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ON A CLASS OF SOBOLEV FUNCTIONS AND ITS APPLICATIONS TO HIGHER-ORDER ELLIPTIC EQUATIONS

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Abstract

It is well known that higher-order linear elliptic equations with measurable coefficients and higher-order nonlinear elliptic equations with analytic coefficients can admit unbounded solutions, unlike their second-order counterparts. In this work we introduce the concept of approximate truncates for functions in higher-order Sobolev spaces and prove that if a solution of a higher-order linear elliptic equation has an approximate truncate somewhere then it is bounded there.

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1. Introduction

The problem of regularity of second-order elliptic equations with bounded measurable coefficients has been thoroughly investigated in the works of De Giorgi [1], Nash [10], Moser [9], Ladyzhenskaya and Ural'tseva [7], Stampacchia [15], Serrin [12] and Trudinger [17]. Issues such as boundedness of solutions, Hölder continuity, Harnack inequality and removability of singularities are now well-established facts.

The hope for the generalization of these results to systems of elliptic equations, to higher-order linear elliptic equations with bounded measurable coefficients and to higher-order nonlinear elliptic equations with analytic coefficients was dealt a blow with the discovery of counter-examples by De Giorgi [2], Freshe [3], and others. We refer to the monograph by Giaquinta [4] for an interesting historical account of these developments. Some results on boundedness and Hölder continuity of the solutions for some isolated cases of higher-order elliptic equations (when the order of the equation

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is equal to the dimension of the underlying space or is in some sense close to it) were obtained by Freshe [3], Widman [18, 19], Skrypnik [13] and Solonnikov [14].

Another stumbling block in the development of the theory is the lack of a suitable concept of truncates for functions in higher-order Sobolev spaces, making it difficult to use the method of De Giorgi or Stampacchia. Moser's iteration process is also hard to implement in this context since the insertion of any test function which contains the power of an expression involving the solution in the weak formulation of the equations leads to some terms which are hard to handle.

It is well known that regularity results for solutions of partial differential equations and their a priori estimates depend on the class of equations as well as the class of function spaces to which the solutions belong. So far efforts have been made mainly in identifying suitable classes of equations whose solutions are bounded. The identification of more restrictive classes of solutions for which boundedness can be established for larger classes of equations seems not to have been undertaken even though the idea is often used by researchers dealing with existence results.

This work deals with a class of functions in some higher-order Sobolev spaces whose super level sets are extension domains in a sense to be made precise and which admit what we refer to here as approximate truncates. Large classes of functions have sufficiently smooth level sets. This smoothness depends in general on the magnitude of the smoothness of the functions. We should mention here that super level sets of singular functions might also be extension domains for Sobolev spaces. As an example $f(x) = 1/|x|$, $x \in \mathbb{R}^n$ with $n \geq 2$, is discontinuous but its super level sets, which are balls with the centre removed, are obviously extension domains for some Sobolev spaces modulo some appropriate conditions on the power of integrability and differentiation order; a ball without its centre is not Lipschitz but the singularity is removable. On the basis of the outstanding results of Jones [6] and Stein [16] on sufficient conditions for a domain to be an extension domain for Sobolev spaces (slightly less than lipschitzity), we see that the extension property for super level sets of functions is a rather natural condition; we note that the corresponding necessary condition is still one of the most challenging problems in analysis. The second condition needed for a function to have an approximate truncate is a more subtle one and seems not to have been previously discovered. This makes our concept of approximate truncate an additional matter of independent interest.

It turns out that solutions of linear higher-order elliptic equations with bounded measurable coefficients satisfying standard growth conditions from the classes of functions with approximate truncates are bounded. The idea is to insert in the integral identity in the weak formulation of the equation a test function constructed with the help of an approximate truncate of the solution. Then some adaptation of the powerful method of a priori estimates of De Giorgi and Stampacchia is possible. We limit ourselves to higher-order linear equations, but generalization to nonlinear equations is possible.

In Section 2, we introduce the concept of approximate truncates for Sobolev functions and provide an example. In Sections 3 and 4, we derive some a priori estimates for the solution of a linear higher-order elliptic equation with bounded measurable coefficients. In Section 5, we give an example of a function not admitting an approximate truncate.

2. Approximate truncates for Sobolev functions

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$; $x = (x_1, \dots, x_n)$ denotes a generic point in $\mathbb{R}^n$. We denote by $W_p^l(\Omega)$ ($p \in (1, \infty)$, l is a nonnegative integer) the usual Sobolev space on Ω; $W_p^0(\Omega) =: L_p(\Omega)$. By $W_{p,\mathrm{loc}}^l(\Omega)$ we denote the space of functions that, together with all their weak derivatives of order up to l, belong to $L_p^l(\Omega')$ for any open set Ω whose closure is a subset of Ω in Ω and we denote by $\mathring{W}_p^l(\Omega)$ the set of functions in $W_p^l(\mathbb{R}^n)$ vanishing off Ω together with their derivatives of order up to l.

Let $u \in W_p^l(\Omega)$ and let $k > 0$. For any subdomain Ω' lying inside Ω ($\bar{\Omega}' \subset \Omega$), we denote by $A_k(\Omega')$ the set $\{x \in \Omega' \mid u(x) > k\}$, the k-super level set of u. For any $\varepsilon > 0$, we set $A_{k\varepsilon}(\Omega') = \{x \in \Omega' \mid \operatorname{dist}(x, \partial A_k) < \varepsilon\}$, where dist stands for distance and $\partial\cdot$ the boundary of a set $\cdot$. By $B(x, R)$ we denote the ball centered at x with radius $R > 0$ and by $K(x, r, R)$ the annulus $\{x : r < |x| < R\}$, where $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. We also set $u_k(x) = u(x) - k$.

DEFINITION 1. Let Ω' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$. We shall say that $u \in W_p^l(\Omega)$ has an approximate truncate in $W_p^l(\Omega')$ if for all $k > 0$ such that meas $A_k(\Omega') \neq 0$, there exits an $\varepsilon_0 > 0$ independent of k such that $\overline{A_{k\varepsilon}(\Omega')} \subset \Omega$ for all $\varepsilon \in (0, \varepsilon_0)$, and there exists an extension operator

$$\mathcal{E}_k^\varepsilon : W_p^l(A_k(\Omega')) \to \mathring{W}_p^l(A_{k\varepsilon}(\Omega'))$$

such that

$$\lim_{\varepsilon \to 0} \|\mathcal{E}_k^\varepsilon u_k - u_k\|_{\mathring{W}_p^l(A_{k\varepsilon}(\Omega'))} = 0. \tag{1}$$

We refer to the function $v_k^\varepsilon = \mathcal{E}_k^\varepsilon u_k$ as an approximate truncate of u in $W_p^l(\Omega')$.

We call Ω an extension domain for W_p^l if there exists a bounded operator $\mathcal{E} : W_p^l(\Omega) \to W_p^l(\mathbb{R}^n)$ and we write $u \in EW_p^l$. Sufficient conditions for a domain to be an extension domain in Sobolev spaces can be found in the works of Jones [6] and Stein [16]. The necessary condition in the general case is still not known. For the existence of an extension operator from $W_p^l(A_k(\Omega'))$ into $\mathring{W}_p^l(A_{k\varepsilon}(\Omega'))$ it is sufficient that $A_k(\Omega') \in EW_p^l$, since due to the fact that $\overline{A_k(\Omega')} \subset A_{k\varepsilon}(\Omega')$ it is enough to multiply any function in $W_p^l(A_k(\Omega'))$ by a test function $\varphi \in \mathring{W}_p^l(A_{k\varepsilon}(\Omega'))$ such that $\varphi(x) = 1$ in $A_k(\Omega')$. For condition (1) to be satisfied it is sufficient that the extension operator $\mathcal{E}_k^\varepsilon$ be uniformly bounded. The fact that $\partial A_k(\Omega')$ does not intersect with $\partial A_{k\varepsilon}(\Omega')$ is important, since otherwise the construction of such an extension operator would have been an extremely nontrivial problem apparently unsolved in general; a corresponding result in some planar domains can be found in [8].

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EXAMPLE 2. Let $u(x) = |x|$. Then $u \in W_2^2(B(0, R))$ for all $R > 0$ and $n \geq 1$. We show that u has an approximate truncate in $W_2^2(B(0, r))$ with $r < R$. Now $A_k(B(0, r)) = K(0, k, r) = \{x : k < |x| < r\}$ and meas $A_k(B(0, r)) \neq 0$ for $k < r$. Let $\varepsilon_0 = \min\{R - r, k\}$; then for all $\varepsilon \in (0, \varepsilon_0)$, $A_{k\varepsilon}(B(0, r)) = K(0, k - \varepsilon, r + \varepsilon) \subset B(0, R)$. Let us construct an extension v_k^ε of $(u - k)$ from $W_2^2(A_k(B(0, r)))$ into $\overset{\circ}{W}{}_2^2(A_{k\varepsilon}(B(0, r)))$ which satisfies the condition (1). We consider the well-known bump function

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{if } k \leq |x| \leq r, \\ 0 & \text{if } 0 \leq |x| < k - \varepsilon \text{ and } |x| > r + \varepsilon, \\ \text{increases from 0 to 1} & \text{for } k - \varepsilon < |x| \leq k, \\ \text{decreases from 1 to 0} & \text{for } r \leq |x| \leq r + \varepsilon, \end{cases}$$

which can be found in numerous references, for example in [11, Lemma 1, Ch. 2, Paragraph 8]. In fact ψ_ε is the difference $\psi_{r,\varepsilon} - \psi_{k,\varepsilon}$ where $\psi_{r,\varepsilon}$ and $\psi_{k,\varepsilon}$ are the bump functions equal to 1 in the balls $B(0, r)$ and $B(0, k - \varepsilon)$ and equal to 0 outside $B(0, r + \varepsilon)$ and $B(0, k)$. Explicit expressions for $\psi_{r,\varepsilon}$ and $\psi_{k,\varepsilon}$ are given in the reference quoted. We easily obtain that ψ_ε is infinitely differentiable and uniformly bounded together with its derivatives irrespective of ε. The function $v_k^\varepsilon(x) = \psi_\varepsilon(x)u(x)$ belongs to $\overset{\circ}{W}{}_2^2(A_{k\varepsilon}(B(0, r)))$ and coincides with u on $A_k(B(0, r))$. Simple calculations using the explicit expression of ψ_ε show that the corresponding extension operator $\mathcal{E}_k^\varepsilon$ is uniformly bounded and thus

$$\lim_{\varepsilon \to 0} \|v_k^\varepsilon - |x| + k\|_{\overset{\circ}{W}{}_2^2(K(0,k-\varepsilon,r+\varepsilon))} = 0.$$

Therefore u has an approximate truncate in $W_2^2(B(0, r))$.

Toward the end of this paper we give an example of a function whose level sets are extension domains but which does not admit an approximate truncate. The postponement of the example is due to the fact that we shall rely on results that will be derived in the next section.

3. A priori estimates for higher-order elliptic equations

We look for a function u defined in Ω and satisfying the equation

$$\sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u) = 0 \quad \text{in } \Omega, \tag{2}$$

where $a_{\alpha\beta}$ are bounded measurable functions and satisfy the condition of ellipticity

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha \xi^\beta \geq \nu \sum_{|\alpha|=m} |\xi^\alpha|^2, \tag{3}$$

for any $\xi = (\xi_1, \ldots, \xi_n)$; here α is a multiindex $(\alpha_1, \ldots, \alpha_n)$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi^{\alpha_n}$, $D^\alpha = D^{\alpha_1} \cdots D^{\alpha_n}$, $D^{\alpha_i} = \partial^{\alpha_i}/\partial x^{\alpha_i}$.

DEFINITION 3. We shall say that $u \in W^m_{2,\text{loc}}(\Omega)$ is a local weak solution of equation (2) if for any subdomain $\bar{\Omega}' \subset \Omega$ and all functions $\varphi \in C^\infty_o(\Omega')$ the integral identity

$$\sum_{|\alpha|=|\beta|=m} \int_{\Omega'} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha \varphi(x)\, dx = 0 \tag{4}$$

holds.

Our main result is the following theorem.

THEOREM 4. *For $2m < n$, let $0 \le u \in W^m_{2,\text{loc}}(\Omega)$ be a local weak solution of (2) and let $B(x_0, R)$ be a ball inside Ω ($\overline{B(x_0, R)} \subset \Omega$). If the function u has an approximate truncate in $W^m_2(B(x_0, R))$, then for any $0 < \rho \le R$ and $\sigma \in (0, 1)$ there exists a constant $K > 0$ depending only on the data such that*

$$\operatorname*{ess\,sup}_{x \in B(x_0, \sigma\rho)} u(x) \le K \left\{ \max\left\{ \rho^{n/\alpha}, \frac{1}{(\sigma\rho)^n} \right\} \int_{B(x_0,\rho)} u^2(x)\, dx \right\}^{1/2}, \tag{5}$$

where α is the positive root of the quadratic equation $2m\alpha^2 - n(\alpha + 1) = 0$.

When $m = 1$, the above theorem has been obtained in most of the works that we refer to at the very beginning of this paper.

From now on we agree to denote by C, C_t all positive constants depending only on the data.

Let $\sigma \in (0, 1)$ and $0 < \rho < R$. We consider an infinitely differentiable function ψ such that $0 \le \psi(x) \le 1$,

$$\psi(x) = \begin{cases} 1 & \text{if } |x - x_0| \le (1 - \sigma)\rho, \\ 0 & \text{if } |x - x_0| \ge \rho, \end{cases} \qquad |D^\alpha \psi| \le \frac{1}{(\sigma\rho)^{|\alpha|}}.$$

Let $A_k(B(x_0, \rho)) = \{x \in B(x_0, \rho) \mid u(x) > k\}$ and let ε_0 be a positive number independent of k such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$A_{k\varepsilon} = \{x \in B(x_0, R) \mid \operatorname{dist}(\partial A_{k\varepsilon}, A_k) < \varepsilon\}.$$

We establish the following result.

LEMMA 5. *Let the conditions of Theorem 4 be satisfied. Then*

$$\sum_{|\alpha|=m} \int_{A_k(B(x_0,(1-\sigma)\rho))} |D^\alpha u(x)|^2\, dx \le C \max\{1, (\sigma\rho)^{-2m}\} \int_{A_k(B(x_0,\rho))} (u - k)^2\, dx, \tag{6}$$

where C is a constant depending only on the data and independent of k.

PROOF. Since u has an approximate truncate in $B(x_0, R)$, there exists an extension operator $\mathcal{E}^\varepsilon_k : W^m_2(A_k) \to \overset{\circ}{W}{}^m_2(A_{k\varepsilon})$ such that $u_{k\varepsilon}(x) = \mathcal{E}^\varepsilon_k u_k$ and

$$\lim_{\varepsilon \to 0} \|u_{k\varepsilon} - u_k\|_{\overset{\circ}{W}{}^m_2(A_{k\varepsilon})} = 0. \tag{7}$$

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From this relation and the fact that

$$\begin{aligned}\|u_k\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} - \|u_{k\varepsilon} - u_k\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} &\le \|u_{k\varepsilon}\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} \\ &\le \|u_{k\varepsilon} - u_k\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} + \|u_k\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})},\end{aligned}$$

and $\lim_{\varepsilon\to\infty} \operatorname{meas}(A_{k\varepsilon}\backslash A_k) = 0$, we get

$$\lim_{\varepsilon\to 0} \|u_{k\varepsilon}\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} = \lim_{\varepsilon\to 0} \|u_k\|_{\overset{\circ}{W}{}_2^m(A_{k\varepsilon})} = \|u_k\|_{\overset{\circ}{W}{}_2^m(A_k)}, \tag{8}$$

where we have used the absolute continuity of integrals.

In the integral identity (4), we substitute $\varphi(x) = u_{k\varepsilon}(x)\psi^l(x)$ for some $l \ge 2m$. By (3) we get

$$\begin{aligned}&\sum_{|\alpha|=m} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^\alpha u_{k\varepsilon}|^2 \psi^l \, dx \\ &\quad\le C_1 \sum_{|\alpha|=m} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^\alpha (u_{k\varepsilon} - u_k)|^2 \psi^l \, dx \\ &\qquad + C_2 \sum_{|\alpha|+|\beta|=2m, |\alpha|\le m-1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^\beta u||D^\alpha u_{k\varepsilon}||D^\beta \psi^l| \, dx.\end{aligned} \tag{9}$$

Owing to the fact that

$$|D^j \psi^l(x)| \le \frac{C}{(\sigma\rho)^j} \psi^{l-j}(x),$$

using Cauchy's inequality, we estimate the second term in the right-hand side of (9) (which we denote by I_1) as

$$\begin{aligned}I_1 &\le \varepsilon' \sum_{|\beta|=2m} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^\beta u|^2 \psi^l \, dx \\ &\quad + C_{\varepsilon'} \sum_{j=0}^{m-1} \frac{1}{(\sigma\rho)^{2(m-j)}} |D^j u_{k\varepsilon}|^2 \psi^{l-2(m-j)} \, dx,\end{aligned} \tag{10}$$

for any $\varepsilon' > 0$. Let us denote the second integral in this inequality by H_1 and show that for all $\delta > 0$,

$$\begin{aligned}&\sum_{j=0}^{m-1} \frac{1}{(\sigma\rho)^{2(m-j)}} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^j u_{k\varepsilon}|^2 \psi^{l-2(m-j)} \, dx \\ &\quad \le \delta \sum_{|\alpha|=2m} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^\alpha u_{k\varepsilon}|^2 \psi^l \, dx \\ &\qquad + C_\delta \max\{1, (\sigma\rho)^{-2m}\} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon}^2 \psi^{l-2m} \, dx.\end{aligned} \tag{11}$$

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We start by showing by induction that, for any $j = 1, \ldots, m-1$,

$$\int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^j u_{k\varepsilon}|^2 \psi^{l-2(m-j)}\,dx \le \delta_j \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^{j+1} u_{k\varepsilon}|^2 \psi^{l-2(m-j-1)}\,dx + C_j \max\{1, (\sigma\rho)^{-2}\}^j \int_\Omega u_{k\varepsilon}^2 \psi^{l-2m}\,dx, \tag{12}$$

with $\delta_j > 0$ sufficiently small.

Let $j = 1$. By an integration by parts,

$$\begin{aligned} H_1 &= \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |Du_{k\varepsilon}|^2 \psi^{l-2(m-1)}\,dx \\ &= \int_{A_{k\varepsilon}\cap B(x_0,\rho)} Du_{k\varepsilon} Du_{k\varepsilon} \psi^{l-2(m-1)}\,dx \\ &= \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon} D(Du_{k\varepsilon} \psi^{l-2(m-1)})\,dx \\ &\le \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |u_{k\varepsilon}||D^2 u_{k\varepsilon}| \psi^{l-2(m-1)}\,dx \\ &\qquad + (l-2(m-1)) \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |u_{k\varepsilon}||Du_{k\varepsilon}|\, \psi^{l-2(m-1)-1}\, |D\psi|\,dx. \end{aligned}$$

Applying Cauchy's inequality to the integrands in the last inequality, we get, for some $\delta_1 > 0$,

$$\begin{aligned} H_1 \le{}& C_{\delta_1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon}^2 \psi^{l-2m}\,dx + \delta_1 \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^2 u_{k\varepsilon}|^2 \psi^{l-2(m-2)}\,dx \\ &+ C_{\delta_1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon}^2 |D\psi|^2 \psi^{l-2m}\,dx \\ &+ \delta_1 \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |Du_{k\varepsilon}|^2 \psi^{l-2(m-1)}\,dx. \end{aligned}$$

Thus for δ_1 sufficiently small, we see that

$$\begin{aligned} H_1 \le{}& \delta_1 \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^2 u_{k\varepsilon}|^2 \psi^{l-2(m-2)}\,dx \\ &+ C_{\delta_1} \max\{1, (\sigma\rho)^{-2}\} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon}^2 \psi^{l-2m}\,dx. \end{aligned}$$

This proves (12) when $j = 1$.

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Suppose that (12) is true for $j = r$, that is,

$$\begin{aligned}\int_{A_{k\varepsilon}\cap B(x_0,\rho)} &|D^r u_{k\varepsilon}|^2 \psi^{l-2(m-r)}\,dx \\ &\le \delta_r \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^{r+1} u_{k\varepsilon}|^2 \psi^{l-2(m-r-1)}\,dx \\ &\quad + C_{\delta_r} \max\{1,(\sigma\rho)^{-2}\}^r \int_{A_{k\varepsilon}\cap B(x_0,\rho)} u_{k\varepsilon}^2 \psi^{l-2m}\,dx. \end{aligned} \tag{13}$$

We show that (12) is true for $j = r + 1$. By integration by parts, and later using Cauchy's inequality, we get

$$\begin{aligned}\int_{A_{k\varepsilon}\cap B(x_0,\rho)} &|D^{r+1} u_{k\varepsilon}|^2 \psi^{l-2(m-r-1)}\,dx \\ &= \int_{A_{k\varepsilon}\cap B(x_0,\rho)} D D^r u_{k\varepsilon} D^{r+1} u_{k\varepsilon} \psi^{l-2(m-r-1)}\,dx \\ &\le \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^r u_{k\varepsilon}||D^{r+2} u_{k\varepsilon}| \psi^{l-2(m-r-1)}\,dx \\ &\quad + (l-2(m-k-1)) \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^r u_{k\varepsilon}||D^{r+1} u_{k\varepsilon}|\, \psi^{l-2(m-r-1)-1}\,|D\psi|\,dx \\ &\le C_{\delta_{r+1}} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^r u_{k\varepsilon}|^2 \psi^{l-2(m-r)}\,dx \\ &\quad + \delta_{r+1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^{r+2} u_{k\varepsilon}|^2 \psi^{l-2(m-r-1)}\,dx \\ &\quad + C_{\delta_{r+1}} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^r u_{k\varepsilon}|^2\, \psi^{l-2(m-r)}\,|D\psi|^2\,dx \\ &\quad + \delta_{r+1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^{r+1} u_{k\varepsilon}| \psi^{l-2(m-r-1)}\,dx, \end{aligned}$$

where δ_{r+1} is a controllable constant.

For $\delta_{r+1} < 1$, we deduce that

$$\begin{aligned}\int_{A_{k\varepsilon}\cap B(x_0,\rho)} &|D^{r+1} u_{k\varepsilon}|^2 \psi^{l-2(m-r-1)}\,dx \\ &\le \delta_{r+1} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^{r+2} u_{k\varepsilon}|^2 \psi^{l-2(m-r-1)}\,dx \\ &\quad + C_{\delta_{r+1}} \max\{1,(\sigma\rho)^{-2}\} \int_{A_{k\varepsilon}\cap B(x_0,\rho)} |D^r u_{k\varepsilon}|^2 \psi^{l-2(m-r)}\,dx \end{aligned}$$

$$\begin{aligned}&\le \delta_{r+1}\int_{A_{k\varepsilon}\cap B(x_0,\rho)}|D^{r+2}u_{k\varepsilon}|^2\psi^{l-2(m-r-1)}\,dx\\&\quad + C_{\delta_{r+1}}\max\{1,(\sigma\rho)^{-2}\}\delta_r\int_{A_{k\varepsilon}\cap B(x_0,\rho)}|D^{r+1}u_{k\varepsilon}|^2\psi^{l-2(m-r-1)}\,dx\\&\quad + C_{\delta_{r+1}}C_{\delta_r}\max\{1,(\sigma\rho)^{-2}\}^{r+1}\int_{A_{k\varepsilon}\cap B(x_0,\rho)}u_{k\varepsilon}\psi^{l-2m}\,dx,\end{aligned}$$

where we have used the hypothesis of induction (13). Requiring further that δ_r satisfies the inequality $C_{\delta_{r+1}}\max\{1,(\sigma\rho)^{-2}\}\delta_r<1$, we can easily see that inequality (12) holds for $j=r+1$. Hence, by induction, (12) is established.

Inequality (11) will follow from the summation of the inequalities (12) for $j=1,\dots,m-1$ with both sides multiplied by $(\sigma\rho)^{2(j-m)}$. The terms on the right-hand side of the resulting inequality, involving derivatives of $u_{k\varepsilon}$ of order between 1 and $m-1$, will be taken to the left-hand side by choosing the constants δ_j small enough. Inequality (11) is thus proved.

Choosing ε' and δ sufficiently small in (10) and (11) respectively, we get from (9) that

$$\begin{aligned}&\sum_{|\alpha|=m}\int_{A_{k\varepsilon}\cap B(x_0,(1-\sigma)\rho)}|D^\alpha u_{k\varepsilon}|^2\,dx\\&\quad\le C_1\sum_{|\alpha|=m}\int_{A_{k\varepsilon}\cap B(x_0,\rho)}|D^\alpha(u_{k\varepsilon}-u_k)|^2\,dx\\&\qquad + C_2\max\{1,(\sigma\rho)^{-2m}\}\int_{A_{k\varepsilon}\cap B(x_0,\rho)}u_{k\varepsilon}^2\,dx.\end{aligned}\tag{14}$$

Lemma 5 follows by passing to the limit in this inequality and using (7) and (8). □

3.1. Proof of Theorem 4 By Hölder's inequality,

$$\begin{aligned}\int_{A_k(B(x_0,(1-\sigma)\rho))}(u-k)^2\,dx &\le C[\mathrm{meas}(A_k(B(x_0,(1-\sigma)\rho)))]^{2m/n}\\&\quad\times\bigg(\int_{A_k(B(x_0,(1-\sigma)\rho))}(u-k)^{2n/(n-2m)}\,dx\bigg)^{2n/(n-2m)}.\end{aligned}$$

Thus by the Sobolev embedding theorem, $W_2^m(\Omega)\hookrightarrow L_{2n/(n-2m)}(\Omega)$, and inequality (6), we get

$$\begin{aligned}&\int_{A_k(B(x_0,(1-\sigma)\rho))}(u-k)^2\,dx\\&\quad\le C[\mathrm{meas}(A_k(B(x_0,(1-\sigma)\rho)))]^{2m/n}\\&\qquad\times\max\{1,(\sigma\rho)^{-2m}\}\int_{A_k(B(x_0,\rho))}(u-k)^2\,dx.\end{aligned}\tag{15}$$

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For $h > k$,

$$(h-k)^2 \operatorname{meas}(A_h(B(x_0, \rho))) = \int_{A_h(B(x_0,\rho))} (h-k)^2\,dx$$
$$\le \int_{A_k(B(x_0,\rho))} (u-k)^2\,dx. \qquad (16)$$

Thus setting

$$a(h, r) = \operatorname{meas}(A_h(B(x_0, r))), \quad u(h, r) = \int_{A_h(B(x_0,r))} (u-h)^2\,dx,$$

and assuming that $(\sigma\rho)^{-2m} \ge 1$, we obtain from (15) and (16) that

$$u^\alpha(h, (1-\sigma)\rho)a(h, (1-\sigma)\rho) \le C\frac{(\sigma\rho)^{-2m\alpha}}{(h-k)^2}[u(k, \rho)]^{\alpha+1}[a(k, \rho)]^{2m\alpha/n}. \qquad (17)$$

Let

$$\Phi(l, r) = u^\alpha(l, r)a(l, r),$$

and let α be the positive solution of the quadratic equation $2m\alpha^2 - n(\alpha+1) = 0$. Then (17) becomes

$$\Phi(h, (1-\sigma)\rho) \le \frac{C}{(\sigma\rho)^{2m\alpha}(h-k)^2}[\Phi(k, \rho)]^{1+1/\alpha}. \qquad (18)$$

By [15, Lemma 5.1] or [5, Proposition 5.1] we get that for any $k_0 \ge 0$,

$$\Phi(d + k_0, \sigma\rho) = 0, \qquad (19)$$

with

$$d^2 = \frac{C^2}{(\sigma\rho)^{2m\alpha}}[\Phi(k_0, \rho)]^{1/\alpha}.$$

Analogously for $(\sigma\rho)^{-2m} \le 1$, we get (19) with

$$d^2 = C^2[\Phi(k_0, \rho)]^{1/\alpha}.$$

Inequality (5) therefore follows from these relations by choosing $k_0 = 0$. This complete the proof of the main theorem.

4. A function with no approximate truncate

In this section we give the example of a function which does not possess an approximate truncate; in particular, we show that unbounded functions might not have approximate truncates in domains containing their singularity. For that purpose, we shall need the following result on sharp estimates of extension operators for Sobolev spaces in small domains obtained in [8, p. 157].

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THEOREM 6. *Let $\Omega \subset \mathbb{R}^n$ be an extension domain for W_p^l and let $G \subset \mathbb{R}^n$ such that Ω and G contain the origin. Let $\Omega_\sigma = \sigma\Omega = \{x \mid x/\sigma \in \Omega\}$, $\sigma \in (0, 1/2)$ and let $\bar{\Omega}_\sigma \subset G_\rho$ where $G_\rho = \rho G$, $\rho \in (0, \infty)$. Then for any extension operator $\mathcal{E} : W_p^l(\Omega_\sigma) \to \overset{\circ}{W}{}_p^l(G_\rho)$,*

$$c\|\mathcal{E}\| \geq \sigma^{-l} \quad \text{if } pl < n. \tag{20}$$

We consider the functions $u(x) = |x|^\lambda$ with $(4 - n)/2 < \lambda < 0$ in $W_2^2(B(0, R))$ for $0 < R < \infty$. We use an indirect argument based upon the fact that u is a weak solution of a fourth-order linear elliptic equation

$$\sum_{|\alpha|+|\beta|\leq 4} D^\alpha(a_{\alpha\beta}(x) D^\beta u(x)) = 0, \tag{21}$$

with bounded measurable coefficients satisfying (3) in any bounded region in $\mathbb{R}^n$ containing the origin; the explicit expressions for the functions $a_{\alpha\beta}$ can be found in [4, pp. 55–56]. Therefore u is a solution of (21) in the ball $B(0, R)$. An extension operator $\mathcal{E}_{k\varepsilon} : W_2^2(A_k(B(0, \rho))) \to \overset{\circ}{W}{}_2^2(A_{k\varepsilon}(B(0, \rho)))$ exists where $A_k(B(0, \rho)) = B(0, k^{1/\lambda})\backslash\{0\}$, $0 < \rho < R$, $\varepsilon \in (0, \varepsilon_0)$ such that $A_{k\varepsilon}(B(0, \rho)) \subset B(0, R)$. Arguing as in the proof of Lemma 5, we get

$$\begin{aligned}\sum_{|\alpha|=2} \int_{A_{k\varepsilon}\cap B(0,(1-\sigma)\rho)} |D^\alpha u_{k\varepsilon}|^2\,dx &\leq C_1 \sum_{|\alpha|=2} \int_{A_{k\varepsilon}\cap B(0,\rho)} |D^\alpha(u_{k\varepsilon} - u_k)|^2\,dx \\ &\quad + C_2 \max\{1, (\sigma\rho)^{-4}\} \int_{A_{k\varepsilon}\cap B(0,\rho)} u_{k\varepsilon}^2\,dx,\end{aligned} \tag{22}$$

where $u_k(x) = u(x) - k$, $u_{k\varepsilon}(x) = \mathcal{E}_{k\varepsilon} u_k(x)$ and C_1 and C_2 are constants independent of k. It is clear that there exists a $k_0 > 0$ such that for all $k \geq k_0$, $k^{1/\lambda} \in (0, 1/2)$ and $A_{k\varepsilon} \subset B(0, (1 - \sigma)\rho)$. Thus $A_{k\varepsilon} \cap B(0, (1 - \sigma)\rho) = A_{k\varepsilon}$. From Theorem 6,

$$\|\mathcal{E}_{k\varepsilon} u_{k\varepsilon}\|_{\overset{\circ}{W}{}_2^2(A_{k\varepsilon})} \geq ck^{-2/\lambda} \|u_k\|_{W_2^2(A_k)},$$

with the constant c independent of ε. Therefore we derive from (22) that

$$ck^{-2/\lambda} \|u_k\|_{\overset{o}{W}{}_2^2(A_k)} \leq C_1 \sum_{|\alpha|=2} \int_{A_{k\varepsilon}} |D^\alpha(u_{k\varepsilon} - u_k)|^2\,dx + C_2 \int_{A_{k\varepsilon}} u_{k\varepsilon}^2\,dx.$$

Let us assume that u has an approximate truncate in $W_2^2(B(0, \rho))$. Then passing to the limit on the right-hand side of this inequality as $\varepsilon \to 0$ we get

$$k^{-2/\lambda} \|u_k\|_{W_2^2(A_k)} \leq C \|u_k\|_{L_2(A_k)}.$$

Explicit calculations show that for sufficiently large k, there exists a positive number μ such that

$$k^\mu \leq C_0,$$

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where C_0 is a constant independent of k. Since meas $A_k \neq 0$ for all $k > 0$, we can pass to the limit in this inequality as $k \to \infty$. This leads to a contradiction since the left-hand side becomes unbounded while the right-hand side is bounded. Therefore u does not have an approximate truncate in $W_2^2(B(0, R))$ for all $0 < R < \infty$.

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References

[1] E. De Giorgi, 'Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari', *Mem. Accad. Sci. Torino* **3** (1957), 25–43.
[2] ———, 'Un esempio di estremali discontinue per un problema variazionale di tipo ellitico', *Boll. Unione Mat. Ital., ser. IV* **1** (1968), 135–137.
[3] J. Frehse, 'On the boundedness of weak solutions of higher-order nonlinear elliptic partial differential equations', *Boll. Unione Mat. Ital., ser. IV* **3** (1970), 607–627.
[4] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, 105 (Princeton University Press, Princeton, NJ, 1983).
[5] ———, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lecture in Mathematics, ETH Zürich (Birkhäuser, Basel, 1994).
[6] P. W. Jones, 'Quasiconformal mappings and extendability of functions in Sobolev spaces', *Acta Math.* **147** (1981), 71–88.
[7] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations* (Academic Press, New York, 1968).
[8] V. G. Maz'ya and S. V. Poborshii, *Differentiable Functions on Bad Domains* (World Scientific, Singapore, 1997).
[9] J. Moser, 'A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations', *Comm. Pure Appl. Math.* **13** (1960), 457–468.
[10] J. Nash, 'Continuity of solutions of parabolic and elliptic equations', *Amer. J. Math.* **80** (1958), 931–954.
[11] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry – Methods and Applications*, Vol. 2 (Springer, New York, 1990).
[12] J. Serrin, 'Local behavior of quasi-linear elliptic equations', *Acta Math.* **111** (1964), 247–302.
[13] I. V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Translations of Mathematical Monographs, 139 (American Mathematical Society, Providence, RI, 1994), Translated from the 1990 Russian original by D. D. Pascali.
[14] V. A. Solonnikov, 'The differential properties of weak solutions of quasilinear elliptic equations', *Zap. Nauchn. Sem. LOMI* **39** (1974), 110–119.
[15] G. Stampacchia, 'Le problème de Dirichlet pour les équations élliptiques du second ordre à coéfficients discontinus', *Ann. Inst. Fourier* **14**(1) (1965), 189–258.
[16] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, NJ, 1970).
[17] N. S. Trudinger, 'On Harnack type inequalities and their application to quasilinear elliptic equations', *Comm. Pure Appl. Math.* **20** (1967), 721–747.

[18] K. O. Widman, 'Hölder continuity of solutions of elliptic systems', *Manuscripta Math.* **5**(4) (1971), 299–308.
[19] ———, 'Local bounds for solutions of higher-order nonlinear partial differential equations', *Math. Z.* **121**(1) (1971), 81–95.

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