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A PRIORI ESTIMATES FOR SOME CLASSES OF DIFFERENCE SCHEMES

NIKOLAI BAKAEV

ABSTRACT. A new approach to the analysis of the well-posedness of difference parabolic problems is proposed, which is based on weaker assumptions than in earlier works. The results are applied to the study of multi-dimensional difference parabolic problems in mesh Lebesgue spaces.

1. **Introduction.** Lately, work devoted to the well-posedness of difference initial-boundary value problems (*e.g.*, [1, 6, 8, 9, 14, 15]), have treated the case where a space connected with the problem need not be Hilbert. In particular, this allows one to analyse the well-posedness of difference problems in the scale of the spaces L_{ph}, $1 \leq p \leq \infty$ (the mesh analogues of the Lebesgue spaces). However, the similar results are based on some assumptions that may be verified only for narrow classes of difference parabolic problems in the spaces L_{ph}, $1 \leq p \leq \infty, p \neq 2$ [2]. Consequently, up to now, we do not have any essential results for wide classes of difference schemes in the spaces L_{ph}, $1 \leq p \leq \infty, p \neq 2$ when nonuniform spatial meshes, input operators with mixed derivatives or discontinuous coefficients in their principal parts, and curvilinear domains are used. On the other hand, the Hilbert case $p = 2$ is well studied by means of the theory of self-adjoint operators [17].

In order to investigate the problem of well-posedness in wider classes of difference parabolic problems, we apply a new approach in the present work. Our main requirements are weaker than in the previous works, although the *a priori* estimates established below are slightly weaker than, for example, in the theory of [1, 8, 9, 13–15]. At the same time, these hypotheses turn out to be easily verified in the applications concerned with multi-dimensional finite-difference operators on nonuniform spatial meshes.

Recently, in [3], the author has considered similar subjects, including applications to the well-posedness analysis of multi-dimensional difference parabolic problems in the scales of the mesh Lebesgue spaces L_{ph}, $1 \leq p \leq \infty$. The tools used in [3] are based on slightly different assumptions and lead to the slightly different *a priori* estimates than in the present work. Namely, the estimates from [3] contain the singular multiplier $\ln^3(2+H^{-1})$, where H is the minimal stepsize of a spatial mesh, and the estimates established below contain the singular multiplier $\ln^{r+1}(2+t_k)^{-1}$, where t_k is a corresponding discrete time moment, and r is some natural number. Note that the multiplier $\ln^3(2+H^{-1})$

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depends on the characteristics of a spatial mesh, and the multiplier $\ln^{r+1}(2+t_k)^{-1}$ depends on the characteristics of a temporal mesh. Moreover, it could be shown that the estimates with $\ln^{r+1}(2+t_k)^{-1}$ are stronger than, for example, those with $\ln^{r+1}(2+\tau)^{-1}$, where τ is a temporal mesh stepsize. At the same time, the estimates from [3] are established under more general assumptions than below.

2. **Abstract formulations.** Unless otherwise noted, we use the symbols C or $C_1, C_2, \ldots$ for different constants in our formulas.

Let a family of Banach spaces E_h, depending on a parameter $h \in \mathcal{H}$, be given. In the family of spaces E_h, $h \in \mathcal{H}$, the following initial value problem with a parameter h is considered:

$$\frac{dy}{dt} + A_h y = f_h(t), \quad 0 < t < \infty; \quad y(t=0) = y_{0h}, \tag{1}$$

where $y = y(t)$ is a (E_h)-valued function interpreted as a solution of (1), A_h is a certain linear bounded operator acting on E_h, $f_h(t)$ is a given (E_h)-valued function determining a forcing term in the differential equation, $y_{0h} \in E_h$ is an initial value.

We introduce on the interval $[0,\infty)$ the mesh $\bar{\omega}_\tau$ with a stepsize $\tau > 0$

$$\bar{\omega}_\tau = \{t_k = k\tau, k = 0, 1, \ldots\}.$$

Let us consider in the family of spaces E_h, $h \in \mathcal{H}$ the following difference scheme corresponding to the problem (1)

$$\begin{aligned} y(t_{k+1}) &= y(t_k) - \tau \sum_{j=1}^{\nu_1} b_j A_h Y_j^{(k)} + \tau \sum_{j=1}^{\nu_2} \bar{b}_j f_h(b_k + \bar{c}_j\tau), \quad k = 0, 1, \ldots \\ y(t_0) &= y_{0h}, \end{aligned} \tag{2}$$

where $Y_j^{(k)} \in E_h$ are determined from the system of equations

$$Y_j^{(k)} = \beta_j y(t_k) - \tau \sum_{l=1}^{\nu_1} a_{jl} A_h Y_l^{(k)} + \tau \sum_{l=1}^{\nu_2} \bar{a}_{jl} f_h(t_k + \bar{c}_l\tau), \quad j = 1, 2, \ldots, \nu_1. \tag{3}$$

Here $y(t_k)$ is a (E_h)-valued function of a discrete argument t_k being a solution of the problem (2), (3); ν_1, ν_2 are some natural numbers and b_j, $\bar{b}_r$, β_j, a_{jl}, $\bar{a}_{jr}$, $\bar{c}_r$; $j, l = 1, 2, \ldots, \nu_1$, $r = 1, 2, \ldots, \nu_2$ are some complex parameters determining a concrete form of the scheme (2), (3).

If the operators $g_{jl}(\tau A_h)$, $j, l = 1, 2, \ldots, \nu_2$ are well defined, where the matrix $\big((g_{jl}(z))\big)$ is inverse for $\big((\delta_{jl} + z a_{jl})\big)$ and δ_{jl} is the Kronecker delta, one can exclude the elements $Y_j^{(k)}$ from (2), (3), and the scheme (2), (3) may be represented in the following canonical form

$$\begin{aligned} y(t_{k+1}) &= [I - \tau \mathfrak{U}_{\tau h}] y(t_k) + \tau F_{\tau h}(t_k), \quad k = 0, 1, \ldots, \\ y(t_0) &= y_{0h}, \end{aligned} \tag{4}$$

where the linear operator $\mathfrak{U}_{\tau h}$ and (E_h)-valued function $F_{\tau h}(t_k)$ are defined by

$$\mathfrak{U}_{\tau h} = A_h \sum_{j=1}^{\nu_1} \sum_{l=1}^{\nu_1} b_j \beta_l g_{jl}(\tau A_h),$$

$$F_{\tau h}(t_k) = \sum_{r=1}^{\nu_2} \Big\{ \bar{b}_r I - \sum_{j=1}^{\nu_1} \sum_{l=1}^{\nu_1} \tau A_h b_j \bar{a}_{lr} g_{jl}(\tau A_h) \Big\} f_h(t_k + \bar{c}_r \tau).$$

Note that the operators $\mathfrak{U}_{\tau h}$ and

$$\omega_r(\tau A_h) = \sum_{j=1}^{\nu_1} \sum_{l=1}^{\nu_1} \tau A_h b_j \bar{a}_{lr} g_{jl}(\tau A_h), \quad r = 1, 2, \ldots, \nu_2$$

may be well defined even this is not valid for $g_{jl}(\tau A_h), j, l = 1, 2, \ldots, \nu_2$. Therefore we can consider the problem (4) as a generalized difference scheme for the problem (1).

DEFINITION 1. The operator $\mathfrak{U}_{\tau h}$ in (4) is called a *generator of scheme*.

The generator of scheme $\mathfrak{U}_{\tau h}$ may be represented as follows

$$\mathfrak{U}_{\tau h} = \tau^{-1} \alpha(\tau A_h),$$

where $\alpha(z)$ is the rational function given by

$$\alpha(z) = z \sum_{j=1}^{\nu_1} \sum_{l=1}^{\nu_1} b_j \beta_l g_{jl}(z).$$

DEFINITION 2. The function $\alpha(z)$ is called a *scheme generator symbol*. The functions $\omega_r(z), r = 1, 2, \ldots \nu$ given by

$$\omega_r(z) = \sum_{j=1}^{\nu_1} \sum_{l=1}^{\nu_1} z b_j \bar{a}_{lr} g_{jl}(z)$$

are called *correcting symbols*.

Schemes like (2) and (3) were introduced and studied in [1]. In the present work we deal with the generalized scheme (4). Our main aim is to analyse the well-posedness of (4).

DEFINITION 3. A bounded linear operator $B_h\colon E_h \longrightarrow E_h$ is said to be *uniformly* (with respect to $h \in \mathcal{H}$) *almost sectorial of power r* (r is some natural number) on E_h if there exist constants $\varepsilon_0 > 0, \varkappa > 0$ such that the set

$$\{z\,; |\arg z| \geq \pi/2 - \varkappa\varepsilon_0, |z| > 0\}$$

belongs to the resolvent set of B for all $h \in \mathcal{H}$ and the inequality

$$\|[R(\lambda, A_h)]^r\|_{E_h} \leq C\varepsilon^{-r}|\lambda|^{\varepsilon - r}$$

holds for all $h \in \mathcal{H}$, $\varepsilon \in (0, \varepsilon_0]$, and λ such that $|\arg \lambda| \geq \pi/2 - \varkappa\varepsilon$.

Note that in [1] and in the other works on the approximations of solution of evolution equations (see, *e.g.*, [8, 9, 13–15]) an input operator is assumed to be (uniformly) sectorial. This assumption is stronger than that of (uniform) almost sectorialness. It is known for some problems that it is difficult to prove that an input operator is (uniformly) sectorial, but there is a possibility of establishing that it is (uniformly) almost sectorial. A similar situation is considered in the final section of this work.

Further, the well-posedness of the scheme (4) will be established under the assumptions that the operator A_h is uniformly almost sectorial of power r on E_h and a discretization method leading to (4) is A-stable.

3. **Auxiliary results.** Here we study the behaviour of the resolvent of the operator $\alpha(\tau A_h)$, where $\alpha(z)$ is the scheme generator symbol. First note that, for the operator A_h to be uniformly almost sectorial of power r on E,

$$\|[R(\lambda,\tau A_h)]^r\|_{E_h} \le C\varepsilon^{-r}\tau^{-\varepsilon}|\lambda|^{\varepsilon-r} \tag{5}$$

holds for all $h \in \mathcal{H}$, $\varepsilon \in (0,\varepsilon_0]$, $\tau > 0$, and λ such that $|\arg\lambda| \ge \pi/2 - \varkappa\varepsilon$.

LEMMA 1. *Let the operator A_h be uniformly almost sectorial of power r on E_h. Moreover, assume the scheme (4) is generated by an A-stable discretization method, and the following conditions are satisfied:*

(i) $\sum_{j=1}^{\nu_1} b_j\beta_j = 1$;

(ii) $|1-\lambda_0| < 1$.

where λ_0 is defined by

$$\lambda_0 = \lim_{|z|\to\infty} \alpha(z) \tag{6}$$

and $\alpha(z)$ is the scheme generator symbol.

Then the estimate

$$\left\|R\big(\lambda,\alpha(\tau A_h)\big)\right\|_{E_h} \le C\varepsilon^{-(r+1)}\tau^{-2\varepsilon}|\lambda|^{2\varepsilon-1} \tag{7}$$

holds for all $h \in \mathcal{H}$, $\varepsilon \in (0,\varepsilon_0/2]$, $\tau > 0$, $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$, where

$$\Lambda_\varepsilon = M_\varepsilon \bigcup \{z; |1-z| \le 1 - d_0\varepsilon\}, \quad d_0 = \text{const},\ 1 - 3\,d_0\varepsilon_0 \ge |1-\lambda_0|, \tag{8}$$

and the set M_ε is the image of the sector $\{z\,; |\arg z| \le \pi/2 - \varkappa\varepsilon\}$ under the map $\alpha(z)$.

PROOF. Since a discretization method leading to the scheme (4) is A-stable, the conditions

$$|1-\alpha(z_0)| = 1 \quad \text{and} \quad \alpha'(z_0) = 0$$

are not satisfied simultaneously for any point z_0 with $\operatorname{Re} z_0 = 0$. It also follows from the A-stability that for any point z_0 such that $\operatorname{Re} z_0 = 0$ and $\alpha(z_0) = 0$ there holds

$$\operatorname{Im}\alpha'(z_0) = 0, \quad \operatorname{Re}\alpha'(z_0) > 0.$$

Furthermore, note that the relations

$$\frac{\partial}{\partial|z|}\alpha(z) = \frac{z\alpha'(z)}{|z|}, \quad \frac{\partial}{\partial \arg z}\alpha(z) = iz\alpha'(z) \tag{9}$$

are valid and complex vectors that correspond to the values

$$\frac{\partial}{\partial|z|}\alpha(z) \quad \text{and} \quad \frac{\partial}{\partial \arg z}\alpha(z)$$

are orthogonal to each other. By the above, we get

$$C_1 \geq |z\alpha'(z)| \geq C_2|\alpha(z)|, \quad C_1, C_2 > 0 \tag{10}$$

for all z: $\operatorname{Re} z = 0$, $d_2 \leq |z| \leq d_1$, $d_1, d_2 = \text{const} > 0$, where d_1 is chosen so that

$$|1 - \alpha(z)| \leq 1 - 2\, d_0\varepsilon_0$$

(d_0 is the constant in (8)) for all z such that $|z| \geq d_1$, $\operatorname{Re} z \geq 0$, and d_2 is chosen so that the equation

$$\alpha(z) = 0 \tag{11}$$

has the only root $z = 0$ for $|z| \leq d_2$. Moreover, using

$$|\alpha'(z)| \geq C > 0$$

for $|z| \leq d_2$, we have

$$C_1|\alpha(z)| \geq |z\alpha'(z)| \geq C_2|\alpha(z)|, \quad C_1, C_2 > 0 \tag{12}$$

for all z: $|z| \leq d_2$. It follows from (9), (10), (12) that

$$|\lambda - \mu| \geq C\varepsilon(|\lambda| + |\mu|), \quad C > 0 \tag{13}$$

for all $\varepsilon \in (0, \varepsilon_0/2]$, $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$, $\mu \in M_{2\varepsilon}$.

In order to study the behaviour of $R\big(\lambda, \alpha(\tau A_h)\big)$ for λ: $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$, we use the representation

$$\begin{aligned} &R\big(\lambda,\alpha(\tau A_h)\big) \\ &\quad = (r-1)(2\pi i)^{-1}\int_\Gamma dz \int_0^z d\xi (z-\xi)^{r-2}\{[\lambda - \alpha(\xi)]^{-1} - (\lambda - \lambda_0)^{-1}\}[R(z,\tau A_h)]^r \\ &\qquad + (\lambda - \lambda_0)^{-1} I, \quad \lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon, \end{aligned} \tag{14}$$

where the contour Γ is defined as

$$\Gamma = \{z\,; \arg z = \pm(\pi/2 - 2\varkappa\varepsilon)\},$$

and the integration path from 0 to z belongs to the contour Γ. The formula (14) is based on the evident relation

$$\frac{d^{r-1}}{dz^{r-1}}\int_0^z d\xi(z-\xi)^{r-2}\{[\lambda-\alpha(\xi)]^{-1} - (\lambda-\lambda_0)^{-1}\}/(r-2)! = [\lambda - \alpha(z)]^{-1} - (\lambda-\lambda_0)^{-1}$$

and follows from the well known generalization of the Cauchy formula

$$f^{(n)}(z) = n!\,(2\pi i)^{-1} \oint f(\zeta)(\zeta - z)^{-(n+1)}\, d\zeta \tag{15}$$

(see, *e.g.* [16, p. 172]). Since the operator $[\tau A_h]$ is bounded for all fixed $\tau > 0$ and $h \in \mathcal{H}$, formula (15) may be easily generalized to the operator case. Therefore, (14) holds if we replace the contour Γ by a closed contour surrounding the spectrum of the operator $[\tau A_h]$. As will be shown below, the norm of the integrand function in (14) tends to zero quickly enough as $|z| \longrightarrow \infty$ so that the closed contour may be transformed into the contour Γ.

By (14), we derive

$$\begin{aligned} &\|R(\lambda, \alpha(\tau A_h))\|_{E_h} \\ &\quad \le C\varepsilon^{-r}\tau^{-2\varepsilon} \int_\Gamma \left| \int_0^z (z-\xi)^{r-2}\{[\lambda - \alpha(\xi)]^{-1} - (\lambda-\lambda_0)^{-1}\}\, d\xi \right| |z|^{2\varepsilon - r} |dz| \end{aligned} \tag{16}$$

for all λ such that $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$. In addition, by (13), we have

$$\begin{aligned} |[\lambda - \alpha(z)]^{-1} - (\lambda - \lambda_0)^{-1}| &= |[\alpha(z) - \lambda_0](\lambda-\lambda_0)^{-1}[\lambda - \alpha(z)]^{-1}| \\ &\le C_1|\alpha(z) - \lambda_0[|\lambda| + |\alpha(z)|]^{-1} \\ &\le C_2(1+|z|)^{-1}[|\lambda| + |\alpha(z)|]^{-1} \\ &\le C_3[|\lambda| + |z|]^{-1} \end{aligned} \tag{17}$$

for all λ such that $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$, z: $\arg z = \pm(\pi/2 - 2\varkappa\varepsilon)$. Finally, using (16), (17), we obtain

$$\begin{aligned} \|R(\lambda, \alpha(\tau A_h))\|_{E_h} &\le C_1\varepsilon^{-r}\tau^{-2\varepsilon} \int_\Gamma |dz|\, |z|^{2\varepsilon-2} \int_0^{|z|} (|\lambda| + x)^{-1}\, dx \\ &= C_2\varepsilon^{-r}\tau^{-2\varepsilon} \int_0^\infty y^{2\varepsilon-2} \ln(1 + y|\lambda|^{-1})\, dy \\ &= C_2\varepsilon^{-r}\tau^{-2\varepsilon}|\lambda|^{2\varepsilon-1} \int_0^\infty y^{2\varepsilon-2}\ln(1+y)\, dy \\ &\le C_3\varepsilon^{-(r+1)}\tau^{-2\varepsilon}|\lambda|^{2\varepsilon-1} \end{aligned}$$

for all λ: $\lambda \bar{\in} \operatorname{Int}\Lambda_\varepsilon$, $\varepsilon \in (0, \varepsilon_0/2]$, $\tau > 0$, $h \in \mathcal{H}$. This completes the proof. ∎

4. **Well-posedness of difference schemes.** Here the conditions of the well-posedness of the difference scheme (4) are established.

THEOREM 1. *Let the conditions of Lemma 1 be satisfied and assume*

$$\deg[\omega_r(z)] \le 0, \quad r = 1, 2, \ldots, \nu_2$$

where $\omega_r(z)$, $r = 1, 2, \ldots, \nu_2$ *are the correcting symbols. Then, for any solution of (4), the a priori estimate*

$$\begin{aligned} &\|\amalg_{\tau h}^{\xi} y(t_k)\|_{E_h} \\ &\quad \le C \ln^{r+2}\big(2 + [(k+1)\tau]^{-1}\big) \\ &\quad\quad \times \Big\{[(k+1)\tau]^{-\xi}\|y_{0h}\|_{E_h} + \tau\sum_{l=1}^{k}[(k-l+1)\tau]^{-\xi} \max_{r=1,2,\ldots,\nu_2} \|f_h(t_{l-1} + \bar{c}_r\tau)\|_{E_h}\Big\}, \\ &k = 0, 1, \ldots \end{aligned} \tag{18}$$

holds for all $\tau > 0$, $h \in \mathcal{H}$, $\xi \in [0,1]$, *where* $\mathfrak{U}_{\tau h} = \tau^{-1}\alpha(\tau A_h)$ *is the generator of scheme,* $\alpha(z)$ *is the scheme generator symbol.*

PROOF. Let us use the representation

$$[I-\alpha(\tau A_h)]^k = (2\pi i)^{-1}\int_{\Gamma_\varepsilon}(1-\lambda)^k R\big(\lambda,\alpha(\tau A_h)\big)\,d\lambda, \quad k=1,2,\ldots \tag{19}$$

where the contour Γ_ε coincides with the boundary of the set Λ_ε (Λ_ε is defined as in the formulation of Lemma 1). It follows from (9), (10), and (12) that

$$|1-\lambda| \le 1 - a_0\varepsilon|\lambda|, \quad a_0 = \text{const} > 0 \tag{20}$$

for all $\varepsilon \in (0,\varepsilon_0/2]$, $\lambda \in \Gamma_\varepsilon$. Taking into account Lemma 1, we obtain from (19) and (20)

$$\begin{aligned}\|[I-\alpha(\tau A_h)]^k\|_{E_h} &\le C_1\varepsilon^{-(r+1)}\tau^{-2\varepsilon}\int_{\Gamma_\varepsilon}\exp(-a_0k\varepsilon|\lambda|)|\lambda|^{2\varepsilon-1}|d\lambda| \\ &\le C_2\varepsilon^{-(r+1)}\tau^{-2\varepsilon}\Big[\int_0^\infty \exp(-a_1k\varepsilon x)x^{2\varepsilon-1}\,dx + \exp(-a_2k\varepsilon)\Big] \\ &\le C_3\varepsilon^{-(r+1+2\varepsilon)}(k\tau)^{-2\varepsilon}, \quad a_1,a_2 = \text{const} > 0,\ k=1,2,\ldots\end{aligned} \tag{21}$$

for all $\tau > 0$, $h \in \mathcal{H}$, $\varepsilon \in (0,\varepsilon_0/2]$. In the same way, we get

$$\|\tau^{-1}\alpha(\tau A_h)[I-\alpha(\tau A_h)]^k\|_{E_h} \le C\varepsilon^{-(r+1+2\varepsilon)}(k\tau)^{-1-2\varepsilon}, \quad k=1,2,\ldots \tag{22}$$

for all $\tau > 0$, $h \in \mathcal{H}$, $\varepsilon \in (0,\varepsilon_0/2]$. Using the moments inequality (5) and taking $\varepsilon = \big[\ln\big(2+(k\tau)^{-1}\big)\big]^{-1}$, we derive from (21), (22) that

$$\|\mathfrak{U}_{\tau h}^{\xi}[I-\alpha(\tau A_h)]^k\|_{E_h} \le C\ln^{r+1}\big(2+(k\tau)^{-1}\big)(k\tau)^{-\xi}, \quad k=1,2,\ldots \tag{23}$$

for all $\tau > 0$, $h \in \mathcal{H}$, $\xi \in [0,1]$. Moreover, taking $k=1$, $\xi=0$ in (23), we get

$$\|\tau^{-1}\alpha(\tau A_h)\|_{E_h} = \|\mathfrak{U}_{\tau h}\|_{E_h} \le C\ln^{r+1}(2+\tau^{-1})\tau^{-1}$$

for all $\tau > 0$, $h \in \mathcal{H}$. This yields the fact that one can substitute $(k+1)$ for k in the right-hand side of (23). Obviously, the estimate (23) will be valid after such substitution for $k = 0$ too. To conclude the proof, one applies the results of [4]. ∎

5. **Concrete families of difference schemes.** Let us consider now the concrete families of difference schemes that may be investigated on the basis of the results obtained above.

Taking $\nu_1 = \nu_2 = \nu$, $b_j = \bar{b}_j$, $a_{jl} = \bar{a}_{jl}$, $\beta_j = 1; j,l = 1,2,\ldots,\nu$ in (2), (3), we obtain a discretization method belonging to the class of ν-stage Runge-Kutta methods [10]. We shall study some methods from this class.

First of all we describe the so called simplifying conditions $B(m)$, $C(m)$, $D(m)$ introduced into the theory of Runge-Kutta methods by Butcher [7]. Consider the condition

$$B(m)\colon \sum_{j=1}^{\nu} b_j\bar{c}_j^{k-1} = k^{-1}, \quad 1 \le k \le m$$

the condition

$$C(m)\colon \sum_{l=1}^{\nu} a_{jl}\bar{c}_l^{k-1} = k^{-1}\bar{c}_j^k, \quad 1 \le k \le m,\ 1 \le j \le \nu,$$

and the condition

$$D(m)\colon \sum_{j=1}^{\nu} b_j\bar{c}_j^{k-1}a_{jl} = k^{-1}b_l(1-\bar{c}_l^k), \quad 1 \le l \le \nu,\ 1 \le k \le m.$$

Let us now consider the ν-stage Radau IA and Radau IIA methods [10]. Both families of methods lead to schemes of the order of accuracy $(2\nu - 1)$. For the Radau IA methods, the abscissae $\bar{c}_j, j = 1,2,\ldots,\nu$ are determined from the equation

$$P_{\nu-1}(2\bar{c}-1) + P_\nu(2\bar{c}-1) = 0$$

and b_j and $a_{jl}, j,l = 1,2,\ldots,\nu$ are determined from the simplifying conditions $B(\nu)$, $D(\nu)$. For the Radau IIA methods, the abscissae $\bar{c}_j, j = 1,2,\ldots,\nu$ are the roots of the equation

$$P_{\nu-1}(1-2\bar{c}) + P_\nu(1-2\bar{c}) = 0,$$

and the other coefficients are founded from the simplifying conditions $B(\nu)$, $C(\nu)$. It follows from [10] that the methods of both families are A-stable and the eigenvalues of the generating matrices $\big((a_{jl})\big)$ belong to the sector $\{z\,;|\arg z| < \pi/2\}$. Since the matrices $\big((a_{jl})\big)$ are non-degenerate [10], the conditions

$$\deg[\omega_j(z)] \le -1, \quad j = 1,2,\ldots,\nu \tag{24}$$

hold, where $\omega_j(z)$ are correcting symbols. Furthermore, the condition

$$\sum_{j=1}^{\nu} b_j = 1 \tag{25}$$

reflects the fact that the corresponding ν-point quadrature formula is exact for constant functions. By the representations for $[1-\alpha(z)]$ given in [10], we easily get that $\lambda_0 = 1$. Thus the schemes (4) based on the Radau IA and the Radau IIA methods satisfy the conditions of Theorem 1.

Let us also carry out an analysis of the ν-stage Lobatto IIIC methods [10]. These methods generate the difference schemes of the order of accuracy $(2\nu - 2)$. Their abscissae $\bar{c}_j, j = 2,3,\ldots,\nu-1$ coincide with the roots of the equation

$$\frac{d}{d\bar{c}}P_{\nu-1}(2\bar{c}-1) = 0,$$

and $\bar{c}_1 = 0$, $\bar{c}_\nu = 1$. The other coefficients are determined from the conditions $B(\nu)$, $C(\nu-1)$ and the additional conditions $a_{j1} = b_1, j = 1,2,\ldots,\nu$. It follows from [10] that the methods of this family are A-stable and (25) is valid for all of them. Moreover, the eigenvalues of the matrices $\big((a_{jl})\big)$ belong to the sector $\{z\,;|\arg z| < \pi/2\}$, (24) holds, and $\lambda_0 = 1$. Hence the schemes based on the Labatto IIIC methods satisfy the conditions of Theorem 1.

Vinokurov has suggested a family of methods not contained in the class of Runge-Kutta methods. At the same time, the schemes generated by these methods are also described by (2), (3). It is significant that the relations $\nu_1 > \nu_2$, $a_{jj} = \bar{a}_{ll}$, $j = l \bmod \nu_2$ hold for Vinokurov's methods. Such methods have an increased economy in calculations. Simple analysis shows that the two methods of this family of higher order described in [20] are A-stable. Moreover, for both methods, $\lambda_0 = 1$, the eigenvalues of the generating matrices $\big((a_{jl})\big)$ belong to the sector $\{z; |\arg z| \leq \pi/2\}$, the conditions

$$\deg[\omega_j(z)] \leq 0, \quad j = 1, 2, \ldots, \nu_2$$

hold, and

$$\sum_{j=1}^{\nu_1} b_j \beta_j = 1.$$

Thus the methods from [20] satisfy the conditions of Theorem 1.

6. **Applications. Analysis of difference schemes approximating initial-boundary value problems for the heat conduction equation.** In the present section, the abstract results established above are applied to the analysis of the difference schemes approximating initial-boundary value problems for the multi-dimensional heat conduction equation. Further, the operator A_h considered above as abstract is the difference operator approximating an elliptic differential operator (with the boundary conditions of the first kind).

Let $\mathbb{D}_n = [0,1] \times \cdots \times [0,1]$ be the n-dimensional unit hypercube, and suppose $x = (x_1, x_2, \ldots, x_n)$ is a sequence of n one-dimensional coordinates. In the segment $\mathbb{D}_1$ we introduce the non-uniform meshes $\hat{\bar{\omega}}_{hl} = \{x_l^{(k)}, k = 0, 1, \ldots, \mathcal{N}_l\}$, $l = 1, 2, \ldots, n$, where $\mathcal{N}_l$, $l = 1, 2, \ldots, n$ are some natural numbers such that $x_l^{(0)} = 0, x_l^{(\mathcal{N}_l)} = 1, l = 1, 2, \ldots, n$. Let the stepsizes $h_l^{(k)}$ of the mesh $\hat{\bar{\omega}}_{hl}$ be

$$h_l^{(k)} = x_l^{(k)} - x_l^{(k-1)}, \quad l = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \mathcal{N}_l.$$

Further, we shall use the notation

$$\bar{h}_l^{(k)} = (h_l^{(k)} + h_l^{(k+1)})/2, \quad l = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \mathcal{N}_l - 1,$$

$$H = \min_{\substack{1 \leq k_1 \leq \mathcal{N}_1 - 1, \\ \vdots \\ 1 \leq k_n \leq \mathcal{N}_n - 1}} (\bar{h}_1^{(k_1)} \cdots \bar{h}_n^{(k_n)}).$$

Denote by $\hat{\omega}_{hl} \setminus (l = 1, 2, \ldots, n)$ the meshes obtained from $\hat{\bar{\omega}}_{hl}$, $l = 1, 2, \ldots, n$ by excluding the boundary nodes $x_l^{(0)}$ and $x_l^{(N_l)}$. In the hypercube $\mathbb{D}_n$ let us introduce the multi-dimensional rectangular mesh

$$\hat{\bar{\omega}}_h = \hat{\bar{\omega}}_{h1} \times \cdots \times \hat{\bar{\omega}}_{hn}$$

and also the mesh of inner nodes

$$\hat{\omega}_h = \hat{\omega}_{h1} \times \cdots \times \hat{\omega}_{hn}.$$

Let γ_h be the set of boundary nodes so that $\gamma_h = \hat{\bar{\omega}}_h \backslash \hat{\omega}_h$. Let $\overset{\circ}{S}_h$ denote the linear space of complex-valued functions $y(x)$ defined on the mesh $\hat{\bar{\omega}}_h$ and such that $y(x) = 0$ for $x \in \gamma_h$. Also, let $a_l(x_l)$, $l = 1, 2, \ldots, n$, $x \in \mathbb{D}_1$ be coefficients such that

$$a_l(x_l) \geq a_0 = \text{const} > 0, \quad l = 1, 2, \ldots, n,\ x_l \in \mathbb{D}_1. \tag{26}$$

Suppose A_h is the multi-dimensional difference operator determined by

$$[A_h y](x) = \begin{cases} -\sum_{l=1}^{n} (\tilde{a}_l y_{\bar{x}_l})_{\hat{x}_l}, & x \in \hat{\omega}_h, \\ 0, & x \in \gamma_h \end{cases} \tag{27}$$

for all $y(x) \in \overset{\circ}{S}_h$ (here and further on the standard notation for difference derivatives is used [17]), where $\tilde{a}_l = \tilde{a}_l(x_l)$, $l = 1, 2, \ldots, n$, $x_l \in \hat{\omega}_{hl}$ are the mesh functions defined as

$$\tilde{a}_l(x_l^{(k_l)}) = a_l(x_l^{(k_l - 1)} + h_l^{(k_l)}/2), \quad l = 1, 2, \ldots, n, \quad k_l = 1, 2, \ldots, \mathcal{N}_l - 1.$$

The operator A_h approximates on the mesh $\hat{\bar{\omega}}_h$ with first-order accuracy the differential operator

$$A = -\sum_{l=1}^{n} \frac{\partial}{\partial x_l} a_l(x_l) \frac{\partial}{\partial x_l}$$

with boundary conditions of the first kind at the boundary of $\mathbb{D}_n$; and the difference problem (4), (27) approximates on the mesh $\bar{\omega}_\tau \times \hat{\bar{\omega}}_h$ the first initial-boundary value problem for the multi-dimensional heat conduction equation.

In addition, we introduce the norms

$$\|y\|_{L_{ph}} = \begin{cases} [\sum_{k_1=1}^{\mathcal{N}_1-1} \cdots \sum_{k_n=1}^{\mathcal{N}_n-1} |y(x_1^{(k_1)}, \ldots, x_n^{(k_n)})|^p \bar{h}_1^{(k_1)} \cdots \bar{h}_n^{(k_n)}]^{1/p}, & 1 \leq p < \infty, \\ \max_{x \in \hat{\omega}_h} |y(x)|, & p = \infty \end{cases}$$

for all $y(x) \in \overset{\circ}{S}_h$ so that the space $\overset{\circ}{S}_h$ with the norm $\|\cdot\|_{L_{ph}}$, $1 \leq p \leq \infty$, is the mesh analogue of the Lebesgue space L_p, $1 \leq p \leq \infty$.

LEMMA 2. *The operator A_h given by (26) satisfies*

$$\|R(-\lambda, A_h)\|_{L_{ph}} \leq \lambda^{-1}, \quad 1 \leq p \leq \infty \tag{28}$$

for all $\hat{\bar{\omega}}_h$, $\lambda \geq 0$.

PROOF. For $p = \infty$, (28) follows from the maximum principle [17]. Using the tools of [5], we can establish that the operator A_h is self-adjoint. By this fact, (28) is valid for $p = 1$ too. To conclude the proof, it remains to apply the Riesz interpolation theorem [19]. ∎

LEMMA 3. *For the operator A_h given by (27) the inequality*

$$\|R(\lambda, A_h)|_{L_{ph}} \leq C\varepsilon^{-1}|\lambda|^{-1}, \quad p = 1, \infty \tag{29}$$

holds for all $\hat{\omega}_h$, $\varepsilon \in (0, \pi/2)$, and λ such that $|\arg \lambda| \geq \pi/2 + \varepsilon$.

PROOF. By Lemma 2, it is sufficient to use the method of an analytical continuation of the resolvent into the corresponding sector [12]. ∎

Let us introduce in the space $\overset{\circ}{S}_h$ the auxiliary norms $\|\cdot\|_{h(j)}$, $0 \leq j \leq n$, defined by

$$\|y(\cdot)\|_{h(j)} = \max_{1 \leq k_{j+1} \leq \mathcal{N}_{j+1}-1} \Big[\sum_{k_1=1}^{\mathcal{N}_1-1} \cdots \sum_{k_j=1}^{\mathcal{N}_j-1} |y(x_1^{(k_1)}, \ldots, x_n^{(k_n)})|^2 \bar{h}_1^{(k_1)} \cdots \bar{h}_n^{(k_j)} \Big]^{1/2}, \quad \text{for } 1 \leq j \leq n-1$$

$$\|y(\cdot)\|_{h(j)} = \|y(\cdot)\|_{L_{\infty h}} \quad \text{for } j = 0,$$

and

$$\|y(\cdot)\|_{h(j)} = \|y(\cdot)\|_{L_{2h}} \quad \text{for } j = n.$$

LEMMA 4. *The imbedding inequalities*

$$\|y(\cdot)\|_{h(j-1)} \leq C\|[A_{hj}^{1/2} y](\cdot)\|_{h(j)}, \quad 1 \leq j \leq n \tag{30}$$

hold for all $\hat{\omega}_h$, $y(x) \in \overset{\circ}{S}_h$, where A_{hj} are the mesh operators given by

$$[A_{hj} y](x) = \begin{cases} -(\tilde{a}_j y_{\bar{x}_j})_{\hat{x}_j}, & x \in \hat{\omega}_h, \\ 0, & x \in \gamma_h, \; j = 1, 2, \ldots, n. \end{cases} \tag{31}$$

PROOF. Let us denote by A_{0hj} the mesh operators given by (31) in the case $a_j(x_j) \equiv 1$. By [18], we have

$$\max_{x_j \in \hat{\omega}_{hj}} |y(x_1, \ldots, x_n)| \leq \frac{1}{2} \Big[\sum_{1 \leq k_j \leq \mathcal{N}_j - 1} |[A_{0hj}^{1/2} y](x_1, \ldots, x_{j-1}, x_j^{(k_j)}, x_{j+1}, \ldots, x_n)|^2 \bar{h}_j^{(k_j)} \Big]^{1/2}, \quad j = 1, 2, \ldots, n \tag{32}$$

for all fixed $x_l \in \hat{\omega}_{hl}$, $l = 1, 2, \ldots, n$, $l \neq j$. Using the tool of difference summation by parts and taking into account (26), we can also derive from (32)

$$\max_{x_j \in \hat{\omega}_{hj}} |y(x_1, \ldots, x_n)| \leq C \Big[\sum_{1 \leq k_j \leq \mathcal{N}_j - 1} |[A_{hj}^{1/2} y](x_1, \ldots, x_{j-1}, x_j^{(k_j)}, x_{j+1}, \ldots, x_n)|^2 \bar{h}_j^{(k_j)} \Big]^{1/2}, \quad j = 1, 2, \ldots, n \tag{33}$$

for all fixed $x_l \in \hat{\omega}_{hl}$, $l = 1,2,\ldots,n$, $l \neq j$. Squaring both sides of (33), multiplying by $\bar{h}_1^{(k_1)}\cdots\bar{h}_{j-1}^{(k_{j-1})}$, and summing over $k_1,\ldots,k_{j-1}$, we have

$$\sum_{1\le k_1\le \mathcal{N}_1-1}\cdots\sum_{1\le k_{j-1}\le \mathcal{N}_{j-1}-1}\bar{h}_1^{(k_1)}\cdots\bar{h}_{j-1}^{(k_{j-1})}\max_{x_j\in\hat{\omega}_{hj}}|y(x_1^{(k_1)},\ldots,x_{j-1}^{(k_{j-1})},x_j,x_{j+1},\ldots,x_n)|^2$$
$$\le C\sum_{1\le k_1\le \mathcal{N}_1-1}\cdots\sum_{1\le k_j\le \mathcal{N}_j-1}|[A_{hj}^{1/2}y](x_1^{(k_1)},\ldots,x_n^{(k_n)})|^2\bar{h}_1^{(k_1)}\cdots\bar{h}_j^{(k_j)} \tag{34}$$
$$j = 1,2,\ldots,n$$

for all fixed k_l: $1 \le k_l \le \mathcal{N}_{l-1}$, $l = j+1,\ldots,n$. Inverting the order of $\sum$ and max on the left-hand side of (34) only decreases the corresponding expression so that

(35)
$$\max_{1\le k_j\le \mathcal{N}_j-1}\Big[\sum_{1\le k_1\le \mathcal{N}_1-1}\cdots\sum_{1\le k_{j-1}\le \mathcal{N}_{j-1}-1}\bar{h}_1^{(k_1)}\cdots\bar{h}_{j-1}^{(k_{j-1})}|y(x_1^{(k_1)},\ldots,x_{j-1}^{(k_{j-1})},x_j,x_{j+1},\ldots,x_n)|^2\Big]^{1/2}$$
$$\le C\Big[\sum_{1\le k_1\le \mathcal{N}_1-1}\cdots\sum_{1\le k_j\le \mathcal{N}_j-1}|[A_{hj}^{1/2}y](x_1^{(k_1)},\ldots,x_n^{(k_n)})|^2\bar{h}_1^{(k_1)}\cdots\bar{h}_j^{(k_j)}\Big]^{1/2}$$
$$j = 1,2,\ldots,n$$

for all fixed k_l: $1 \le k_l \le \mathcal{N}_{l-1}$, $l = j+1,\ldots,n$. In order to obtain (30), it remains to take maximum values over k_l: $1 \le k_l \le \mathcal{N}_l - 1$, $l = j+1,\ldots,n$ on both sides of (35). ∎

LEMMA 5. *For any fixed natural number r such that $r \ge n/2$, the estimate*

$$\|[R(\lambda,A_h)]^r\|_{L_{ph}} \le C|\lambda|^{n/2-r}, \quad p = 1,\infty$$

holds for all $\hat{\omega}_h$ and λ such that $|\arg\lambda| \ge \pi/4$, where A_h is the operator given by (27).

PROOF. By Lemma 4, we have

(36)
$$\|[R(\lambda,A_h)]^r y\|_{L_{\infty h}}$$
$$\le C_1\|A_{h1}^{1/2}[R(\lambda,A_h)]^r y\|_{h(1)}$$
$$\le C_2\|A_{h2}^{1/2}A_{h1}^{1/2}[R(\lambda,A_h)]^r y\|_{h(2)} \le \cdots \le C\|A_{hn}^{1/2}\cdots A_{h1}^{1/2}[R(\lambda,A_h)]^r y\|_{L_{2h}}$$

for all $y(x) \in \overset{\circ}{S}_h$, $\hat{\omega}_h$, λ: $|\arg\lambda| \ge \pi/4$.

Since the operators A_{hj}, $j = 1,2,\ldots,n$ are the self-adjoint positive-definite in the space L_{2h} and commute with each other, we have instead of (36)

$$\|[R(\lambda,A_h)]^r y\|_{L_{\infty h}} \le C_1\|(A_{h1}+A_{h2}+\cdots+A_{hn})^{n/2}[R(\lambda,A_h)]^r y\|_{L_{2h}}$$
$$= C_1\|A_h^{n/2}[R(\lambda,A_h)]^r y\|_{L_{2h}} \le C_2|\lambda|^{n/2-r}\|y\|_{L_{2h}} \le C_3|\lambda|^{n/2-r}\|y\|_{L_{\infty h}}$$

for all $y(x) \in \overset{\circ}{S}_h$, $\hat{\omega}_h$ and λ such that $|\arg\lambda| \ge \pi/4$, that is equivalent to

$$\|[R(\lambda,A_h)]^r\|_{L_{\infty h}} \le C|\lambda|^{n/2-r} \tag{37}$$

for all $\hat{\omega}_h$ and λ such that $|\arg\lambda| \geq \pi/4$.

Going over to adjoint operators on the left-hand side of (37) and taking into account the symmetry of the operator A_h, we also obtain

$$\|[R(\lambda, A_h)]^r\|_{L_{1h}} \leq C|\lambda|^{n/2-r}$$

for all $\hat{\omega}_h$ and λ such that $|\arg\lambda| \geq \pi/4$.

This completes the proof. ∎

LEMMA 6. *The operator A_h given by (27) is uniformly (with respect to $h_j \in (0,1]$, $j = 1,2,\ldots,n$) almost sectorial of power r in the spaces L_{ph}, $p = 1,\infty$.*

PROOF. By Lemmas 3 and 5, we can write

$$\|[R(\lambda, A_h)]^r\|_{L_{ph}} \leq C\varepsilon^{-r}|\lambda|^{-r}, \quad p = 1,\infty \tag{38}$$

for all $\hat{\omega}_h$, $\varepsilon \in (0,\pi/2)$ and λ such that $|\arg\lambda| \geq \pi/2+\varepsilon$, as well

$$\|[R(\lambda, A_h)]^r\|_{L_{ph}} \leq C|\lambda|^{n/2-r}, \quad p = 1,\infty \tag{39}$$

for all $\hat{\omega}_h$ and λ such that $|\arg\lambda| \geq \pi/4+\varepsilon$, $\varepsilon \in (0,\pi/4)$. Using the Phragmen-Lindelof principle [11, p. 214], we have in view of (38), (39)

$$\|[R(\lambda, A_h)]^r\|_{L_{ph}} \leq C\varepsilon^{-r}|\lambda|^{-(r-4n\varepsilon/\pi)}, \quad p = 1,\infty \tag{40}$$

for all $\hat{\omega}_h$, $\varepsilon \in (0,\pi/8]$, and λ such that $|\arg\lambda| \geq \pi/2-\varepsilon$. Finally, it is enough to substitute $\pi\varepsilon/(4n)$ for ε in (40). ∎

THEOREM 2. *Let the operator A_h be given by (27), and let the scheme (4) satisfy the conditions of Theorem 1. Then, for any solution of (4), (27), the a priori estimate*

$$\begin{aligned}
&\|\mathfrak{U}^{\xi}_{\tau h}y(t_k)\|_{L_{ph}} \\
&\quad\leq C\Big\{\ln\big(2+[(k+1)\tau]^{-1}\big)\Big\}^{(r+1)|1-2p^{-1}|} \\
&\qquad\times\Big\{[(k+1)\tau]^{-\xi}\|y_{0h}\|_{L_{ph}} + \tau\sum_{l=1}^{k}[(k-l+1)\tau]^{-\xi}\max_{r=1,2,\ldots,\nu_2}\|f_h(t_{l-1}+\bar{c}_r\tau)\|_{L_{ph}}\Big\}, \\
&\qquad\qquad 1 \leq p \leq \infty,\ k = 0,1,\ldots
\end{aligned}$$

holds uniformly with respect to $\tau > 0$, $\hat{\omega}_h$ and $\xi \in [0,1]$, where $\mathfrak{U}_{\tau h}$ is the generator of scheme, $\alpha(z)$ is the scheme generator symbol.

PROOF. By Theorem 1 and Lemma 6, the needed assertion is true for $p = 1,\infty$. For $p = 2$ this follows from [1], since the original assumptions of that work are satisfied for self-adjoint operators in Hilbert spaces. To prove the theorem for all p, $1 \leq p \leq \infty$, we use the Riesz interpolation theorem [19]. ∎

In conclusion, note that all the techniques may be easily extended for the case of the third initial-boundary value problem.

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Millionschikova, 11, Kv. 82
Moscow 115487
Russia
e-mail: bakaev@vviamath.msk.su