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ON POSITIVE DEFINITE FUNCTIONS OVER A LOCALLY COMPACT GROUP

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In this note we are concerned with several questions on positive definite functions over a Hausdorff locally compact group. The main result, Theorem A, gives some necessary and sufficient conditions for $\mu * \tilde{\mu}$ to be a positive definite function when μ is a (complex Radon) measure. In particular, $\mu * \tilde{\mu}$ is a positive definite function if and only if $\mu \in L^2$, and Theorem B then follows by giving a complete characterization of functions of the type $f * \tilde{f}$, where $f \in L^2$. Perhaps the most interesting aspect of these results is that they provide further examples of results over a non-abelian, non-compact group, which otherwise are simple consequences (with μ a bounded measure in Theorem A) of the theorems of Plancherel and Bochner.

Unless otherwise specified, all notation and definitions will follow [**1**;**2**]. The underlying group will always be G, a Hausdorff locally compact group with identity e, and with left Haar measure dx. Corresponding to each measure μ such that there exists a locally integrable function f satisfying $\mu = f \cdot dx$, we select and fix a function g equal to f locally almost everywhere and define $\mu^0 = g$. For each $f \in C_c$, the set of continuous complex-valued functions on G with compact supports, define $\tilde{f}$ by $\tilde{f}(x) = \overline{f(x^{-1})}$. To extend this definition consistently to measures in such a manner that $(\tilde{\mu})^0 = (\mu^0)^\sim$ locally almost everywhere, we define $\tilde{\mu}$ as the measure which satisfies $\tilde{\mu}(f) = [\mu(\Delta\tilde{f})]^-$ for each $f \in C_c$, where Δ is the modular function of G (see [**2**, 4.18.3]).

A function ϕ on G is said to be *positive definite* if it is continuous and satisfies

$$f * \phi * \tilde{f}(e) \geqq 0 \tag{1}$$

for each $f \in C_c$. (This definition is known to be equivalent to Bochner's original definition [**2**, (10.3.1)] of positive definiteness for continuous functions. One of the implications is straightforward, and the more difficult implication is resolved by showing that if ϕ is positive definite in the sense of continuity and (1), then ϕ satisfies [**2**, (10.3.9)].)

The definition of convolution, and the various criteria for its existence, used in the following can all be found in [**2**, especially §4.19].

LEMMA 1. *Let μ be a measure such that $\mu * \tilde{\mu}$ is defined. Then $f * (\mu * \tilde{\mu}) * \tilde{f}$ is defined and continuous everywhere on G, for all f in C_c, and moreover*

$$f * (\mu * \tilde{\mu}) * \tilde{f}(e) = (\|f * \mu\|_{L^2})^2.$$

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Proof. When $f \in C_c$, it is easily shown that $f * (\mu * \tilde{\mu}) * \tilde{f}$ is defined and continuous everywhere on G, and that it is immaterial whether the first or last convolution product is performed first.

Since every function in C_c and every measure has a "minimal decomposition" [**2**, p. 202] in terms of positive functions in C_c and positive measures, respectively, it is sufficient to prove the second part of this lemma for f positive and in C_c, and μ a positive measure.

In the following calculation, the interchanges of the order of integration will always be valid since the corresponding integrand is always positive and continuous; see [**2**, 4.17.2].

$$\begin{aligned} f * (\mu * \tilde{\mu}) * \tilde{f}(e) &= \int dx \int f(x) f(xy)\, d\mu * \tilde{\mu}(y) \\ &= \int dx \int d\mu(y) \int f(x) f(xyz)\, d\tilde{\mu}(z) \\ &\qquad \text{(from the definition of the convolution } \mu * \tilde{\mu}) \\ &= \int d\mu(y) \int d\tilde{\mu}(z) \int f(x) f(xyz)\, dx \\ &= \int d\mu(y) \int d\tilde{\mu}(z) \int f(xy^{-1}) f(xz) \Delta(y)\, dx \end{aligned}$$

(by making the right translation $x \mapsto xy^{-1}$ in the inner integrand)

$$\begin{aligned} &= \int dx \{\textstyle\int f(xy^{-1}) \Delta(y)\, d\mu(y)\} \{\textstyle\int f(xz)\, d\tilde{\mu}(z)\} \\ &= \int dx \{\textstyle\int f(xy^{-1}) \Delta(y)\, d\mu(y)\} \{\textstyle\int f(xz^{-1}) \Delta(z)\, d\mu(z)\} \\ &= \|f * \mu\|_{L^2}^2. \end{aligned}$$

THEOREM A. *Let μ be a measure such that $\mu * \tilde{\mu}$ and $(\mu * \tilde{\mu})^0$ are defined. Then the following three conditions are equivalent:*

(a) *$(\mu * \tilde{\mu})^0$ is equal locally almost everywhere to a positive definite function;*

(b) *$(\mu * \tilde{\mu})^0$ is essentially bounded on some neighbourhood of e in G;*

(c) *μ^0 exists and belongs to L^2.*

Proof. It is easily proved that (c) implies (a) and (b). Assume (b) and let $f \in C_c$. Then $f * (\mu * \tilde{\mu})^0 * \tilde{f}(e)$ is defined and equals $f * (\mu * \tilde{\mu}) * \tilde{f}(e)$. Apply Lemma 1 to obtain

$$f * (\mu * \tilde{\mu}) * \tilde{f}(e) = \int |f * \mu(x)|^2\, dx.$$

But this final term is always non-negative, so that $(\mu * \tilde{\mu})^0$ satisfies (1). A corollary of a standard theorem due to Gelfand and Raikov (see [**2**, 10.3.3 and 10.3.4]) completes the proof that (b) implies (a).

The cycle of implications is closed by showing that (a) implies (c) or, rather that $(\mu * \tilde{\mu})^0 \in L^\infty$ implies (c), since every positive definite function is bounded. Assume (a), and hence $(\mu * \tilde{\mu})^0 \in L^\infty$. Let $\{\delta_\alpha\}$ denote an approximate identity consisting of positive functions in C_c such that $\|\delta_\alpha\|_{L^1} = 1$. Then $\{\delta_\alpha * \mu\}$ is a set of continuous functions uniformly bounded in L^2 since

$$\begin{aligned}\|\delta_\alpha * \mu\|_{L^2}{}^2 &= \delta_\alpha * (\mu * \tilde{\mu}) * \tilde{\delta}_\alpha(e) \quad \text{by Lemma 1}\\ &\leqq \|\delta_\alpha * (\mu * \tilde{\mu}) * \tilde{\delta}_\alpha\|_{L^\infty}\\ &\leqq \|\delta_\alpha\|_{L^1} \|(\mu * \tilde{\mu})^0\|_{L^\infty} \|\Delta\tilde{\delta}_\alpha\|_{L^1}\\ &= \|(\mu * \mu)^0\|_{L^\infty}.\end{aligned}$$

Thus $\{\delta_\alpha * \mu\}$ possesses a weak limit point in L^2 which must be μ^0.

Counterexample. The implication (b) implies (c) is best possible in the sense that we can construct a function, f say, such that $f * \tilde{f}$ is continuous everywhere except at the origin and that $f * \tilde{f}$ is bounded in the complement of every neighbourhood of the origin, but $f \notin L^2$. (However, $f \in L^p$ for $1 \leqq p < 2$.)

On the real line, define $f: x \to 1/x^{\frac{1}{2}}$ for $0 < x \leqq 1$, and 0 otherwise. Then $f * \tilde{f}$ is defined and continuous everywhere except at the origin. But it is unbounded in every neighbourhood of the origin since, when $0 < x \leqq 1$,

$$\begin{aligned} f * \tilde{f}(x) &= \int_{-\infty}^{\infty} f(y) f(y - x)\, dy\\ &= \int_x^1 y^{-\frac{1}{2}} (y - x)^{-\frac{1}{2}}\, dy\\ &= 2 \operatorname{arcsinh} x^{-\frac{1}{2}}.\end{aligned}$$

Let A denote the "Fourier algebra" of G defined and discussed by Eymard [**3**]. Algebraically, A can be identified with the space $L^2 * (L^2)^\sim$ under pointwise multiplication [**3**, p. 218]. It is given a norm under which (a) it becomes a Banach algebra, and (b) its normed dual is isometrically isomorphic to $\mathrm{VN}(G)$, the von Neumann algebra of continuous linear endomorphisms of L^2 which commute with convolution on the right by functions in C_c. (When G is abelian, A is the set of functions f on G whose Fourier transforms $\hat{f}$ belong to L^1, with the L^1-norm of $\hat{f}$ taken as the norm of f.) Our aim is to show that the set of positive definite functions in A is precisely $\{f * \tilde{f}: f \in L^2\}$. The latter half of this proof will be essentially the same as Dixmier's proof of "$A = L^2 * (L^2)^\sim$" found in [**3**, p. 218].

Let H denote a complex Hilbert space, and $\mathscr{L}(H)$ the space of continuous linear endomorphisms of H. If $Z \subset \mathscr{L}(H)$, the Z^+ denotes the set of positive self-adjoint operators in Z. No difficulty should arise from the fact that both the operator norm on $\mathscr{L}(H)$ and the Hilbert space norm will be denoted by $\|\cdot\|$.

LEMMA 2. *Let Y be the von Neumann algebra generated by the $*$-subalgebra $X \subset \mathscr{L}(H)$, and suppose that X is strongly dense in Y (that is, that X is dense in Y when Y is equipped with the strong operator topology). Then each T in Y^+ may be strongly approximated by operators of the form S^2 where S belongs to X^+.*

Proof. Let $T \in Y^+$; then there exists a unique $T^{\frac{1}{4}} \in Y^+$ such that $(T^{\frac{1}{4}})^4 = T$ [**4**, p. 485, Theorem (C.35)]. Select a net $\{S_\alpha\} \subset X$ such that $\|S_\alpha\| \leqq \|T^{\frac{1}{4}}\|$ and $\lim_\alpha S_\alpha = T^{\frac{1}{4}}$ in the strong operator topology; that this is possible follows from the hypothesis of the theorem and the density theorem of Kaplansky [**1**, p. 46].

Since $T^{\frac{1}{4}}$ is self-adjoint, it is clear that $\lim_\alpha S_\alpha^* = T^{\frac{1}{4}}$ in the weak operator topology. Now $S_\alpha^* S_\alpha \in X^+$ and we will show that $\lim S_\alpha^* S_\alpha = T^{\frac{1}{2}}$ in the weak operator topology. Let $x, y \in H$; then

$$\begin{aligned}
&|\langle (S_\alpha^* S_\alpha x - T^{\frac{1}{2}}x), y\rangle| \\
&\quad \leqq |\langle (S_\alpha^* S_\alpha x - S_\alpha^* T^{\frac{1}{4}}x), y\rangle| + |\langle (S_\alpha^* T^{\frac{1}{4}}x - T^{\frac{1}{2}}x), y\rangle| \\
&\quad \leqq \|S_\alpha^*\| \cdot \|S_\alpha x - T^{\frac{1}{4}}x\| \cdot \|y\| + |\langle (S_\alpha^* T^{\frac{1}{4}}x - T^{\frac{1}{4}}T^{\frac{1}{4}}x), y\rangle|
\end{aligned}$$

which converges to 0, since $\|S_\alpha^*\| = \|S_\alpha\| \leqq \|T^{\frac{1}{4}}\|$. By [**1**, p. 41, *théorème* 1 (iv)], we may choose a net $\{U_\beta\}$ in X^+, consisting of convex combinations of the $\{S_\alpha^* S_\alpha\}$, which converges to $T^{\frac{1}{2}}$ in the strong operator topology.

Since $\|U_\beta\| \leqq \|T^{\frac{1}{2}}\|$ for each β, and since

$$\begin{aligned}
\|U_\beta^2 x - Tx\| &\leqq \|U_\beta(U_\beta x) - U_\beta T^{\frac{1}{2}}x\| + \|U_\beta T^{\frac{1}{2}}x - Tx\| \\
&\leqq \|U_\beta\| \cdot \|U_\beta x - T^{\frac{1}{2}}x\| + \|U_\beta T^{\frac{1}{2}}x - T^{\frac{1}{2}}T^{\frac{1}{2}}x\|
\end{aligned}$$

for each $x \in X$, U_β^2 is strongly convergent to T.

LEMMA 3. *If $g, h \in L^2$, and $u = g * \tilde{h}$ is positive definite, then the functional*

$$T \mapsto \langle Tg, h\rangle = \int Tg(x)\overline{h(x)}\, dx$$

on VN(G) *is positive and normal.*

Proof. Let $T \in \mathrm{VN}(G)^+$. Since the set of operators of the form $f \mapsto \mu * f$, where $\mu \in C_c$, is strongly dense in $\mathrm{VN}(G)$ (see Dixmier [**1**, p. 44, *corollaire* 1]), we may use Lemma 2 to define $\{\mu_\alpha\} \subset C_c$ such that the operators $S_\alpha: f \mapsto \mu_\alpha * f$ belong to $\mathrm{VN}(G)^+$ and the weak limit of $T_\alpha = S_\alpha^2$ is T. Then

$$\begin{aligned}
\langle Tg, h\rangle &= \lim_\alpha \langle S_\alpha^2 g, h\rangle \\
&= \lim_\alpha \langle S_\alpha g, S_\alpha h\rangle \\
&= \lim_\alpha \langle \mu_\alpha * g, \mu_\alpha * h\rangle.
\end{aligned}$$

But, by repeated application of the theorems of Lebesgue-Fubini and Tonelli, we have

$$\begin{aligned}
\langle \mu_\alpha * g, \mu_\alpha * h\rangle &= \int \left(\int \mu_\alpha(y) g(y^{-1}x)\, dy\right)\left(\int \bar{\mu}_\alpha(z)\bar{h}(z^{-1}x)\, dz\right) dx \\
&= \iint \mu_\alpha(y)\bar{\mu}_\alpha(z)\, dydz \int g(y^{-1}x)\bar{h}(z^{-1}x)\, dx \\
&= \iint \mu_\alpha(y)\bar{\mu}_\alpha(z)\, dydz \int g(x)\, \bar{h}(z^{-1}yx)\, dx \\
&= \iint u(y^{-1}z)\mu_\alpha(y)\bar{\mu}_\alpha(z)\, dydz \\
&= \mu_\alpha * u * \tilde{\mu}_\alpha(e).
\end{aligned}$$

Since u is positive definite, $\mu_\alpha * u * \tilde{\mu}_\alpha(e) \geqq 0$ for all α, and so $T \mapsto \langle Tg, h\rangle$ is a positive functional. It is also ultraweakly continuous [**3**, *théorème* 3.10] and so is normal [**1**, p. 54, *théorème* 1], which completes the proof.

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Suppose that G is separable. Then, as in [**3**, p. 218, part 1 of the proof of the characterization theorem], we have that all linear positive normal functionals on VN(G) are of the form $T \mapsto \langle Tf, f\rangle$, with $f \in L^2$. Thus, since $u \in A$ implies that there exist $g,\ h \in L^2$ such that $u = g * \tilde{h}$, if $u \in A$ is also positive definite, Lemma 3 shows that there exists $f \in L^2$ such that

$$\langle Tg, h\rangle = \langle Tf, f\rangle$$

for all $T \in$ VN(G). Allowing T to vary over operators of the form $\phi \mapsto \mu * \phi$, where $\mu * C_c$, it follows that $u = g * \tilde{h} = f * \tilde{f}$. By imitating parts 2 and 3 of the above-mentioned characterization theorem, we have the following result.

THEOREM B. *Let G be any Hausdorff locally compact group. Then the set of positive definite functions in A is precisely $\{f * \tilde{f}: f \in L^2\}$.*

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