 soils (70.25%).

The Kolmgorov-Smirnov test (DeGroot and Schervish, 2002) was used to compare the goodness of fit of calculated PSD data to real soil texture data. For each model, 3570 *D*-statistic values were obtained: 21 possible different data sets for each soil (170 soils), and 3318 log-selfsimilar calculated PSDs (92.94%) did not differ from the real soil data at the 95% confidence level. An illustration compares the cumulative PSD of real soil data to PSDs calculated using the two models (Figure 1). The previous results suggest that log-selfsimilarity can be viewed as a statistical regularity law that is useful in describing soil PSDs and for estimating PSDs by using the proposed cascade model to small amounts of soil texture data. Also, log-selfsimilarity might be useful in estimating other soil properties closely related to PSD.

Heterogeneity quantification

The mean and variance of the heterogeneity parameter $H = \alpha(1)/\alpha(0)$ were computed from soil data (170 soils) for soils in each textural class (Table 1). The textural classes that occupy large areas in the USDA textural triangle group together soils with very different textures. As expected, heterogeneity parameter, *H*, variances for large-area textural classes were greater than small-area textural classes and the *H* index gives a quantitative characterization of soil textural classes. As an illustrative example, particle masses between different size limits (mm) for two sandy loam soils and two loam soils (Table 2) and calculated heterogeneity parameters for four soils are shown (Table 3). The results clearly indicate that this parameter (*H*) is useful in identifying different soil textures within a textural class. The small (0.023) *H* index values (*i.e.* signifying homogeneity) for soil samples that contain >99% of a single particle-size separate (sand, silt, clay) indicate low heterogeneity (Table 4). In contrast, the large (0.903) *H* value for a clay loam soil with 33% sand, 33% silt, and 33% clay indicates a large degree of heterogeneity.

Table 3. Values of the heterogeneity parameter $H = \alpha(1)/\alpha(0)$ for the soils in Table 2.

Soil	Textural class	*H*
Soil A (498a)	Sandy loam	0.702
Soil B (600)	Sandy loam	0.892
Soil C (558a)	Loam	0.780
Soil D (638a)	Loam	0.948

Because the proposed model effectively described PSD, heterogeneity parameters derived from the model might be expected to yield reliable numerical estimates of soil texture. Hopefully, these parameters will correlate with other soil properties and might be useful for predicting other soil properties.

CONCLUSIONS

A new way to conceive scale invariance in soil PSDs called log-selfsimilarity is proposed. Based on this regularity law, a model for calculating PSDs using random multiplicative cascades was developed. The PSD model permits the calculation of intermediate particle-size values from common texture data. The model was tested using soil data and found to be an improvement over previously reported models for calculating PSDs

Table 4. Heterogeneity parameter, *H*, for relatively pure sand, silt, and clay textures.

Soil	*H*
Clay 99%	0.014
Silt 99%	0.023
Sand 99%	0.019

from equivalent input data. The proposed model provides heterogeneity parameters that facilitate quantitative characterization of textures. The model and the parameters provided with it might also be used for estimating other soil properties closely related to soil particle-size distribution. Hopefully, basic ideas from this work might be applied to other fine-grained minerals and materials. Also, these ideas might be applied to important mass-size distributions in Earth and Life Sciences that share essential features with soil PSD, specifically to relationships between body size and biomass distribution of living organisms.

ACKNOWLEDGMENTS

This research has been partially supported by Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica (I+D+I) under ref. AGL2004-04079, Spain, and and by DGUI (Comunidad de Madrid) – UPM (Technical University of Madrid), Ref.: M0700204135.

REFERENCES

Anderson, A.N., McBratney, A.B., and Crawford, J.W. (1998) Applications of fractals to soil studies. *Advances in Agronomy*, **63**, 1–76.

Arya, L.M. and Paris, J.F. (1981) A physicoempirical model to predict the soil moisture characteristic from particle-size distribution and bulk density data. *Soil Science Society of America Journal*, **45**, 1023–1030.

Caniego, F.J., Martín, M.A., and San José, F. (2001) Singularity features of pore-size soil distribution: singularity strength analysis and entropy spectrum. *Fractals*, **9**, 305– 316.

Chhabra, A. and Jensen, R.V. (1989) Direct determination of the α singularity spectrum. *Physical Review Letters*, **62**, 1327–1330.

DeGroot, M.H. and Schervish, M.J. (2002) *Probability and Statistics* (3rd edition). Addison-Wesley, New York.

Everstz, C.J.G. and Mandelbrot, B.B. (1992) Multifractal measures. Pp. 921–953 in: *Chaos and Fractals* (H. Peitgen, H. Jürgens, and D. Saupe, editors). Springer, Berlin.

Falconer, K.J. (1994) The multifractal spectrum of statistically self-similar measures. *Journal of Theoretical Probability* **7**, 681–702.

Falconer, K.J. (1997) *Techniques in Fractal Geometry*. Wiley & Sons, Chichester, UK.

Haverkamp, R. and Parlange, J.Y. (1986) Predicting the water-retention curve from particle-size distribution: I. Sandy soils without organic matter. *Soil Science*, **142**, 325–339.

Kolmogorov, A.N. (1992) On the logarithmic normal distribution of particle sizes under grinding. Pp. 281–284 in: *Selected Works of A.N. Kolmogorov* (A.N. Shiryayev, editor). Vol. II, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Mandelbrot, B.B. (1982) *The Fractal Geometry of Nature*. W. H. Freeman & Co., New York.

Martín, M.A. and Taguas, F.J. (1998) Fractal modelling, characterization and simulation of particle-size distributions in soil. *Proceedings of the Royal Society of London Series A*, **454**, 1457–1468.

Martín, M.A., Rey, J.M., and Taguas, F.J. (2001) An entropy-based parametrization of soil texture via fractal modelling of particle-size distribution. *Proceedings of the Royal Society of London Series A*, **457**, 937–947.

Martín, M.A., Pachepsky, Y.A., Rey, J.M., Taguas, J., and Rawls, W.J. (2005) Balanced entropy index to characterize soil texture for soil water retention estimation. *Soil Science*, **170**, 759–766.

Montero, E. and Martín, M.A. (2003) Hölder spectrum of dry grain volume-size distributions in soil. *Geoderma*, **112**, 197–204.

Soil Taxonomy (1975) *A Basic System of Soil Classification for Making and Interpreting Soil Surveys*. Agriculture handbook No. **436**, Soil Survey Staff, Soil Conservation Service, U.S. Department of Agriculture.

Taguas, F.J. (1995) Modelización fractal de la distribución del tamaño de las partículas en el suelo. PhD thesis, Universidad Politécnica de Madrid, Spain.

Taguas, F.J., Martín, M.A., and Perfect, E. (1999) Simulation and testing of self-similar structures for soil particle-size distributions using iterated function systems. *Geoderma*, **88**, 191–203.

Turcotte, D.L. (1986) Fractals and fragmentation. *Journal of Geophysical Research*, **91**, 1921–1926.

Turcotte, D.L. (1992) *Fractals and Chaos in Geology and Geophysics.* Cambridge University Press, Cambridge, UK.

Tyler, S.W. and Wheatcraft, S.W. (1989) Application of fractal mathematics to soil water retention estimation. *Soil Science Society of America Journal*, **53**, 987–996.

Tyler, S.W. and Wheatcraft, S.W. (1992) Fractal scaling of soil particle-size distributions: analysis and limitations. *Soil Science Society of America Journal*, **56**, 362–369.

Vrscay, E.R. (1991) Moment and collage methods for the inverse problem of fractal construction with iterated function systems. Pp. 443–461 in: *Fractals in the Fundamental and Applied Sciences* (H.-O. Peitgen, J.M. Henriques, and L.F. Penedo, editors). North Holland, The Netherlands.

Wu, Q., Borkovec, M., and Sticher, H. (1993) On partice-size distributions in soils. *Soil Science Society of America Journal*, **57**, 883–890.

(*Received 1 June 2007; revised 27 March 2008; Ms. 0029; A.E. W. Jaynes*)

https://doi.org/10.1346/CCMN.2008.0560308 Published online by Cambridge University Press