1 Introduction

1.1 Arbitrary Objects and the Philosophy of Mathematics

Mathematics is the science of structure. Such is the battle cry of structuralism in the philosophy of mathematics.

Mathematical structuralism emerged on the scene as a major position in the philosophy of mathematics in the writings of Dedekind and Hilbert at the end of the nineteenth century. Since then, the discussion of mathematical structuralism has renewed itself several times. Around 1940, Bourbaki connected mathematical structuralism with set theory. In 1965, Benacerraf articulated an enormously influential argument for mathematical structuralism. In the 1980s, new forms of mathematical structuralism were developed: nominalistic ones (Field, Hellman) and platonistic ones (Resnik, Shapiro).

Structuralism became the dominant view in the philosophy of mathematics during the 1980s. Since then, and as a result of this development, the philosophical discussion about mathematical structures has become more inward looking: it has become to a very considerable extent an *internal* debate between different forms of mathematical structuralism. In recent years, despite a growing influence of ideas from category theory on the philosophical debate, the discussion about mathematical structuralism seems to be running out of steam. New ideas are needed if we are to rejuvenate it once more.

The discussion among mathematical structuralists has come to revolve to a large extent around a question of realism. Should we conceive of mathematical structures as platonic entities, or are mathematical structures reducible to systems of objects that can be recognised from a non-platonistic standpoint to exist?

This concentration on questions of mathematical realism has taken our eye off the question of the *nature* of mathematical structures. It has come to stand in the way of developing deeper insight in the metaphysical nature of mathematical structures. This book is an attempt to move away from the question of realism and to strike out in a new direction. One of my aims is to explore a new way of understanding mathematical structures.

https://doi.org/10.1017/9781139600293.003 Published online by Cambridge University Press

Well, the idea of how to do this is not completely new. It is inspired by a metaphysical theory that takes the idea of *arbitrary objects* at face value. The thought is that there are specific objects, such as you and I, but there are also arbitrary objects such as *the man on the Clapham omnibus*. This view goes back at least to the nineteenth century, but it was developed into a metaphysical theory by Kit Fine in the 1980s. In the philosophy of mathematics he mainly applied it in his theory of Cantorian abstraction, but in a last brief section of an important article he even applied it to mathematical structuralism.

The idea of bringing arbitrary object theory to bear on the question of the nature of mathematical structures is, very roughly, this: mathematical structures stand to systems of objects that instantiate them as arbitrary objects stand to the specific objects that can be their values. Just as we make a metaphysical distinction between specific and arbitrary objects, we distinguish between specific systems and *generic systems*, which is what structures are.

Fine's theory of arbitrary objects did attract attention in the natural language semantics community, but less so in the metaphysical community, although now it is gradually becoming clear how much philosophical potential his theory has. Fine's suggestions for applying this theory to mathematical structuralism have attracted even less attention by the philosophical community. Given that Fine is widely recognised as one of the leading metaphysicians of our days, this is surprising.

In fairness to the philosophical community, I think the scant attention given to Fine's theory of arbitrary objects is due in part to the fact that Fine has not worked out his theory of arbitrary objects in sufficient metaphysical detail. In his (overall very positive) review of Fine's book *Reasoning with arbitrary objects*, Santambrogio (1988, p. 634) writes:

> *The main criticism I have is that a richer discussion at an informal level of the philosophical status of arbitrary objects could have been more interesting than some of the technical results he has produced.*

Santambrogio is right. Logical questions about reasoning with arbitrary objects are important, and so are philosophical applications of arbitrary object theory. But the metaphysics of arbitrary objects is difficult and subtle, and Fine's discussion of it is dense at places. Readers have expressed difficulties in understanding aspects of Fine's theory of arbitrary objects.

Most philosophers today still find the arguments of Frege against arbitrary objects convincing, and they find the proposal of taking the idea of arbitrary objects at face value a bit mad. But, equally, analytic philosophers generally do not object to a philosophical view being crazy – especially if it is a metaphysical view – as long as they feel that they understand it. Therefore I think that the metaphysics of arbitrary objects should be given a more extensive, in-depth treatment, and deserves a more perspicuous and systematic development than one finds in Fine's writings. Taking this to heart, I will devote much attention to the embedding of my discussion of generic systems of mathematical objects in the general framework of the theory of arbitrary objects.

In any case, I believe that Fine's ideas in this area deserve a better fate and want to play a role in bringing this about. I aim to develop Fine's ideas into a more fully developed metaphysical theory of mathematical structure. My view of the nature of mathematical structures is not the same as that of Fine. This is to a significant extent because Fine and I disagree about the way in which the underlying theory of arbitrary objects should be de developed. Nonetheless, you can find most of the basic ideas that are developed in this book already in Fine's work.

1.2 Scratching the Surface

Arbitrary object theory is a very young sub-field of analytical metaphysics. The philosophical and logical literature on the subject is very limited: philosophical unclarity abounds, and there are myriad unsolved logical questions. This makes writing a monograph on arbitrary object theory a daunting enterprise. There are moments when one has a sense of being overwhelmed – although some would say that this is simply an illusion: there is nothing there to be seen at all.

My objectives in this monograph must therefore necessarily be modest. They are threefold. In the first place, I want to develop a metaphysical perspective on the nature of arbitrary entities and to defend it against objections. Secondly, I want to develop a few applications of arbitrary object theory to some extent. Thirdly, I want to develop a perspective on how various themes in the contemporary literature on arbitrary objects are connected with each other.

In his book *Reasoning about Arbitrary Objects*, Fine has provided robust philosophical replies to the strongest objections against arbitrary objects (most of which were first articulated by Frege). Moreover, Fine has also worked out elements of a metaphysical theory of arbitrary objects. But the metaphysical theory remains underdeveloped: the nature of arbitrary objects is presently not well understood. Articulating the metaphysical picture of arbitrary objects in some detail is therefore my main objective.

In broad outlines my theory agrees with that of Fine. But I want to develop it further, and there are important aspects in which my account differs with Fine's.

Turning to the second objective, I will be mainly interested in the way in which arbitrary object theory can be applied not only to objects but also to *systems* of objects. I will be concerned with *arbitrary* systems of objects, which, it turns out, can also be seen as systems of arbitrary objects. Moreover, I will also discuss the extent to which arbitrary objects can be said to be more or less *likely* to have some property. As far as attributing probabilities to arbitrary objects is concerned, we will see that there are problems relating to arbitrary objects that can take on infinitely many values that I am not able to solve in a completely satisfactory manner.

Concerning the third objective, I aim to show how some of the main strands in the small body of literature about arbitrary objects from the past four decades are related to each other. Again, I cannot claim to have completely succeeded. In particular, I regret not having been able fully to integrate the interesting theory that was developed by Santambrogio (1987) in this book. Nonetheless, I believe that the bibliography at the end of this book is at present fairly complete concerning philosophical thought on arbitrary objects from around 1983 until 2018 – and hope that this bibliography will soon be labelled as dated.

On the whole, I am only able to scratch the surface. I seek an understanding of what will be my stock example: the generic system of the natural numbers. I will extrapolate from this example to other examples and to the shape that a general theory of arbitrary objects could take. But I will not be able to go far beyond an understanding of the arbitrary natural numbers and the structures to which they belong. So I will fall far short of developing a general metaphysical and logical theory of arbitrary objects and generic systems.

In sum, the ambition of this book is to lay out some of the groundwork for a theory of arbitrary objects. It is a *prolegomenon to a future theory*. Even if my account does not go too far off track, there is an immense amount of work that remains to be done. And there is indeed a great risk that I have gone off track at crucial junctures. Most mistakes in arbitrary object theory have not yet been made, so I am very likely to make some of them.

1.3 Structure and Method

This book is divided into 11 chapters, but at a higher level it can be roughly be subdivided into three parts. The first part discusses arbitrary object theory as an exercise in what Fine calls *naive metaphysics* (Chapters 2–4).

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The second part investigates the relation between the notion of mathematical structure and arbitrary object theory (Chapters 5–9). The third part is shorter: it explores the connection between arbitrary objects and random variables, and the concept of probability that goes with it (Chapter 10). (The brief Chapter 11 contains a discussion of open problems and avenues for future research.) Throughout the book, the natural numbers and the structure to which they belong are taken to constitute the paradigmatic test case for the problems and theories that are discussed.

Let me now turn to a more detailed overview of the structure and methodology of the book.

The fact that questions of realism have come to play a central role in the philosophy of mathematics, as they have done in other philosophical disciplines, such as philosophy of science and philosophy of mind, is no accident. It is the result of a methodology that has become prevalent in analytic philosophy, largely through the work and influence of Quine. I start in the next chapter (Chapter 2) by advocating an alternative methodology in metaphysics. Inspiration for this effort can be found in Fine's writings. I rely on his distinction between naive and foundational metaphysics. Naive metaphysics is concerned with the way things metaphysically appear to us, whereas foundational metaphysics is concerned with which entities exist. In a word, Fine's slogan is:

Naive metaphysics first.
Foundational metaphysics second.

I go further than this and argue that we should not merely postpone questions of existence until after inquiring into their metaphysical nature; questions of existence should be dismissed altogether. In other words, I argue that a version of the forms of quietism that have been proposed in realism debates in other areas of philosophy (such as philosophy of science and philosophy of mathematics) can be and should be adopted *in* metaphysics.

As a naive metaphysician, I am sceptical about most ontological reductionist programmes in philosophy. In particular, I strongly reject the thesis that arbitrary entities can somehow be reduced ontologically to classes of specific objects. This does not mean, however, that I am sceptical about the use of mathematical methods in metaphysics. Quite on the contrary: I see no contradiction between naive metaphysics and 'mathematical philosophy'. Indeed, the success of the whole enterprise will to a significant extent hinge on how well the set theoretical models that I will use capture salient logical properties of and relations between arbitrary entities. Understanding the nature of arbitrary entities and being able to model them well go hand in hand.

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I want to practice what I preach by *doing* naive metaphysics. In particular, Chapter 3 contains an investigation into the *nature* of arbitrary objects. The account that I propose differs in key points from that of Fine, but a detailed comparison with his account is postponed until Chapter 7. Instead, in Chapter 3, we go further back in history. I show how key elements of the theory of arbitrary objects (as I develop it) are already present in the theory of *variables* that was developed by Russell in his *Principles of Mathematics* (1903).

The paradigmatic examples of arbitrary objects have always come mostly from mathematics. Chapter 4 contains an application of arbitrary object theory to metaphysical questions about mathematical objects. Special attention is given in this chapter, as in the remainder of the book, to the natural numbers. Moreover, I investigate to what extent arbitrary object theory is connected with the theory of what Charles Parsons calls *quasi-concrete* objects such as linguistic expressions.

In Chapter 5 I turn to philosophical questions about mathematical structures. If my arguments in Chapter 2 are sound, then the realism debate about mathematical structures is misguided. Nonetheless, my aim cannot be achieved without a thorough discussion of an ongoing realism debate, viz. the dispute between platonistic and nominalist versions of mathematical structuralism. These accounts contain essential insights into aspects of the nature of mathematical structures. They are a source of adequacy conditions that any good theory of mathematical structure will have to satisfy. In particular, a good theory of mathematical structure should leave room for a concomitant theory of mathematical objects, it should not be vulnerable to a variant of Benacerraf's identification problem, and it should admit a plausible account of what it means to compute on the natural numbers. Later in the book, my account of mathematical structures is tested against these conditions.

One central claim of this book is that in the same way as there are, beside specific objects, also arbitrary objects, there are also, beside specific systems of objects, arbitrary systems of objects. Arbitrary objects stand to specific objects in roughly the same way as arbitrary systems of objects stand to specific systems of objects:

$$\frac{\text{arbitrary object}}{\text{specific object}} \approx \frac{\text{arbitrary system}}{\text{specific system}}.$$

Moreover, there is a special subclass of the class of arbitrary systems, which I will call *generic* systems. Another central thesis of this book is then that generic systems can be identified with mathematical structures and

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vice versa. In this way, in Chapter 6 I connect the theory of arbitrary entities with the question of the nature of mathematical structures.

I will propose an account according to which many mathematical theories – algebraic theories as well as non-algebraic theories – can be said to describe one or more generic systems. Foundational mathematical theories, however, turn out to be exceptions. If a discipline such as set theory can be said to describe arbitrary entities at all, then it must be in a sense that is different from the way in which non-foundational mathematical theories can be taken to be about arbitrary entities.

Chapter 7 compares Kit Fine's metaphysical theory of arbitrary objects and generic systems with the account that I propose in earlier chapters of the book. It will emerge that the most important difference between Fine's account and mine concerns the relation of *dependence* between arbitrary objects. On Fine's account, dependence is a directional and irreducible cornerstone notion of arbitrary object theory. In my account dependence between arbitrary objects is a definable relation.

Chapter 8 is concerned with the relation between my conception of mathematical structures as generic systems and existing forms of mathematical structuralism. The natural number structure functions as a paradigmatic test case. I show how generic systems such as the natural number structure can be taken to be composed of mathematical objects. I argue that my account is invulnerable to Benacerrafian identification problems and that it can make philosophical sense of the fact that we compute *on* the natural numbers.

However, it will emerge that my account should not be seen as a rival to existing forms of mathematical structuralism. I do not take a stance in the realism debate about mathematical structures: '-isms' in general have too much of a reductive flavour for my taste. Moreover, there are good reasons for resisting the rather too broad claim that the subject matter of mathematics *is* mathematical structures.

The remaining two chapters in the book are somewhat more technical. They can be seen as mainly concerned with following up two logical suggestions that were made by Saul Kripke.

In Chapter 9 I carry out a logical investigation of one particular generic system: the structure of the natural numbers. This turns out to be a very rich mathematical structure. Following a suggestion of Kripke, it is argued that *Carnapian quantified modal logic* is the appropriate formal framework for expressing facts about specific and arbitrary numbers, for reasoning about them, and for formulating and investigating key concepts pertaining to arbitrary objects (such as indistinguishability and dependence). The main result of this chapter is that arbitrary numbers can play the role of *sets* of natural numbers, i.e. of real numbers. There is a precise sense in which the theory of the natural number structure, seen as a generic system, is exactly as expressive as the framework of second-order number theory.

Kripke once suggested that *random variables* can be seen as Carnapian individual concepts. Carnapian individual concepts will be seen to be closely related to, but not quite the same as, arbitrary objects. In Chapter 10, the analogy between random variables and arbitrary objects is probed. The leading thought is that the theory of random variables suggests a natural way in which probabilities can be associated with arbitrary objects. For arbitrary objects with a finite associated value range of specific objects, it is not hard to see how one should proceed. For arbitrary objects with infinite value ranges, however, it is less clear what a satisfactory notion of associated probabilities would look like. In this context, I will turn to *non-Archimedean probability theory* as the best account that I can come up with.

In a short final chapter, I try to step back and reflect on the *scope* of the arbitrary object theory that I have developed. Moreover, some possible avenues for further research are highlighted and briefly discussed. The list of problems that I discuss is far from exhaustive.

By the time that the reader has arrived at the end of the book, she will be keenly aware of the fact the metaphysics and logic of arbitrary objects still remain in an underdeveloped state, and that many of the issues that have been touched upon in this book have not been given the attention that they ultimately deserve. I can only express the hope that this will motivate the reader to try to develop the theory further and to correct some of my mistakes.

1.4 Intended Audience

This is a research monograph rather than a textbook or a reference text. It is aimed at graduate (or post-graduate, in UK terminology) students in metaphysics, philosophy of mathematics, and/or philosophical logic and at professional philosophers with a research interest in these domains.

Not every reader may want to read the whole book. If you are only interested in the metaphysics of arbitrary objects and generic systems, then you may may simply stop reading after Chapter 8. If you only want to see the methodology that I use practiced rather than discussed, then you can skip

Chapter 2. If you are already sufficiently familiar with the intricacies of the debate between eliminative and non-eliminative structuralists in the philosophy of mathematics, then you may not want to read Chapter 5.

If you are someone who is primarily interested in the logico-mathematical part of this book, then the quickest way is to start with Chapter 6 before going on to Chapters 9 and 10. But this may leave you wondering why the logical framework that is used is the right one for the task at hand. So in this case I recommend you to skip Chapters 2 and 5, but at least have a cursory look at Chapters 3, 7, and 8 before moving to the technical part of this monograph.

I attempt to be as clear as possible and to make this monograph reasonably self-contained – I found complete self-containment unachievable. If you want to get the most out of this book, and especially if you want to adopt a healthy critical attitude towards its content, it helps if you are able to situate it in the wider context in which it is embedded.

On the philosophical side, I assume first of all familiarity with contemporary literature on the philosophy of mathematics. Especially, I assume readers to be at least somewhat familiar with the main themes in the contemporary debate on mathematical structuralism. I expect them to be acquainted with what is covered in a good introduction to the philosophy of mathematics such as Linnebo (2017). Familiarity with the ground that is covered by more advanced works such as Shapiro (1997) or Parsons (2008) – which cannot really be called textbooks – is more than sufficient. Secondly, a significant part of the action in this book takes place in and around the intersection of philosophy of mathematics and logic. An excellent work that covers all the background that is relevant here (and more), is Button and Walsh (2018). Thirdly, I also assume a good understanding of the main concepts and themes of contemporary analytical metaphysics. Familiarity with what is covered in a book such as Loux and Zimmerman (2003) definitely suffices.

On the logical side, I presuppose what is covered in basic and intermediate courses on mathematical and philosophical logic. As far as mathematical logic is concerned, I rely on knowledge of basic proof theory, model theory, and the relations between the two. Boolos et al. (2007) covers almost all of this, except that I will also make use of arguments involving ultrafilters. For an introduction to the latter, the reader is referred to Bell and Slomson (2006). As far as philosophical logic is concerned, I assume knowledge of intermediate philosophical logic, in particular knowledge of the main concepts and techniques of quantified modal logic. Knowledge of the material in Garson (2006) is more than enough.

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To conclude, familiarity is assumed with basic elements of some mathematical disciplines. Only very elementary set theory is presupposed: Schoenfield (1977) more than suffices. Concerning the foundations of number theory and analysis, Truss (1997) is recommended, and some basic knowledge of non-standard analysis as is contained, for instance, in the first three chapters of Goldblatt (1998), is useful. Concerning probability theory, Blitzstein and Hwang (2015) more than suffices. In addition, familiarity is assumed with basic elements of graph theory: the first chapters of Diestel (2006) will do.

1.5 On Notation and Technical Matters

Since metaphysical and logical issues are intertwined in this book, it is written in a mixture of prose and technical notation. Throughout the book, I try to use standard logical and mathematical notation. I try to be careful in describing new notation clearly at the point where it is introduced. A glossary of symbols and abbreviations can be found at the end of the preamble to this book.

I use standard terminology for logical concepts:

- $\wedge$ stands for conjunction;
- $\vee$ stands for disjunction;
- $\neg$ stands for negation;
- $\rightarrow$ stands for material implication;
- $\leftrightarrow$ stands for material equivalence;
- $=$ stands for numerical equality;
- $\Diamond$ stands for possibility (and $\Box$ stands for necessity).

As is also usual, I use the symbols $\wedge$, $\vee$, $\overline{(\ldots)}$ for the corresponding operations of Boolean algebras. It will be clear from the context when for instance $\vee$ stands for the 'join' operation in some Boolean algebra rather than the more specific logical operation of disjunction.

The distinction between specific and arbitrary objects will be of fundamental importance. In prose contexts, I mostly use lowercase letters from the end of the Roman alphabet ($m, n, \ldots$) to refer to *specific* objects and lowercase letters from the beginning of the Roman alphabet ($a, b, c, \ldots$) to refer to *arbitrary* objects.

When I *model* arbitrary objects in set theory as functions – as will often happen in technical contexts – I refer to them by lowercase Greek letters

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($\alpha, \beta, \gamma, \delta, \dots$). I use the usual symbols for elementary set theoretic operations and relations:

- $A \cap B$ stands for the intersection of the sets A and B;
- $A \cup B$ stands for the union of the sets A and B;
- $A \times B$ stands for the Cartesian product of A and B;
- $A \subseteq B$ expresses that A is a subset of B;
- $A \backslash B$ stands for the set of elements of A that do not belong to B;
- $\emptyset$ stands for the empty set.

When I use functions that are not intended to refer to arbitrary objects, I use the ordinary *math font* symbols $f, g, \dots$ The expression $dom(f)$ stands for the domain of the function f; $ran(f)$ stands for the range of f.

I use capital *sans serif* letters ($\mathsf{A}, \mathsf{B}, \dots$) to refer to *sets* of arbitrary objects. And I use capital boldface letters ($\mathbf{N}, \mathbf{G}, \dots$) to refer to arbitrary *systems* of objects, whereas I use *math font* capital letters (S, T,...) to refer to *specific* systems of objects.

Upper case Latex *math font* letters $A, B, \dots$ stand for sets. Lowercase Greek letters are used not only for arbitrary objects modeled as functions (as explained above) but also for ordinal numbers. I rely on the context to make it clear when a lowercase Greek letter is used for an ordinal and when it is used for an arbitrary object. The expression B^A stands for the collection of functions from the set A to the set B. In particular, $\omega^{<\omega}$ stands for the collection of functions from finite sets of numbers to ω. Cardinal numbers are identified with the smallest ordinals equinumerous with them. The ordinal ω, for instance, will stand for the smallest infinite cardinal number, and I will identify the cardinal of 2^ω, i.e. of the continuum, with the smallest ordinal of that cardinality. The cardinality of a set A is denoted as $|A|$. (But sometimes I will be sloppy. For instance, I will often write 2^ω for $|2^\omega|$.) The symbol On stands for the class of ordinals. The symbol V stands for the set theoretic universe, and V_α stands for the ordinal rank α thereof. The symbol $\mathcal{P}$ refers to the full power set operation, and $[S]^{<\kappa}$ refers to the subsets of the set S that are of cardinality $<\kappa$.

Concerning proof theoretic notation, I use the lowercase letters $x, y, z, \dots$ as first-order variables, and the uppercase letters $X, Y, Z, \dots$ as second-order variables. Calligraphic font $\mathcal{L}, \mathcal{L}_1, \dots$ is used to refer to formal languages, and Greek letters $\varphi, \psi, \dots$ are used to refer to formulas of a formal language. $\varphi[y \backslash x]$ stands for the uniform substitution of all free occurrences of the variable x by occurrences of the variable y in the formula φ. As usual, the symbol $\vdash$ stands for the derivability relation. I use uppercase Roman letters to refer to theories, which are intended

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to be closed under logical derivation. For instance, PA stands for first-order Peano arithmetic, PA^2 stands for second-order Peano arithmetic, ZFC stands for first-order Zermelo-Fraenkel set theory with the Axiom of Choice, and ZFC^2 stands for second-order Zermelo-Fraenkel set theory with the Axiom of Choice.

The *fraktur font* $\mathfrak{A}, \mathfrak{B}, \ldots$ is used to refer to models. As is common, the symbol $\models$ stands for the model theoretic 'making true' relation. The notion $\mathfrak{A} \models \varphi[x/d]$ expresses that in the model $\mathfrak{A}$, the formula φ is made true on the assignment that assigns the element d of the domain of $\mathfrak{A}$ to free occurrences of the first-order variable x.

Familiar mathematical structures will be referred to in familiar ways. For instance, $\mathbb{N}$ refers to the standard natural number structure, $\mathbb{Q}$ refers to the structure of the rational numbers, $\mathbb{R}$ refers to the structure of the real numbers, and $\mathbb{C}$ refers to the structure of the complex numbers.

State spaces will play an important role in my account of arbitrary objects and in my application of arbitrary object theory to random variables. I will mostly use the symbol Ω to refer to state spaces, and I will use variables $X, Y, \ldots$ for random variables. $\mathsf{Pr}(A)$ will stand for the (unconditional) probability of the event A, and $\mathsf{Pr}(A \mid B)$ will stand for the probability of the event A *conditional* on the event B.

Even though technical aspects cannot and will not be side-stepped, this is primarily intended to be a philosophical work. Conceptual and metaphysical questions take precedence over mathematical questions. Therefore the reader will not find many logical or mathematical theorems in this monograph, and even fewer proofs. Only simple propositions are proved, sometimes in a sketch way. For non-elementary proofs of theorems, the reader is referred to the literature.