SEMIGROUPS IN WHICH EACH IDEAL IS A RETRACT

EDWARD J. TULLY, Jr.

(Received 7 July 1967)

A non-empty subset I of a semigroup S is called an *ideal* if ab, $ba \in I$ whenever $a \in I$, $b \in S$. A subset R of S will be called a *retract* if there exists a *retraction* of S onto R, that is a homomorphism of S onto R which leaves each element of R fixed. The purpose of this paper is to study semigroups in which every ideal is a retract. For convenience we shall call such semigroups *retractable*. Such semigroups seem to arise naturally; for example, it is easy to show that if the lattice of congruence relations on S is a complemented lattice then S is retractable.

In Section 1 we give three examples of retractable semigroups. The first example begins with an arbitrary branching tree-like diagram, and defines the semigroup product of two elements to be the greatest lower bound of those elements. In the second example, following a method of construction introduced by Clifford [1941] in his study of semigroups which are unions of groups, we associate an arbitrary group with each element of a tree, take S to be the union of these groups, and define the semigroup multiplication with the aid of certain homomorphisms between the groups. The third example is formed by starting as in Example 2, and then adjoining extra elements, each of which acts like one of the elements already present.

In Section 2 we begin the general study of retractable semigroups. The structure of such a semigroup is reduced (Lemma 2) to that of a certain subsemigroup, which, with the aid of ideas introduced by Green [1951] in the general theory of semigroups, can be shown to be a union of non-zero parts of *0-simple semigroups* (that is, semigroups with 0 in which not every product is 0 and $xay = b$ is solvable for x and y whenever $a \neq 0$). Finally (Lemma 5) the semigroup multiplication is described in terms of certain mappings between the non-zero parts of the 0-simple semigroups.

In Section 3 we show that every retractable semigroup can be constructed by a generalization of Example 3, in which, instead of arbitrary groups, we use arbitrary 0-simple semigroups. This result, in effect, reduces the study of retractable semigroups to that of 0-simple semigroups. It is not completely satisfactory since 0-simple semigroups are not yet completely known. For the case of finite retractable semigroups, the result is more satisfactory, since Suschkewitsch [1928] and Rees [1940] have determined the structure of finite 0-simple semigroups. An easy corollary of the main theorem states that every commutative retractable semigroup is isomorphic to a semigroup constructed as in Example 3; this, in effect, reduces the study of such semigroups to that of groups.

1. Examples of retractable semigroups

EXAMPLE 1. Let T be a partly ordered set satisfying the following conditions

(1) For all $a, b \in T$ there exists $x \in T$ such that $a \geqq x$ and $b \geqq x$

(2) If A is a non-empty subset of T having an upper bound in T then A actually contains a greatest element.

In the case where T is finite, the effect of these conditions is to assume that T can be represented by a branching tree-like diagram rising from a unique least element. We define multiplication in T by letting ab be the (necessarily unique) greatest lower bound of a and b, that is the greatest element in the set $\{x : a \geqq x, b \geqq x\}$. It is easy to see that the ideals are those subsets $I \neq \emptyset$ of S for which $a \leqq b \in I$ always implies $a \in I$. Moreover, if I is any ideal we can define a retraction π of S onto I by letting $a\pi$ be the greatest element in the set $\{x \in I : a \geqq x\}$. From now on, we shall call a semigroup constructed as in this example a *tree*.

EXAMPLE 2. Let T be any tree. For each $i \in T$, let G_i be an arbitrary group, all the groups being disjoint. For each $i, j \in T$ with $i > j$, let π_{ij} be a homomorphism of G_i into G_j satisfying the condition that $a\pi_{ij}\pi_{jk} = a\pi_{ik}$ whenever $a \in G_i$ and $i > j > k$. For each $i \in T$, define π_{ii} to be the identity automorphism on G_i. Let S be the union of all the groups. Define multiplication in S as follows. If $a \in G_i$ and $b \in G_j$, let k be the product ij formed in the tree T. Then let ab be the product $(a\pi_{ik})\ (b\pi_{jk})$ formed in the group G_k. It is easy to check that each ideal of S can be formed by letting I be an ideal of T, and taking the union of all groups G_i for $i \in I$. Moreover, we can define a retraction φ of S onto this ideal by letting π be a retraction of T onto I, as in Example 1, and then defining $g\varphi = g\pi_{i,i\pi}$ whenever $g \in G_i$.

EXAMPLE 3. Let S be formed as in Example 2. Let D be any *dilation* of S. That is, let $D = (\bigcup_{x \in S} S_x) \cup S$, where, for each $x \in S$, S_x is an arbitrary set, all of these sets being mutually disjoint and disjoint from S. Define multiplication in D as follows. If $a, b \in S$, $c \in S_a$, $d \in S_b$, then the products ab, cb, ad and cd are all defined to be the product ab formed in S. One can check that each ideal K of D arises by taking an ideal J of S, as in Example 2, together with an arbitrary subset of $\bigcup_{x\in J} S_x$. Finally, we can define a retraction ψ of D by starting with a retraction φ of S onto J as in Example 2, and extending φ to all of D by: $c\psi = a\varphi$ if $c \in S_a$ and $c \notin K$, $c\psi = c$ if $c \in S_a \cap K$.

2. Preliminary results

Throughout this section let S be a retractable semigroup. An ideal generated by a single element a is called a *principal ideal* and denoted $J(a)$. Thus $J(a)$ consists of a itself together with all its left, right and two-sided multiples.

LEMMA 1. *The principal ideals of S, partly ordered by set inclusion, form a tree..*

PROOF. We must show that the set of principal ideals satisfies (1) and (2). For (1), let $J(a)$ and $J(b)$ be given, and note that $J(a) \supseteqq J(ab)$ and $J(b) \supseteqq J(ab)$. For (2), let $\{J(a_i)\}$ be a set of principal ideals having $J(a)$ as an upper bound. Thus $J(a) \supseteqq J(a_i)$ for all i. Note that the set union $\cup J(a_i)$ is itself an ideal. Let π be a retraction onto $\cup J(a_i)$. Since π is a homomorphism, it preserves the divisibility relation, and hence preserves inclusion of ideals. Hence from $J(a) \supseteqq J(a_i)$ follows $J(a\pi) \supseteqq J(a_i\pi) = J(a_i)$ for all i. On the other hand, $a\pi \in \cup J(a_i)$. Hence $a\pi \in J(a_j)$ for some j. Hence $J(a_j) \supseteqq J(a\pi)$. By transitivity of inclusion we have $J(a_j) \supseteqq J(a_i)$ for all i. This proves (2).

Green [1951] defined the *principal factors* of any semigroup to be those semigroups obtained by taking a principal ideal $J(a)$, letting I be the set of those elements of $J(a)$ which do not generate $J(a)$, and then forming the semigroup $J(a)/I$ (that is, the homomorphic image of the semigroup $J(a)$ modulo the congruence relation having I as one congruence class, every other congruence class being a single element). Green showed that each principal factor is either a 0-simple semigroup, or a *simple* semigroup (that is, one in which $xay = b$ is always solvable for x and y), or a *null* semigroup (that is, one in which every product is equal to the zero element). He called a semigroup *semisimple* if each of its principal factors is either 0-simple or simple. Thus, semisimple semigroups are characterized by the property that every principal ideal has two generators whose product is a generator. As pointed out by Clifford and Preston [1961, p. 76, exercise 7(a)], a semigroup is semisimple if and only if whenever I is an ideal and $a \in I$, there exist $x, y \in I$ such that $xy = a$.

LEMMA 2. *S is a dilation (as defined in Example 3) of a semisimple retractable semigroup.*

https://doi.org/10.1017/S1446788700005838 Published online by Cambridge University Press

PROOF. The set S^2 of all products of elements of S is an ideal of S. Hence there is a retraction π of S onto S^2. For each $a \in S^2$, define $S_a = \{x \in S : x \notin S^2, x\pi = a\}$. Now suppose $a, b \in S^2$, $c \in S_a$, $d \in S_b$. Then $cd \in S^2$, so that π leaves cd fixed. Hence $cd = (cd)\pi = (c\pi)(d\pi) = ab$. Similarly $cb = (cb)\pi = (c\pi)(b\pi) = ab$ and $ad = (ad)\pi = (a\pi)(d\pi) = ab$. Thus, we have shown that S is a dilation of S^2. Now let I be any ideal of S^2. Since S is a dilation of S^2, it is easy to see that I is also an ideal of S. Hence there exists a retraction φ of S onto I. The restriction of φ to S^2 is a retraction of S^2 onto I. Thus, we have shown that S^2 is retractable. Finally, given $a \in I$, write a in the form xy for $x, y \in S$. Then $a = a\varphi = (xy)\varphi = (x\varphi)(y\varphi)$. Thus, we have written a as a product of two elements of I, so that S^2 must be semisimple.

For $a \in S$, we shall denote by J_a the set of all generators of $J(a)$. Thus, the principal factor associated with $J(a)$ is either J_a itself (in the case where $J(a)$ is a minimal ideal) or J_a with a zero element adjoined. For each $b \in S$ we choose a retraction φ_b of S onto $J(b)$. For all $a, b \in S$ with $J(a) \supseteqq J(b)$, we define π_{ab} to be the restriction of φ_b to J_a.

LEMMA 3. *π_{ab} is a partial homomorphism of J_a into J_b, that is a mapping of J_a into J_b which satisfies $(x\pi_{ab})(y\pi_{ab}) = (xy)\pi_{ab}$ whenever x, y and xy are all in J_a.*

PROOF. Let $x \in J_a$. Then $J(x) = J(a) \supseteqq J(b)$. Since φ_b preserves the inclusion relation for ideals, $J(x\pi_{ab}) = J(x\varphi_b) = J(a\varphi_b) \supseteqq J(b\varphi_b) = J(b)$. On the other hand, since $x\varphi_b \in J(b)$, we have $J(x\pi_{ab}) = J(x\varphi_b) \subseteqq J(b)$. Hence $J(x\pi_{ab}) = J(b)$. This shows that $x\pi_{ab} \in J_b$, so that π_{ab} maps J_a into J_b. It is obvious that π_{ab} is a partial homomorphism, since it is a restriction of a homomorphism.

LEMMA 4. *If $J(a) \supseteqq J(b) \supseteqq J(c)$, then*

(3) $(a\pi_{ab}\pi_{bc})c = (a\pi_{ac})c$, *and*

(4) $c(a\pi_{ab}\pi_{bc}) = c(a\pi_{ac})$.

PROOF. First note that $c, ac \in J(c) \subseteqq J(b)$. Hence both φ_b and φ_c (and also their composite $\varphi_b\varphi_c$) leave fixed the elements c and ac. In particular $(ac)\varphi_c = (ac)\varphi_b\varphi_c$. But $(ac)\varphi_c = (a\varphi_c)(c\varphi_c) = (a\varphi_c)c$ and $(ac)\varphi_b\varphi_c = (a\varphi_b\varphi_c)(c\varphi_b\varphi_c) = (a\varphi_b\varphi_c)c$. Combining the last three equations, we conclude $(a\varphi_c)c = (a\varphi_b\varphi_c)c$. But by the definition of the π's in terms of the φ's, this is precisely (3). (4) is proved similarly.

LEMMA 5. For all $a, b \in S$, $ab = (a\pi_{ac})(b\pi_{bc})$, where $J(c)$ is the largest principal ideal for which

(5) $(a\pi_{ac})(b\pi_{bc}) \in J_c$.

https://doi.org/10.1017/S1446788700005838 Published online by Cambridge University Press

PROOF. First note that (5) is satisfied if we replace c by ab, since $(a\pi_{a,ab})(b\pi_{b,ab}) = (a\varphi_{ab})(b\varphi_{ab}) = (ab)\varphi_{ab} = ab \in J_{ab}$. Every principal ideal $J(c)$ satisfying (5) must be contained in $J(a)$, since π_{ac} is defined only when $J(a) \supseteqq J(c)$. Hence by Lemma 1 there exists a largest such principal ideal $J(c)$. In particular $J(c) \supseteqq J(ab)$. Hence $ab \in J(c)$ so that $(ab)\varphi_c = ab$. Now note that π_{ac} and π_{bc} are restrictions of φ_c. Thus $ab = (ab)\varphi_c = (a\varphi_c)(b\varphi_c) = (a\pi_{ac})(b\pi_{bc})$.

3. Main theorem

EXAMPLE 4. Let T be any tree. If T contains a (necessarily unique) least element i_0, let G_{i_0} be an arbitrary simple semigroup. For each $i \in T$ which is not a least element of T, let G_i be an arbitrary 0-simple semigroup. For each $i, j \in T$ with $i > j$, let π_{ij} be a partial homomorphism of the non-zero part J_i of G_i into the non-zero part J_j of G_j, subject to the following conditions:

(6) $(a\pi_{ij}\pi_{jk})c = (a\pi_{ik})c$ whenever $i > j > k$, $a \in J_i$, $c \in J_k$

(7) $c(a\pi_{ij}\pi_{jk}) = c(a\pi_{ik})$ whenever $i > j > k$, $a \in J_i$, $c \in J_k$

(8) If $a \in J_i$ and $b \in J_j$, where neither i nor j is a least element of T, then there exists $k \in T$ such that $i > k$, $j > k$ and the product $(a\pi_{ik})(b\pi_{jk})$ formed in G_k is not the zero element of G_k.

For each $i \in T$, define π_{ii} to be the identity mapping on J_i. Let $S = \bigcup_{i \in T} J_i$. Define multiplication $\circ$ in S as follows. If $a \in J_i$ and $b \in J_j$, let k be the largest element of T such that $(a\pi_{ik})(b\pi_{jk}) \neq 0$ in G_k, and define $a \circ b = (a\pi_{ik})(b\pi_{jk})$.

EXAMPLE 5. Let S be formed as in Example 4. Let D be any dilation (as defined in Example 3) of S.

THEOREM. *A semigroup is retractable if and only if it is isomorphic to a semigroup D constructed as in Example* 5.

PROOF. First we show that S constructed as in Example 4 is a retractable semigroup. Associativity may be shown by using the definition of $\circ$ and applying (6) and (7). One can check (as in Example 2) that every ideal of S is of the form $\bigcup_{i \in I} J_i$ for some ideal I of T, and that the mapping which takes each $g \in J_i$ to $g\pi_{i,i\varphi}$, where φ is a retraction of T onto I, is a retraction of S onto the ideal $\bigcup_{i \in I} J_i$. Next note (as in Example 3) that a dilation of a retractable semigroup must be a retractable semigroup. This proves the 'if' part of the theorem. For the converse, suppose D is a retractable semigroup. By Lemma 2, D is a dilation of a semisimple retractable semigroup S. Let T be the set of principal ideals of S, which by Lemma 1 form a tree. For each $i \in T$, let G_i be the principal factor of S corresponding to the principal ideal i.

https://doi.org/10.1017/S1446788700005838 Published online by Cambridge University Press

Since S is semisimple, each G_i is either a 0-simple semigroup or a simple semigroup. Moreover, we pointed out in Section 2 that simple principal factors correspond to minimal ideals; thus G_i is simple if i is the least element of the tree T, and 0-simple in all other cases. Let J_i be the non-zero part of G_i. For each $j \in T$ choose a retraction φ_j of S onto the ideal j. Define (for all $i > j$) π_{ij} to be the restriction of φ_j to J_i. Thus π_{ij} is the same thing as the mapping π_{ab} (where a and b are generators of i and j respectively) defined in Section 2. Note that π_{ii} is automatically the identity mapping on J_i, since π_{ii} is derived from a retraction onto the ideal i, which contains J_i. By Lemma 2, π_{ij} is a partial homomorphism of J_i into J_j. (6) and (7), being merely restatements of (3) and (4), follow from Lemma 4. To prove (8), let $a \in J_i$ and $b \in J_j$ be given, where neither i nor j is a least element of T. By Lemma 5, there is an ideal $J(c)$ for which (5) holds. Thus we can take k to be either $J(c)$ itself or (if it should happen that $J(c)$ coincides with i or j) any principal ideal properly contained in $J(c)$, and obtain (8). Now we assert that S is isomorphic to the semigroup constructed as in Example 4 using the T, G_i and π_{ij} which we have just defined. First note that each element of S generates a principal ideal, and hence can be regarded as an element of some J_i. This gives a $1-1$ correspondence between S and $\bigcup_{i \in T} J_i$. By Lemma 5, the product ab in S is $(a\pi_{ac})(b\pi_{bc})$, where $J(c)$ is the largest ideal satisfying (5). This is the same thing as the product $a \circ b$ defined in Example 4. Thus we have shown that S is isomorphic to a semigroup constructed as in Example 4. We immediately conclude that D itself, being a dilation of S, is isomorphic to a semigroup constructed as in Example 5.

If we were interested only in finite semigroups, several simplifications could be made in Example 4. The tree T and the semigroups G_i could, of course, be taken to be finite. From (6) and (7) we could conclude

(9) $a\pi_{ij}\pi_{jk} = a\pi_{ik}$ whenever $i > j > k$, $a \in J_i$

since one can show (using the structure theory of Suschkewitsch [1928] and Rees [1940]) that two elements of a finite [0]-simple semigroup must be equal if they always act the same as left and right multipliers. Since (9) obviously implies (6) and (7), we could replace them by (9). Also, (8) could be omitted, since T being finite contains a least element i_0, and the product $(a\pi_{ii_0})(b\pi_{ji_0})$ in the simple semigroup G_{i_0} is automatically non-zero.

If we were interested only in commutative semigroups, we could take each G_i to be either an abelian group or an abelian group with zero adjoined (since a commutative [0]-simple semigroup must be a group [with zero adjoined]). Thus each J_i would be an abelian group. From (6) and (7), (9) would follow by cancellation in the group J_k. (8) could be omitted, since every product of elements of the group J_k is itself in J_k and hence not 0.

https://doi.org/10.1017/S1446788700005838 Published online by Cambridge University Press

Moreover, the definition of multiplication could be simplified as follows. $a \circ b$ was defined in Example 4 to be $(a\pi_{ik})(b\pi_{jk})$ for the largest $k \in T$ for which $(a\pi_{ik})(b\pi_{jk}) \neq 0$. Since, as we have just pointed out, the product is never 0, k is merely the greatest lower bound in T of i and j. If we apply all the simplifications in this paragraph, we note that we have reduced Example 4 to Example 2. Hence, Example 5 is reduced to Example 3, so that we have proved the following:

COROLLARY. *A commutative semigroup is retractable if and only if it is isomorphic to a semigroup constructed as in Example* 3, *with all the groups* G_i *used in the construction being abelian.*

References

[1] A. Clifford, [1941] 'Semigroups admitting relative inverses', *Ann. of. Math.* 42 (1941), 1037–1049.
[2] A. Clifford and G. B. Preston, [1961] *The algebraic theory of semigroups, Volume I* (Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961).
[3] J. Green, [1951] 'On the structure of semigroups', *Ann. of Math.* 54 (1951), 163–172.
[4] D. Rees, [1940] 'On semigroups', *Proc. Cambridge Philos. Soc.* 36 (1940), 387–400.
[5] A. K. Suschkewitsch, [1928] 'Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit', *Math. Ann.* 99 (1928), 30–50.

University of California
Davis, California