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ABSTRACT

For a closed d-dimensional subvariety X of an abelian variety A and a canonically metrized line bundle L on A, Chambert-Loir has introduced measures $c_1(L|_X)^{\wedge d}$ on the Berkovich analytic space associated to A with respect to the discrete valuation of the ground field. In this paper, we give an explicit description of these canonical measures in terms of convex geometry. We use a generalization of the tropicalization related to the Raynaud extension of A and Mumford's construction. The results have applications to the equidistribution of small points.

1. Introduction

Let K be a field with a discrete valuation v, valuation ring K° and residue field $\tilde{K}$. We denote the completion of the algebraic closure of the completion of K by $\mathbb{K}$. This algebraically closed complete field is used for analytic considerations on algebraic varieties defined over K. For the analytic facts, we refer the reader to § 2.

In non-archimedean analysis, there is no analogue known for the first Chern form of a metrized line bundle. However, Chambert-Loir [Cha06] has introduced measures $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ on the Berkovich analytic space X^{an} associated to a d-dimensional projective variety X. The analogy to the corresponding forms in differential geometry comes from Arakelov geometry. These measures are best understood in the case of metrics induced by line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_d$ on a projective K°-model $\mathscr{X}$ of X, with generic fibres $L_1, \ldots, L_d$. In this standard situation from Arakelov geometry, $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ is a discrete measure on X^{an} with support and multiplicities determined by the irreducible components of $\tilde{\mathscr{X}}$ and their degrees with respect to $\mathscr{L}_1, \ldots, \mathscr{L}_d$. However, the canonical metric on an ample line bundle of an abelian variety A over K is given by such models only if A has potential good reduction. In general, a variation of Tate's limit argument shows that the canonical metric is a uniform limit of roots of model metrics and hence the corresponding canonical measure is given as a limit of discrete measures. We recall the theory of Chambert-Loir's measures in § 3.

We consider an irreducible d-dimensional closed subvariety X of the abelian variety A. Using the Raynaud extension of A, there are a complete lattice Λ in $\mathbb{R}^n$ and a map $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$, where n is the torus rank of A. We call $\overline{\mathrm{val}}(X^{\mathrm{an}})$ the tropical variety associated to X. This analytic analogue of tropical algebraic geometry is described in § 4. Let b be the dimension of the abelian part of good reduction in the Raynaud extension of A and hence $\dim(A) = b + n$. For a simplex Δ in $\mathbb{R}^n$, we denote by $\delta_{\overline{\Delta}}$ the Dirac measure in $\overline{\Delta}$, i.e. the push-forward of the Lebesgue

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measure on Δ to $\mathbb{R}^n/\Lambda$. The main result of this paper is the following explicit description of canonical measures in terms of convex geometry.

Theorem 1.1. *There are rational simplices $\overline{\Delta}_1, \dots, \overline{\Delta}_N$ in $\mathbb{R}^n/\Lambda$ with the following properties:*

(a) *for $j = 1, \dots, N$, we have $\dim(\overline{\Delta}_j) \in \{d-b, \dots, d\}$;*

(b) $\overline{\mathrm{val}}(X^{\mathrm{an}}) = \bigcup_{j=1}^N \overline{\Delta}_j$;

(c) *for canonically metrized line bundles $\overline{L_1}, \dots, \overline{L_d}$ on A, there are $r_j \in \mathbb{R}$ with*

$$\overline{\mathrm{val}}_*(c_1(\overline{L_1}|_X) \wedge \cdots \wedge c_1(\overline{L_d}|_X)) = \sum_{j=1}^N r_j \cdot \delta_{\overline{\Delta}_j};$$

(d) *if all line bundles in (c) are ample, then $r_j > 0$ for $j \in \{1, \dots, N\}$.*

Erratum. In the preprint version [Gub08b] of this paper, it was claimed that the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is of pure dimension. However, the referee has found a gap in the argument (see Remark 4.16) and so this question remains open. As a consequence, in [Gub08a, Theorem 1.2], one should omit claiming that the tropical variety is of pure dimension. All other claims remain valid.

Theorem 1.1 was proved in [Gub07a] for abelian varieties which are totally degenerate at the place v. This special case is equivalent to $b = 0$, which makes the arguments easier. In particular, the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is of pure dimension d. In the general case, we can still show in Theorem 4.15 that the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a polytopal set with the above properties (a) and (b). The most serious problem is that the tropical dimension may be strictly smaller than d. This leads to the unpleasant fact that the canonical measure in Theorem 1.1(c) may have singular parts in lower dimensions, which is in sharp contrast to the totally degenerate case.

Using a semi-stable alteration, we will give in § 6 an explicit description of the canonical measure $c_1(\overline{L_1}|_X) \wedge \cdots \wedge c_1(\overline{L_d}|_X)$ on X^{an} in terms of convex geometry. It relies on our study of Mumford models of A in § 4 and on the properties of the skeleton of the strictly semi-stable model from the alteration given in § 5. A Mumford model is associated to a rational Λ-periodic polytopal decomposition of $\mathbb{R}^n$ such that the reduction modulo v brings toric varieties and convex geometry into play. In Theorem 6.12, we show that the support of this canonical measure is a canonical subset of X^{an} which does not depend on the choice of the ample line bundles L_j and which has a canonical piecewise linear structure. Finally, the proof of Theorem 1.1 will be finished in § 7 and we will show in two examples what these canonical measures can look like. In the appendix, we study building blocks of strongly non-degenerate strictly pluristable formal models. This is the background for the generalization of our results in § 5 to such models, which is required only in the proof of Theorem 6.12.

Theorem 1.1 has the following application to diophantine geometry. Let K be either a number field or the function field of an irreducible projective variety B of positive dimension over a field k. In the latter case, we assume that B is regular in codimension one and we count the prime divisors v of B with multiplicity $\deg_{\mathbf{c}}(v)$ for a fixed ample class $\mathbf{c}$ on B. In any case, K satisfies the product formula and hence we get absolute heights on projective varieties over K (see [BG06]). In particular, we have the Néron–Tate height $\hat{h}$ on the abelian variety A with respect to a fixed ample symmetric line bundle L. Note that $\hat{h}$ is a positive semi-definite quadratic form on $A(\overline{K})$ and hence defines a semi-distance. By Arakelov geometry, there is an extension of the Néron–Tate height to all closed subvarieties of A defined over $\overline{K}$ (see [Gub03]).

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Let X be an irreducible d-dimensional closed subvariety of the abelian variety A over K. We choose a small generic net $(P_m)_{m \in I}$ in $X(\overline{K})$. Here, small means

$$\lim_m \hat{h}(P_m) = \frac{\hat{h}(X)}{(d+1)\deg_L(X)}$$

and generic means that for every proper closed subset Y of X, there is an $m_0 \in I$ such that $P_m \notin Y$ for all $m \geqslant m_0$. The absolute Galois group $G := \mathrm{Gal}(\overline{K}/K)$ acts on $X(\overline{K})$ and $O(P_m)$ denotes the orbit of P_m.

We fix a discrete valuation v on K and we form X_v^{an} and the tropical variety $\overline{\mathrm{val}}(X_v^{\mathrm{an}})$ with respect to v. We fix an embedding $\overline{K} \hookrightarrow \mathbb{K}_v$ to identify $A(\overline{K})$ with a subset of A_v^{an}. On $\overline{\mathrm{val}}(X_v^{\mathrm{an}})$, we consider the discrete probability measures

$$\nu_m := \frac{1}{|O(P_m)|} \cdot \sum_{P_m^\sigma \in O(P_m)} \delta_{\overline{\mathrm{val}}(P_m^\sigma)}.$$

Tropical equidistribution theorem. *There is a regular probability measure ν on $\mathbb{R}^n$ with support equal to the tropical variety $\overline{\mathrm{val}}(X_v^{\mathrm{an}})$ such that $\nu_m \xrightarrow{w} \nu$ as a weak limit of Borel measures. If we endow L with a canonical metric $\| \ \|_v$, then we have $\nu = \deg_L(X)^{-1}\overline{\mathrm{val}}_*(c_1((L|_X, \| \ \|_v))^{\wedge d})$.*

Note that this statement is only useful if we have the positivity of ν from the explicit description in Theorem 1.1. The tropical equidistribution theorem follows from the equidistribution theorem

$$\frac{1}{|O(P_m)|} \cdot \sum_{P_m^\sigma \in O(P_m)} \delta_{P_m^\sigma} \xrightarrow{w} \frac{1}{\deg_L(X)} \cdot c_1(L|_X, \| \ \|_v)^{\wedge d} \tag{1}$$

on X_v^{an}. For an archimedean place v of a number field K and for a metrized ample line bundle with positive curvature on a smooth projective variety, the equidistribution (1) was proved by Szpiro, Ullmo and Zhang [SUZ97]. This was generalized by Yuan [Yua08, Theorem 5.1] to semi-positively metrized ample line bundles on projective varieties over a number field and also to non-archimedean places. In [Gub08a, Theorem 1.1], Yuan's generalization was proved in the function field case.

The potential applications of the tropical equidistribution theorem are related to the Bogomolov conjecture. The latter claims that the Néron–Tate height has a positive lower bound on $X(\overline{K})$ outside an explicit exceptional set. In the number field case, the Bogomolov conjecture was proved by Ullmo [Ull98] for curves and by Zhang [Zha98] in general. The main tool was the archimedean version of (1). For function fields, the Bogomolov conjecture is still open. In [Gub07b], it was proved for abelian varieties which are totally degenerate with respect to a place v. The proof relied on the tropical equidistribution theorem for totally degenerate abelian varieties [Gub07b, Theorem 5.5]. For an arbitrary abelian variety, it is clear that the tropical equidistribution theorem cannot imply the Bogomolov conjecture since the dimension of the tropical variety may decrease. However, it is plausible that it can be used once the case of abelian varieties with everywhere good reduction is understood.

1.2 Terminology

In $A \subset B$, A may be equal to B. The complement of A in B is denoted by $B\backslash A$. The zero is included in $\mathbb{N}$ and in $\mathbb{R}_+$.

All occurring rings and algebras are commutative with 1. If A is such a ring, then the group of multiplicative units is denoted by $A^\times$. A variety over a field is a separated reduced scheme of finite type over that field. However, a formal analytic variety is not necessarily reduced (see § 2). For the degree of a map $f : X \to Y$ of irreducible varieties, we use either $\deg(f)$ or $[X : Y]$.

Let Y be a variety over a field. Following [Ber99, § 2], we use the following canonical stratification of Y. We start with $Y^{(0)} := Y$. For $r \in \mathbb{N}$, let $Y^{(r+1)} \subset Y^{(r)}$ be the complement of the set of normal points in $Y^{(r)}$. Since the set of normal points is open and dense, we get a chain of closed subsets:

$$Y = Y^{(0)} \supsetneq Y^{(1)} \supsetneq Y^{(2)} \supsetneq \cdots \supsetneq Y^{(s)} \supsetneq Y^{(s+1)} = \emptyset.$$

The irreducible components of $Y^{(r)} \backslash Y^{(r+1)}$ are called the *strata* of Y. The set of strata is denoted by $\mathrm{str}(Y)$. It is partially ordered by $S \leqslant T$ if and only if $\overline{S} \subset \overline{T}$. A *strata subset* is a union of strata. A *strata cycle* is a cycle whose components are strata subsets.

For $\mathbf{m} \in \mathbb{Z}^n$, let $\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$ and $|\mathbf{m}| := m_1 + \cdots + m_n$. The standard scalar product of $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$ is denoted by $\mathbf{u} \cdot \mathbf{u}' := u_1 u_1' + \cdots + u_n u_n'$. For the notation used from convex geometry, we refer the reader to 4.4.

2. Analytic and formal geometry

In this section, we recall results from Berkovich analytic spaces and formal geometry. Our base field $\mathbb{K}$ is algebraically closed with a non-trivial, non-archimedean complete absolute value $|\ |$, valuation ring $\mathbb{K}^\circ$ and residue field $\tilde{\mathbb{K}}$.

2.1 The *Tate algebra* $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ is the completion of $\mathbb{K}[x_1, \ldots, x_n]$ with respect to the Gauss norm. Its elements are the power series in the variables $x_1, \ldots, x_n$ with coefficients $a_{\mathbf{m}} \in K$ such that $|a_{\mathbf{m}}| \to 0$ for $m_1 + \cdots + m_n \to \infty$. A $\mathbb{K}$-*affinoid algebra* $\mathscr{A}$ is isomorphic to $\mathbb{K}\langle x_1, \ldots, x_n\rangle / I$ for some ideal I in $\mathbb{K}\langle x_1, \ldots, x_n\rangle$. The maximal spectrum $\mathrm{Max}(\mathscr{A})$ corresponds to the zero set of I in the closed unit ball $\mathbb{B}^n := \{\mathbf{x} \in \mathbb{K}^n \mid \max_j |x_j| \leqslant 1\}$. The supremum semi-norm of $\mathscr{A}$ on $\mathrm{Max}(\mathscr{A})$ is denoted by $|\ |_{\mathrm{sup}}$. We define

$$\mathscr{A}^\circ := \{a \in \mathscr{A} \mid |a|_{\mathrm{sup}} \leqslant 1\}, \quad \mathscr{A}^{\circ\circ} := \{a \in \mathscr{A} \mid |a|_{\mathrm{sup}} < 1\}$$

and the finitely generated reduced $\tilde{\mathbb{K}}$-algebra $\tilde{\mathscr{A}} := \mathscr{A}^\circ / \mathscr{A}^{\circ\circ}$ (see [BGR84]).

2.2 For an affinoid algebra $\mathscr{A}$, the *Berkovich spectrum* $X = \mathscr{M}(\mathscr{A})$ is the set of semi-norms p on $\mathscr{A}$ with $p(ab) = p(a)p(b)$, $p(1) = 1$ and $p(a) \leqslant |a|_{\mathrm{sup}}$ for all $a, b \in \mathscr{A}$. We use the coarsest topology on X such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathscr{A}$. The affine $\tilde{\mathbb{K}}$-variety $\tilde{X} = \mathrm{Spec}(\tilde{\mathscr{A}})$ is called the *reduction* of X. The reduction map $X \to \tilde{X}$, $p \mapsto \tilde{p} := \{p < 1\}/\mathscr{A}^{\circ\circ}$, is surjective. If Y is an irreducible component of $\tilde{X}$, then there is a unique $\xi_Y \in X$ with $\tilde{\xi}_Y$ equal to the generic point of Y. For details, we refer the reader to Berkovich[Ber90]. Note that our definition of an affinoid algebra is the same as in [BGR84], but Berkovich calls them strictly affinoid algebras.

2.3 A *rational subdomain* of $X := \mathscr{M}(\mathscr{A})$ is defined by

$$X\left(\frac{\mathbf{f}}{g}\right) := \{x \in X \mid |f_j(x)| \leqslant |g(x)|, j = 1, \ldots, r\},$$

where $g, f_1, \ldots, f_r \in \mathscr{A}$ are without common zero. It is the Berkovich spectrum of the affinoid algebra

$$\mathscr{A}\left\langle \frac{\mathbf{f}}{g} \right\rangle := \mathbb{K}\langle \mathbf{x}, y_1, \ldots, y_r\rangle / \langle I, g(\mathbf{x})y_j - f_j \mid j = 1, \ldots, r\rangle,$$

where we use the description $\mathscr{A} = \mathbb{K}\langle \mathbf{x}\rangle / I$ from 2.1.

More generally, one defines an *affinoid subdomain* of X as the Berkovich spectrum of an affinoid algebra characterized by a certain universal property (see [BGR84, 7.2.2]). By a theorem of Gerritzen and Grauert, an affinoid subdomain is a finite union of rational domains. For more details, we refer the reader to [BGR84, ch. 7] and [Ber90, § 2.2].

2.4 An *analytic space* X over $\mathbb{K}$ is given by an atlas of affinoid subdomains $U = \mathscr{M}(\mathscr{A})$. For the precise definition, we refer the reader to [Ber93, § 1] (where they are called strictly analytic spaces).

2.5 The *formal topology* on $X = \mathscr{M}(\mathscr{A})$ is given by the preimages of the open subsets of $\tilde{X}$ with respect to the reduction map. This quasi-compact topology is generated by affinoid subdomains and hence we get a canonical ringed space called a *formal affinoid variety* over $\mathbb{K}$, which we denote by $\mathrm{Spf}(\mathscr{A})$. By definition, a morphism of affinoid varieties over $\mathbb{K}$ is induced by a reverse homomorphism of the corresponding $\mathbb{K}$-affinoid algebras (see [Bos77] for details).

A *formal analytic variety* over $\mathbb{K}$ is a $\mathbb{K}$-ringed space $\mathfrak{X}$ which has a locally finite open atlas of formal affinoid varieties $\mathrm{Spf}(\mathscr{A}_i)$ over $\mathbb{K}$ called a *formal affinoid atlas*. The *generic fibre* $\mathfrak{X}^{\mathrm{an}}$ (respectively the *reduction* $\tilde{\mathfrak{X}}$) is locally given by $\mathscr{M}(\mathscr{A}_i)$ (respectively $\mathrm{Spec}(\tilde{\mathscr{A}_i})$). By 2.2, we get a surjective reduction map $\mathfrak{X}^{\mathrm{an}} \to \tilde{\mathfrak{X}}$, $x \mapsto \tilde{x}$. For every irreducible component Y, there is a unique $\xi_Y \in \mathfrak{X}^{\mathrm{an}}$ such that $\tilde{\xi}_Y$ is the generic point of Y.

2.6 An *admissible* $\mathbb{K}^\circ$*-algebra* is a $\mathbb{K}^\circ$-algebra A without $\mathbb{K}^\circ$-torsion isomorphic to $\mathbb{K}^\circ\langle x_1, \ldots, x_n\rangle / I$ for an ideal I. An *admissible formal scheme* $\mathscr{X}$ over $\mathbb{K}^\circ$ is a formal scheme over $\mathbb{K}^\circ$ which has a locally finite atlas of open subsets $\mathrm{Spf}(A_i)$ for admissible $\mathbb{K}^\circ$-algebras A_i (see [BL93a, BL93b] for details).

The *special fibre* $\tilde{\mathscr{X}}$ of $\mathscr{X}$ is a scheme over $\tilde{\mathbb{K}}$ locally given by $\mathrm{Spec}(\tilde{A}_i)$. It is locally of finite type over $\tilde{\mathbb{K}}$ and not necessarily reduced. The latter is in sharp contrast to the reduction of formal analytic varieties. These categories are related by the following functors.

The formal analytic variety $\mathscr{X}^{\mathrm{f-an}}$ associated to $\mathscr{X}$ is locally given by $\mathrm{Spf}(\mathscr{A}_i)$ for the affinoid algebra $\mathscr{A}_i := A_i \otimes_{\mathbb{K}^\circ} \mathbb{K}$. The canonical morphism $(\mathscr{X}^{\mathrm{f-an}})^\sim \to \tilde{\mathscr{X}}$ is finite and surjective (see [BL86, § 1]).

The generic fibre of $\mathscr{X}^{\mathrm{f-an}}$ is also called the *generic fibre* of $\mathscr{X}$ and is denoted by $\mathscr{X}^{\mathrm{an}}$. Note that $\mathscr{X}^{\mathrm{f-an}}$ and $\mathscr{X}^{\mathrm{an}}$ are equal as a set but $\mathscr{X}^{\mathrm{an}}$ has a finer topology. Using composition of the reduction map for $\mathscr{X}^{\mathrm{f-an}}$ (see 2.5) with the canonical morphism above, we get a surjective *reduction map* $\pi : \mathscr{X}^{\mathrm{an}} \to \tilde{\mathscr{X}}$.

If $\mathfrak{X}$ is a formal analytic variety over $\mathbb{K}$ given by the formal affinoid atlas $\mathrm{Spf}(\mathscr{A}_i)$, then the *associated formal scheme* $\mathfrak{X}^{\mathrm{f-sch}}$ is locally given by $\mathrm{Spf}(\mathscr{A}_i^\circ)$.

It is often useful to flip between formal analytic varieties over $\mathbb{K}$ and admissible formal schemes over $\mathbb{K}^\circ$. This is possible because the functors $\mathscr{X} \to \mathscr{X}^{\mathrm{f-an}}$ and $\mathfrak{X} \to \mathfrak{X}^{\mathrm{f-sch}}$ give an equivalence between the category of admissible formal schemes over $\mathbb{K}^\circ$ with reduced special fibre and the category of reduced formal analytic varieties over $\mathbb{K}$. Moreover, the canonical map $(\mathscr{X}^{\mathrm{f-an}})^\sim \to \tilde{\mathscr{X}}$ is an isomorphism. For details, see [BL86, § 1] and [Gub97, § 1].

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2.7 For a scheme X of finite type over a subfield K of $\mathbb{K}$, there is an analytic space X^{an} over $\mathbb{K}$ associated to X. The construction is similar to that for complex varieties. Moreover, most algebraic properties hold also analytically and, conversely, there is a GAGA principle. For details, we refer the reader to [Ber90, 3.4].

If $\mathcal{X}$ is a flat scheme of finite type over the valuation ring K° with generic fibre X, then the associated formal scheme $\hat{\mathcal{X}}$ over $\mathbb{K}^\circ$ is obtained by the π-adic completion of X for any $\pi \in K$ with $|\pi| < 1$. The special fibre of $\hat{\mathcal{X}}$ is the base change of the special fibre of $\mathcal{X}$ to $\tilde{\mathbb{K}}$. The generic fibre $\hat{\mathcal{X}}^{\mathrm{an}}$ is an analytic subdomain of X^{an} such that $\hat{\mathcal{X}}^{an}(\mathbb{K}) = \mathcal{X}(\mathbb{K}^\circ)$. If $\mathcal{X}$ is proper over K°, then $\hat{\mathcal{X}}^{\mathrm{an}} = X^{\mathrm{an}}$. For details, we refer the reader to [Gub07a, 2.7].

For convenience of the reader, we summarize here the *notational policy* of the paper: $\mathcal{X}$ denotes a flat algebraic scheme over $\mathbb{K}^\circ$, $\mathscr{X}$ is used for an admissible formal scheme over $\mathbb{K}^\circ$ and $\mathfrak{X}$ denotes a formal analytic variety over $\mathbb{K}$. The generic fibre in any case is usually denoted by X.

3. Chambert-Loir's measures

In this section, $\mathbb{K}$ denotes an algebraically closed field which is complete with respect to the non-trivial, non-archimedean absolute value $|\ |$. Let K be a subfield of $\mathbb{K}$ such that the valuation $v := -\log |\ |$ restricts to a discrete valuation on K. We will assume, as in our applications later on, that varieties are defined over K and we will perform analytic considerations over $\mathbb{K}$ using 2.7.

First, we will characterize admissible metrics on a line bundle by their fundamental properties. As in Arakelov geometry, metrics associated to $\mathbb{K}^\circ$-models are admissible and we want to include also canonical metrics on an abelian variety. Then we will recall the basic properties of Chambert-Loir's measures with respect to line bundles endowed with admissible metrics. These analogues of top-dimensional wedge products of first Chern forms were introduced in [Cha06] and later generalized in [Gub07a].

3.1 We recall some facts about metrics on line bundles. Let X be a proper scheme over K and let L be a line bundle on X. We consider metrics $\|\ \|$, $\|\ \|'$ on L which are continuous with respect to the analytic topology on L^{an}. Then we have the *distance of uniform convergence*

$$d(\|\ \|, \|\ \|') := \sup_{x \in X^{\mathrm{an}}} |\log(\|s(x)\|/\|s(x)\|')|.$$

The definition is independent of the choice of a non-zero $s(x) \in L_x$.

3.2 A *formal* $\mathbb{K}^\circ$*-model* of X is an admissible formal scheme over $\mathbb{K}^\circ$ together with an isomorphism $\mathscr{X}^{\mathrm{an}} \cong X^{\mathrm{an}}$. A *formal* $\mathbb{K}^\circ$*-model of* L *on* $\mathscr{X}$ is a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with an isomorphism $\mathscr{L}^{\mathrm{an}} \cong L^{\mathrm{an}}$.

For notational simplicity, we usually ignore the isomorphism between the generic fibre $\mathscr{X}^{\mathrm{an}}$ and X^{an}. We simply identify $\mathscr{X}^{\mathrm{an}}$ with X^{an}.

An *algebraic model* $\mathcal{X}$ of X over the discrete valuation ring K° of K is a scheme $\mathcal{X}$ which is flat and proper over K° and which has generic fibre (isomorphic to) X. We will also use formal $\mathbb{K}^\circ$-models for analytic spaces and line bundles in the same sense as above.

Example 3.3. If $\mathscr{L}$ is a formal $\mathbb{K}^\circ$-model of L on $\mathscr{X}$, then the *associated formal metric* $\|\ \|_{\mathscr{L}}$ on L is defined in the following way: if $\mathscr{U}$ is a formal trivialization of $\mathscr{L}$ and if $s \in \Gamma(\mathscr{U}, \mathscr{L})$ corresponds to $\gamma \in \mathscr{O}_{\mathscr{X}}(\mathscr{U})$, then $\|s(x)\| = |\gamma(x)|$ on $\mathscr{U}^{\mathrm{an}}$. Obviously, $\|\ \|_{\mathscr{L}}$ is continuous and independent of the choice of the trivialization. By 2.7, every algebraic model of L over K° induces a formal $\mathbb{K}^\circ$-model and hence an associated formal metric.

PROPOSITION 3.4. *For every line bundle L on a proper scheme X over K, there is a set $\hat{\mathfrak{g}}_X^+(L)$ of continuous metrics on L^{an} with the following properties:*

(a) *if $\| \ \|_i$ is a $\hat{\mathfrak{g}}_X^+(L_i)$-metric for $i=1,2$, then $\| \ \|_1 \otimes \| \ \|_2$ is a $\hat{\mathfrak{g}}_X^+(L_1 \otimes L_2)$-metric;*

(b) *for any $n \in \mathbb{N}\backslash\{0\}$, a metric $\| \ \|$ on L is a $\hat{\mathfrak{g}}_X^+(L)$-metric if and only if $\| \ \|^{\otimes n}$ is a $\hat{\mathfrak{g}}_X^+(L)^{\otimes n}$-metric;*

(c) *if $\varphi : Y \to X$ is a morphism of proper schemes over K and $\| \ \|$ is a $\hat{\mathfrak{g}}_X^+(L)$-metric, then $\varphi^*\| \ \|$ is a $\hat{\mathfrak{g}}_Y^+(\varphi^* L)$-metric;*

(d) *if φ is also surjective and $\| \ \|$ is any metric on L such that $\varphi^*\| \ \|$ is a $\hat{\mathfrak{g}}_Y^+(\varphi^* L)$-metric, then $\| \ \|$ is a $\hat{\mathfrak{g}}_X^+(L)$-metric;*

(e) *the set $\hat{\mathfrak{g}}_X^+(L)$ is closed with respect to uniform convergence;*

(f) *if $\mathscr{L}$ is a formal $\mathbb{K}^\circ$-model of L with numerically effective reduction $\tilde{\mathscr{L}}$, then the associated formal metric $\| \ \|_{\mathscr{L}}$ is in $\hat{\mathfrak{g}}_X^+(L)$.*

Proof. See [Gub03, Remark 10.3 and Proposition 10.4]. □

3.5 Taking the intersection over all possible $\hat{\mathfrak{g}}_X^+(L)$ in Proposition 3.4, we get a smallest set of continuous metrics on L^{an} satisfying the properties of Proposition 3.4. Such a metric is called a *semi-positive admissible metric*. A (continuous) metric $\| \ \|$ on L^{an} is called an *admissible metric* if and only if there is a surjective morphism $\varphi : X' \to X$ of proper schemes over K, line bundles M, N on X' with $\varphi^*(L) \cong M \otimes N^{-1}$ and semi-positive admissible metrics $\| \ \|_1$, $\| \ \|_2$ on M, N such that $\varphi^*\| \ \| = \| \ \|_1 / \| \ \|_2$.

PROPOSITION 3.6. *Admissible metrics of line bundles on a proper scheme X over K have the following properties.*

(a) *The tensor product of admissible metrics is again admissible.*

(b) *The dual metric of an admissible metric is admissible.*

(c) *The pull-back of an admissible metric with respect to a morphism $\varphi : Y \to X$ of proper schemes over K is an admissible metric.*

(d) *Every formal metric is admissible.*

Proof. The base change of a proper surjective morphism is again proper and surjective, which proves easily (a) and (c). Property (b) is trivial and (d) follows from [Gub03, Proposition 10.4]. □

Example 3.7. Let L be a line bundle on an abelian variety A over K. We will define a canonical metric on L and then we will show that it is admissible.

We choose a rigidification ρ of L, i.e. $\rho \in L_0(K)\backslash\{0\}$. We assume first that L is even. Then the theorem of the cube yields a canonical identification $[m]^*(L) = L^{\otimes m^2}$ of rigidified line bundles for every $m \in \mathbb{Z}$. There is a unique bounded metric $\| \ \|_\rho$ on L such that for all $m \in \mathbb{Z}$, we have $[m]^*\| \ \|_\rho = \| \ \|_\rho^{\otimes m^2}$. In fact, a variation of Tate's limit argument shows that

$$\| \ \|_\rho = \lim_{m \to \infty} [m]^*\| \ \|^{\otimes 1/m^2} \tag{2}$$

for every continuous metric $\| \ \|$ on L^{an} (see [BG06, Theorem 10.5.7]). If L is odd, then the same applies with m^2 replaced by m. Since any line bundle on A is the tensor product of an even one with an odd one, unique up to 2-torsion, we get a canonical metric $\| \ \|_\rho$ on every rigidified line bundle (L, ρ) of A.

A metric $\| \ \|$ on L is said to be *canonical* if there is a rigidification ρ of L such that $\| \ \|$ is equal to $\| \ \|_\rho$. A canonical metric is unique up to positive rational multiples [BG06, Remark 9.5.9] and we usually denote it by $\| \ \|_{\mathrm{can}}$. We claim that $\| \ \|_{\mathrm{can}}$ is admissible.

To see this, we assume first that L is ample and even. By Proposition 3.4(f), there is a semi-positive admissible metric $\| \ \|$ on L. Then Proposition 3.4(b) and (2) yield that $\| \ \|_{\mathrm{can}}$ is a semi-positive admissible metric. If L is just even, then there are ample even line bundles M, N on A with $L \cong M \otimes N^{-1}$ and we deduce that $\| \ \|_{\mathrm{can}}$ is admissible from the special case above and from Proposition 3.6.

If L is odd, then L is algebraically equivalent to 0. By definition, the latter means that we have K-rational points P and P_0 on an irreducible smooth projective curve C over K and a correspondence E in $C \times A$ such that the line bundle associated to the divisor $E([P] - [P_0])$ is isomorphic to L. Here, we use $E([P] - [P_0]) := (p_2)_*(E.p_1^*([P] - [P_0]))$, where p_i are the projections of $C \times A$. There are semi-stable $\mathbb{K}^\circ$-models $\mathscr{C}$ and $\mathscr{A}$ of C and A (for curves, this is well known and, for abelian varieties, see Examples 7.2 and 7.4). They are defined over the valuation ring F° of a finite extension F over the completion K_v. More precisely, there are semi-stable algebraic F°-models $\mathcal{C}$ and $\mathcal{A}$ of C_F and A_F such that the associated formal schemes over $\mathbb{K}^\circ$ are $\mathscr{C}$ and $\mathscr{A}$, respectively (see 2.7).

There is a divisor $\mathcal{D}$ on $\mathcal{C}$ with horizontal part $[P] - [P_0]$ and whose vertical part has rational coefficients such that the intersection numbers $\mathcal{D} \cdot Y$ are 0 for all irreducible components Y of $\tilde{\mathcal{C}}$. There is an F°-model $\mathcal{Z}$ of $C \times A$ with $\mathbb{K}^\circ$-morphisms $p_1 : \mathcal{Z} \to \mathcal{C}$ and $p_2 : \mathcal{Z} \to \mathcal{A}$ (extending the corresponding projections) such that the correspondence E extends to a correspondence $\mathcal{E}$ of $\mathcal{Z}$. We define $\mathcal{E}(\mathcal{D}) := (p_2)_*(\mathcal{E}.p_1^*(\mathcal{D}))$ as a $\mathbb{Q}$-divisor on $\mathcal{A}$. It is well known that $\mathcal{E}(\mathcal{D})$ induces the canonical metric $\| \ \|_{\mathrm{can}}$ of L. More precisely, if N is a common denominator for the coefficients of the vertical part of $\mathcal{D}$, then the line bundle associated to the divisor $\mathcal{E}(\mathcal{D})$ induces a formal $\mathbb{K}^\circ$-model of $L^{\otimes N}$ and $\| \ \|_{\mathrm{can}}^{\otimes N}$ is the associated formal metric. Moreover, we deduce that $\| \ \|_{\mathrm{can}}$ is a semi-positive admissible metric. For more details about these constructions, we refer the reader to [Gub03, Theorem 8.9 and Example 10.11].

If L is any line bundle on A, then we deduce that $\| \ \|_{\mathrm{can}}$ is admissible by linearity and by the special cases above.

In non-archimedean analysis, no good definition of Chern forms of metrized line bundles is known. However, Chambert-Loir has introduced a measure which is the analogue of top-dimensional wedge products of such Chern forms.

Proposition 3.8. *There is a unique way to associate to any d-dimensional geometrically integral proper variety X over K and to any family of admissibly metrized line bundles $\overline{L_1}, \ldots, \overline{L_d}$ on X a regular Borel measure $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ on X^{an} such that the following properties hold.*

(a) *$c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ is multilinear and symmetric in $\overline{L_1}, \ldots, \overline{L_d}$.*

(b) *If $\varphi : Y \to X$ is a morphism of d-dimensional geometrically integral proper varieties over K, then*

$$\varphi_*(c_1(\varphi^*\overline{L_1}) \wedge \cdots \wedge c_1(\varphi^*\overline{L_d})) = \deg(\varphi) c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d}).$$

(c) *If the metrics of* $\overline{L_1}, \dots, \overline{L_d}$ *are semi-positive and* $g \in C(X^{\mathrm{an}})$, *then*

$$\left| \int_{X^{\mathrm{an}}} g\, c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d}) \right| \leqslant |g|_{\sup} \deg_{L_1,\dots,L_d}(X).$$

(d) *If* $\mathscr{X}$ *is a formal* $\mathbb{K}^\circ$*-model of* X *with reduced special fibre and if the metric of* $\overline{L_j}$ *is induced by a formal* $\mathbb{K}^\circ$*-model* $\mathscr{L}_j$ *of* L *on* $\mathscr{X}$ *for every* $j = 1, \dots, d$, *then*

$$c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d}) = \sum_Y \deg_{\tilde{\mathscr{L}}_1,\dots,\tilde{\mathscr{L}}_d}(Y)\delta_{\xi_Y},$$

where Y *ranges over the irreducible components of* $\tilde{\mathscr{X}}$ *and* δ_{ξ_Y} *is the Dirac measure in the unique point* ξ_Y *of* X^{an} *which reduces to the generic point of* Y.

(e) *If* $L_1, \dots, L_d$ *are endowed with semi-positive admissible metrics* $\| \ \|_j$, *then* $\mu = c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ *is a positive Borel measure with* $\mu(X^{\mathrm{an}}) = \deg_{L_1,\dots,L_d}(X)$.

(f) *If we endow the set of positive regular Borel measures on* X^{an} *with the weak topology and if we fix the line bundles* $L_1, \dots, L_d$ *on* X, *then* $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ *is continuous with respect to the vector* $(\| \ \|_1, \dots, \| \ \|_d)$ *of semi-positive admissible metrics on* $L_1, \dots, L_d$.

Proof. For existence, we refer the reader to [Gub07a, § 3]. Uniqueness for formal metrics is clear by (d). The general case will be skipped. It follows from a repeated application of the minimality of semi-positive admissible metrics in 3.4. □

3.9 We consider a d-dimensional geometrically integral closed subvariety X of the abelian variety A and canonically metrized line bundles $\overline{L_1}, \dots, \overline{L_d}$ on A. Then $\mu := c_1(\overline{L}|_X) \wedge \cdots c_1(\overline{L}|_X)$ is called a *canonical measure* on X. It does not depend on the choice of the canonical metrics. Moreover, if one line bundle is odd, then $\mu = 0$ (see [Gub07a, 3.15]).

Remark 3.10. By finite base change of K and then using linearity in the irreducible components, we may extend Chambert-Loir's measures to all proper schemes X over K of pure dimension d.

4. Raynaud extensions and Mumford models

In this section, $\mathbb{K}$ denotes an algebraically closed field with a non-trivial, non-archimedean complete absolute value $| \ |$, valuation $v := -\log | \ |$ and value group Γ. We fix an abelian variety A over $\mathbb{K}$.

First, we recall some results of Bosch and Lütkebohmert about the Raynaud extension of A. To simplify the exposition, we will replace cubical line bundles by the use of metrics. Then we explain Mumford's construction, which gives an admissible formal $\mathbb{K}^\circ$-model $\mathscr{A}$ associated to certain polytopal decompositions of $\mathbb{R}^n$. Moreover, we will relate ample line bundles on A and their models on $\mathscr{A}$ to affine convex functions. At the end, we will define the tropical variety of a closed subvariety of A, which is a periodic polytopal set in $\mathbb{R}^n$.

4.1 To define the Raynaud extension of A, we will follow the rigid analytic presentation of Bosch and Lütkebohmert [BL91, § 1] and adapt it to Berkovich analytic spaces as in [Ber90, § 6.5]. There is a formal group scheme $\mathscr{A}_1$ over $\mathbb{K}$ with generic fibre A_1 and a homomorphism $A_1 \to A^{\mathrm{an}}$ of analytic groups over $\mathbb{K}$ inducing an isomorphism onto an analytic subdomain of A^{an} such that $\mathscr{A}_1$ has semi-abelian reduction. Moreover, $\mathscr{A}_1$ and the homomorphism $A_1 \to A^{\mathrm{an}}$ are unique up to isomorphism and hence we may identify A_1 with a compact subgroup of A^{an}.

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It is convenient here to work in the category of formal analytic varieties, as we may identify the objects with their generic fibres using a coarser topology (see 2.5). Since $\mathscr{A}_1$ has semi-stable reduction, the special fibre is reduced and A_1 has the structure of a formal analytic group. Let T_1 be the maximal formal affinoid subtorus in A_1. Then semi-stable reduction means that there is a unique formal abelian scheme $\mathscr{B}$ over $\mathbb{K}^\circ$ with generic fibre B such that we have an exact sequence

$$1 \to T_1 \to A_1 \xrightarrow{q_1} B \to 0 \tag{3}$$

of formal analytic groups. Note that we may identify T_1 with the compact analytic subgroup $\{|x_1| = \cdots = |x_n| = 1\}$ of $T = (\mathbb{G}_m^n)^{\mathrm{an}}$. The *uniformization* E of A is given as an analytic group by $E := (A_1 \times T)/T_1$, where T_1 acts on $A_1 \times T$ by $t_1 \cdot (a,t) := (t_1 + a, t_1^{-1} \cdot t)$. Using the canonical maps, we get an exact sequence

$$1 \to T \to E \xrightarrow{q} B \to 0 \tag{4}$$

of analytic groups. The closed immersion $T_1 \to A_1$ extends uniquely to a homomorphism $T \to A^{\mathrm{an}}$ of analytic groups and hence we get a unique extension of $A_1 \hookrightarrow A^{\mathrm{an}}$ to a homomorphism $p : E \to A^{\mathrm{an}}$ of analytic groups. The kernel M of p is a discrete subgroup of $E(\mathbb{K})$ and the homomorphism $E/M \to A^{\mathrm{an}}$ induced by p is an isomorphism. The exact sequences (3) and (4) are called the *Raynaud extensions* of A. We will write the group structure on the uniformization E multiplicatively. We call n the *torus rank* of A.

By [BGR84, Theorem 6.13], the formal abelian scheme $\mathscr{B}$ is algebraizable and the GAGA principle shows that the same is true for the Raynaud extension (4).

There are two extreme cases of abelian varieties over K. First, we have abelian varieties of *good reduction* at v, which means that the torus part T of the Raynaud extension is trivial. On the other hand, we have the abelian varieties with *totally degenerate reduction* at v, which means that the abelian part B of the Raynaud extension is trivial.

4.2 The Raynaud extension (3) is locally trivial, i.e. there is an open atlas $\mathfrak{T}$ of $\mathscr{B}^{\mathrm{f-an}}$ by formal affinoid varieties V such that $q_1^{-1}(V) \cong V \times T_1$. This follows easily from the corresponding fact for semi-abelian varieties applied to the reduction of (3) (see [BL91, p. 655]). For every V, we fix such a trivialization given by a section $s_V : V \to A_1$. The transition functions $g_{VW} := s_V - s_W$ are maps from $V \cap W$ to T_1. As usual, we fix coordinates $x_1, \ldots, x_n$ on $T = (\mathbb{G}_m^n)^{\mathrm{an}}$. The functions $x_1, \ldots, x_n$ are defined on the trivialization $V \times T$ of E by pull-back, but they do not extend to E. However, the functions $|x_1|, \ldots, |x_n|$ are well-defined on E independently of the choice of the formal affinoid atlas $\mathfrak{T}$. Using $p(x_j) = |x_j|(p)$, we get a well-defined continuous surjective map

$$\mathrm{val} : E \longrightarrow \mathbb{R}^n, \quad p \mapsto (-\log p(x_1), \ldots, -\log p(x_n)).$$

We will see at the end of this section that this map has similar properties as in tropical algebraic geometry, where one considers the special case $T = E$ and where no abelian variety is behind the construction. (In tropical algebraic geometry, this map is called the tropicalization map and it is also denoted by val to emphasize that it is obtained on rational points by applying the valuation to the coordinates.) Note that val maps the discrete subgroup M of A^{an} isomorphically onto a complete lattice Λ in $\mathbb{R}^n$ [BL91, Theorem 1.2] and hence val induces a continuous surjective map

$$\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda.$$

We will construct in Example 7.2 a natural homeomorphism ι of $\mathbb{R}^n/\Lambda$ onto a compact subset $S(A)$ (called the skeleton) of A^{an}. By [Ber90, § 6.5], $\overline{\mathrm{val}} \circ \iota$ gives a proper deformation retraction of A onto $S(A)$.

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If χ is an element of the character group $\check{T}$ of T, then the units $\chi^{-1}(g_{VW})$ are transition functions of a formal line bundle $\mathscr{O}_\chi$ on $\mathscr{B}$. Obviously, s_V induces a trivialization $s_{V+b}(x) = s_V(x-b)+a$ of A_1 for all $a \in A_1$ with $q_1(a)=b$ and $x \in V+b$. Hence, O_χ is a translation-invariant line bundle proving that $O_\chi \in \mathrm{Pic}^\circ(B)$ and the same argument shows that the special fibre $\tilde{\mathscr{O}}_\chi \in \mathrm{Pic}^\circ(\tilde{\mathscr{B}})$. The translation invariance of $\mathscr{O}_\chi$ can also be seen from the fact that $\mathscr{O}_\chi$ is given by the formal group scheme extension of $\mathscr{B}$ by the formal multiplicative group obtained from the push-forward of the Raynaud extension by the character χ. We have the description

$$E = \mathrm{Spec}\left(\bigoplus_{\chi \in \check{T}} O_\chi\right)$$

of the Raynaud extension, which is easily obtained by using the Laurent series development on the trivialization $V \times T$. Note that $q^* O_\chi$ is trivial on E with canonical nowhere vanishing section e_χ given by the function χ on the trivialization $V \times T$ of E. Additional information for this and the next paragraph can also be found in the book by Fresnel and van der Put [FvdP04, ch. 6].

4.3 Next, we describe a line bundle L on A using the uniformization E. By $A^{\mathrm{an}} = E/M$, we see that $p^*(L^{\mathrm{an}})$ is equipped with an M-action α such that L^{an} may be recovered from $p^*(L^{\mathrm{an}})$ by passing to the quotient with respect to α. There is a formal line bundle $\mathscr{H}$ on $\mathscr{B}$ with generic fibre denoted by H such that $q^*(H)$ is isomorphic to $p^*(L^{\mathrm{an}})$ (see [BL91, Proposition 4.4]). We fix such an isomorphism to get the identification $q^*(H) = p^*(L^{\mathrm{an}})$. Then $q^*(\mathscr{H})$ is a formal $\mathbb{K}^\circ$-model of $p^*(L^{\mathrm{an}})$ and, as in Example 3.3, we get a formal metric $q^*\| \ \|_{\mathscr{H}}$ on $p^*(L^{\mathrm{an}})$. There is a cocycle Z of $H^1(M, (\mathbb{R}^\times)^E)$ such that

$$(q^*\|\alpha_\gamma(w)\|_{\mathscr{H}})_{\gamma \cdot x} = Z_\gamma(x)^{-1} \cdot (q^*\|w\|_{\mathscr{H}})_x$$

for all $\gamma \in M$, $x \in E$ and $w \in (p^* L^{\mathrm{an}})_x$. By the description of the action α given in [BL91, Proposition 4.9], it is easy to deduce that $Z_\gamma(x)$ depends only on $\mathrm{val}(x)$. For $\lambda \in \Lambda$, we get a unique function $z_\lambda : \mathbb{R}^n \to \mathbb{R}$ with

$$z_\lambda(\mathrm{val}(x)) = -\log(Z_\gamma(x)) \quad (\gamma \in M, x \in E, \lambda = \mathrm{val}(\gamma)).$$

Moreover, the same consideration shows that

$$z_\lambda(\mathbf{u}) = z_\lambda(\mathbf{0}) + b(\mathbf{u}, \lambda) \quad (\mathbf{u} \in \mathbb{R}^n, \lambda \in \Lambda)$$

for a symmetric bilinear form $b : \Lambda \times \Lambda \to \mathbb{Z}$. By [BL91, Theorem 6.13], L is ample if and only if H is ample on B and b is positive definite on Λ. We note also that the bilinear form b is trivial if $L \in \mathrm{Pic}^\circ(A)$ (use [BL91, Corollary 4.11]).

4.4 We now fix the notation used from convex geometry (see [Gub07a, § 6.1 and Appendix A] for more details). A polytope Δ of $\mathbb{R}^n$ is called Γ-*rational* if it may be given as an intersection of half-spaces of the form $\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{m} \cdot \mathbf{u} \geqslant c\}$ for suitable $\mathbf{m} \in \mathbb{Z}^n$ and $c \in \Gamma$. If $\Gamma = \mathbb{Q}$, then such a polytope is called *rational*. The relative interior of Δ is denoted by $\mathrm{relint}(\Delta)$. A *closed face* of Δ is either the polytope Δ itself or is equal to $H \cap \Delta$, where H is the boundary of a half-space of $\mathbb{R}^n$ containing Δ. An *open face* of Δ is the relative interior of a closed face.

A *polytopal decomposition* of $\Omega \subset \mathbb{R}^n$ is a locally finite family $\mathscr{C}$ of polytopes contained in Ω which includes all faces, which is face to face and which covers Ω. A *subdivision* $\mathscr{D}$ of $\mathscr{C}$ is a polytopal decomposition of Ω such that every $\Delta \in \mathscr{C}$ has a polytopal decomposition in $\mathscr{D}$.

We use the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$, $\mathbf{u} \mapsto \overline{\mathbf{u}}$ to translate the above notions. A *polytope* $\overline{\Delta}$ in $\mathbb{R}^n/\Lambda$ is given by a polytope Δ in $\mathbb{R}^n$ which maps bijectively onto $\overline{\Delta}$. A *polytopal decomposition* $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$ is a finite family of polytopes in $\mathbb{R}^n/\Lambda$ induced by a Λ-periodic polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^n$.

We define convex functions as in analysis (and not as in the theory of toric varieties). A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is called *strongly polyhedral* with respect to the polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^n$ if the n-dimensional polytopes in $\mathscr{C}$ are the maximal subsets of $\mathbb{R}^n$, where f is affine.

4.5 A Γ-rational polytope Δ induces a *polytopal domain* $U_\Delta := \mathrm{val}^{-1}(\Delta)$ of the torus T with affinoid algebra

$$\mathbb{K}\langle U_\Delta\rangle := \left\{ \sum_{\mathbf{m}\in\mathbb{Z}^n} a_{\mathbf{m}} x_1^{m_1}\cdots x_n^{m_n} \;\middle|\; \lim_{|\mathbf{m}|\to\infty} v(a_{\mathbf{m}}) + \mathbf{m}\cdot\mathbf{u} = \infty\ \forall \mathbf{u}\in\Delta \right\}$$

(see [Gub07a, Proposition 4.1]). We need the following generalization.

LEMMA 4.6. *Let V be an affinoid variety with affinoid algebra $\mathscr{O}(V)$. Then every $h \in \mathscr{O}(V \times U_\Delta)$ has a Laurent series development*

$$h = \sum_{\mathbf{m}\in\mathbb{Z}^n} a_{\mathbf{m}} x_1^{m_1}\cdots x_n^{m_n} \tag{5}$$

for uniquely determined $a_{\mathbf{m}} \in \mathscr{O}(V)$ and the supremum semi-norm is given by

$$|h|_{\sup} = \sup_{\mathbf{u}\in\Delta,\mathbf{m}\in\mathbb{Z}^n} |a_{\mathbf{m}}|_{\sup} e^{-\mathbf{m}\cdot\mathbf{u}}. \tag{6}$$

The supremum in (6) is a maximum achieved in a vertex $\mathbf{u}$ of Δ. If V is connected, then h is a unit in $\mathscr{O}(V \times U_\Delta)$ if and only if there is $\mathbf{m}_0 \in \mathbb{Z}^n$ such that $|a_{\mathbf{m}_0}(y)\mathbf{x}^{\mathbf{m}_0}| > |a_{\mathbf{m}}(y)\mathbf{x}^{\mathbf{m}}|$ for all $\mathbf{x} \in U_\Delta$, $y \in V$ and $\mathbf{m} \neq \mathbf{m}_0$.

Conversely, a Laurent series as in (5) is in $\mathscr{O}(V \times U_\Delta)$ if and only if $-\log\|a_{\mathbf{m}}\| + \mathbf{m}\cdot\mathbf{u}$ tends to ∞ for $|\mathbf{m}| \to \infty$, where $\|\ \|$ is any Banach norm on the affinoid algebra $\mathscr{O}(V)$.

Proof. The description of $\mathscr{O}(V \times U_\Delta)$ as the set of Laurent series (5) is a direct generalization of [Gub07a, Proposition 4.1]. The proof follows the same arguments and will be omitted. It remains to prove the characterization of the units.

If $h \in \mathscr{O}(V)$ has such a dominant term $a_{\mathbf{m}_0}(y)\mathbf{x}^{\mathbf{m}_0}$, then $a_{\mathbf{m}_0}$ has no zeros on V and hence Hilbert's *Nullstellensatz* for affinoid algebras [BGR84, Proposition 7.1.3/1] shows that $a_{\mathbf{m}_0} \in \mathscr{O}(V)^\times$. We may assume that $\mathbf{m}_0 = \mathbf{0}$ and $a_{\mathbf{0}} = 1$. Then we have $h = 1 - h_1$ for $h_1 \in \mathscr{O}(V \times U_\Delta)$ with $|h_1|_{\sup} < 1$ and hence

$$h^{-1} = \sum_{m=0}^{\infty} h_1^m \in \mathscr{O}(V \times U_\Delta).$$

If $h \in \mathscr{O}(V \times U_\Delta)$ has no such dominant term, then there are $\mathbf{x} \in U_\Delta$, $y \in V$ and $\mathbf{m}_0 \neq \mathbf{m}_1 \in \mathbb{Z}^n$ with

$$|a_{\mathbf{m}_0}(y)\mathbf{x}^{\mathbf{m}_0}| = |a_{\mathbf{m}_1}(y)\mathbf{x}^{\mathbf{m}_1}| = |h|_{\sup}.$$

Let $\mathbf{u} := \mathrm{val}(\mathbf{x})$ and let W be the affinoid subdomain of $V \times U_\Delta$ given by $\mathrm{val}^{-1}(\mathbf{u})$. It is isomorphic to $V \times T_1$ for the affinoid torus $T_1 = \{|x_1| = \cdots = |x_n| = 1\}$ in T. Since the restriction of h to W has no dominant term as well, we get $h|_W \notin \mathscr{O}(W)^\times$ [BGR84, Lemma 9.7.1/1] and hence $h \notin \mathscr{O}(V \times U_\Delta)^\times$. □

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4.7 Next, we define a formal $\mathbb{K}^\circ$-model $\mathscr{A}$ of A associated to a Γ-rational polytopal decomposition $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$. In the algebraic framework, this is a construction of Mumford [Mum72] which is useful for compactifying moduli spaces of abelian varieties (see [FC89]). We denote by $\mathscr{C}$ the Γ-rational Λ-periodic polytopal decomposition of $\mathbb{R}^n$ which induces $\overline{\mathscr{C}}$.

We choose a formal affinoid atlas $\mathfrak{T}$ as in 4.2. For $V \in \mathfrak{T}$ with trivialization $q_1^{-1}(V) \cong V \times T_1$ and $\Delta \in \mathscr{C}$, we define the affinoid subdomain

$$U_{V,\Delta} := q^{-1}(V) \cap \mathrm{val}^{-1}(\Delta) \cong V \times U_\Delta \tag{7}$$

of E, where the term on the right is in the trivialization $q^{-1}(V) \cong V \times T$. The sets $U_{V,\Delta}$ form a formal analytic atlas on E inducing a formal analytic variety $\mathscr{E}^{\mathrm{f-an}}$ with corresponding formal $\mathbb{K}^\circ$-model $\mathscr{E}$ of E. We note that $\mathscr{E}$ has a formal affine open covering by the sets $\mathscr{U}_{V,\Delta} := U_{V,\Delta}^{\mathrm{f-sch}}$.

We may assume that $\mathfrak{T}$ is closed under translation with elements of $q(M)$. We may form the quotient of $\mathscr{E}^{\mathrm{f-an}}$ by M, leading to a formal analytic structure on A^{an}. The associated formal $\mathbb{K}^\circ$-model $\mathscr{A}$ of A (see 2.6) is called the *Mumford model* associated to $\overline{\mathscr{C}}$. It has a covering by formal affine open subsets $\mathscr{U}_{[V,\Delta]}$ obtained by gluing $\mathscr{U}_{V+q(\gamma),\Delta+\mathrm{val}(\gamma)}$ for all $\gamma \in M$. Obviously, $\mathscr{A}$ is independent of the choice of $\mathfrak{T}$. Note that we have canonical morphisms $q : \mathscr{E} \to \mathscr{B}$ and $p : \mathscr{E} \to \mathscr{A}$ extending the corresponding maps on generic fibres.

Recall that the strata of a variety were introduced in § 1.2. The next result describes the strata of the special fibre of a Mumford model.

Proposition 4.8. *Let $\mathscr{A}$ be the Mumford model of A associated to the Γ-rational polytopal decomposition $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$. Let $\mathscr{E}$ be the formal $\mathbb{K}^\circ$-model of E associated to the polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^n$ which was used in 4.7 to construct $\mathscr{A}$.*

(a) *The formal torus $\mathbb{T}_1 = \mathrm{Spf}(\mathbb{K}^\circ\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}\rangle)$ acts canonically on $\mathscr{E}$ inducing a $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-action on the special fibre $\tilde{\mathscr{E}}$.*

(b) *There is a bijective order reversing correspondence between strata Z of $\tilde{\mathscr{E}}$ and open faces τ of $\mathscr{C}$. It is given by*

$$\tau = \mathrm{val}(\pi^{-1}(Z)), \quad Z = \pi(\mathrm{val}^{-1}(\tau)),$$

where $\pi : E \to \tilde{\mathscr{E}}$ is the reduction map. We have $\dim(Z) + \dim(\tau) = \dim(A)$.

(c) *There is a bijective order reversing correspondence between strata W of $\tilde{\mathscr{A}}$ and open faces $\overline{\tau}$ of $\overline{\mathscr{C}}$. It is given by*

$$\overline{\tau} = \overline{\mathrm{val}}(\pi^{-1}(W)), \quad W = \pi(\overline{\mathrm{val}}^{-1}(\overline{\tau})),$$

where $\pi : A \to \tilde{\mathscr{A}}$ is the reduction map. We have $\dim(W) + \dim(\overline{\tau}) = \dim(A)$.

(d) *Every irreducible component Y' of $\tilde{\mathscr{E}}$ is mapped isomorphically onto an irreducible component Y of $\tilde{\mathscr{A}}$. By (c), we get a bijective correspondence between irreducible components of $\tilde{\mathscr{A}}$ and vertices of $\overline{\mathscr{C}}$. Moreover, $\tilde{q} : Y' \to \tilde{\mathscr{B}}$ is a fibre bundle whose fibre is a $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-toric variety.*

Proof. By construction, $\mathbb{T}_1^{\mathrm{f-an}}$ acts on $\mathscr{E}^{\mathrm{f-an}}$ and (a) follows. To prove (b), we note that strata are compatible with localization and hence we may consider a formal affinoid chart $U_{V,\Delta} \cong V \times U_\Delta$ as in (7). By [Gub07a, Proposition 4.4], it follows that the strata of $\tilde{U}_\Delta$ are the same as the $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-orbits and they correspond to the open faces of Δ. The strata of $\tilde{U}_{V,\Delta}$ are the preimages of the strata of $\tilde{U}_\Delta$, leading to the desired correspondence. The other claims in (b) follow also from the corresponding statements for U_Δ given in [Gub07a, Proposition 4.4].

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By (b) and the construction of $\mathscr{A}$, $\tilde{p}$ maps a stratum of $\tilde{\mathscr{E}}$ isomorphically onto a stratum of $\tilde{\mathscr{A}}$ and hence (c) follows easily from (b). To prove (d), let $\mathbf{u}$ be the vertex of $\mathscr{C}$ corresponding to the irreducible component Y' by (b). Let $\Delta \in \mathscr{C}$ with vertex $\mathbf{u}$. In the trivialization (7), Y' is given by $\tilde{V} \times Y_{\Delta,\mathbf{u}}$, where $Y_{\Delta,\mathbf{u}}$ is the affine toric variety given by the local cone of Δ in $\mathbf{u}$ (see [Gub07a, Proposition 4.4(d)]). If Δ ranges over $\mathscr{C}$, we see that Y' has over V the form $V \times Y_{\mathbf{u}}$, where $Y_{\mathbf{u}}$ is the $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-toric variety given by the fan of local cones of the polytopes $\Delta \in \mathscr{C}$ in the vertex $\mathbf{u}$. This can be done for every $V \in \mathfrak{T}$ to cover Y'. We note that Y' is the union of the strata corresponding to the open faces τ of $\mathscr{C}$ with vertex $\mathbf{u}$. Since Y' is locally isomorphic to Y and no gluing arises with respect to the M-action, we easily deduce (d). $\square$

Remark 4.9. Let $\Delta \in \mathscr{C}$ with relative interior τ. We denote by $\mathbb{L}_\Delta$ the linear subspace of $\mathbb{R}^n$ generated by $\Delta - \mathbf{u}$, $\mathbf{u} \in \Delta$. Then $N_\Delta := \mathbb{L}_\Delta \cap \mathbb{Z}^n$ is a complete lattice in $\mathbb{L}_\Delta$ and we get a subtorus H_Δ of $\tilde{T} = (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ with $H_\Delta(\tilde{\mathbb{K}}) = N_\Delta \otimes_{\mathbb{Z}} \tilde{\mathbb{K}}^\times$. It follows from the above proof and [Gub07a, Remark 4.8] that the stratum of $\tilde{\mathscr{E}}$ associated to τ is a $\tilde{T}/H_\Delta$-torsor over $\tilde{\mathscr{B}}$ with respect to $\tilde{q}$.

4.10 Next, we describe $\mathbb{K}^\circ$-models of the line bundle L on A. They should be defined on a given Mumford model $\mathscr{A}$ of A associated to the Γ-rational polytopal decomposition $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$. As in 4.3, we choose a formal line bundle $\mathscr{H}$ on $\mathscr{B}$ with $q^*(H) = p^*(L^{\mathrm{an}})$ on the uniformization E of A such that $L^{\mathrm{an}} = q^*(H)/M$ on $A^{\mathrm{an}} = E/M$ and which leads to a cocycle $z_\lambda(\mathbf{u})$ with respect to $\lambda \in \Lambda$. We fix a formal affine atlas of $\mathscr{B}$ which trivializes the line bundle $\mathscr{H}$ and which induces a formal affinoid trivialization $\mathfrak{T}$ for $q_1 : A_1 \to B$.

Proposition 4.11. *There is a bijective correspondence between isomorphism classes of formal $\mathbb{K}^\circ$-models $\mathscr{L}$ of L on $\mathscr{A}$ which are trivial over the formal open subsets $\mathscr{U}_{[V,\Delta]}$, where $\Delta \in \mathscr{C}$, $V \in \mathfrak{T}$, and continuous real functions f on $\mathbb{R}^n$ satisfying the following two conditions.*

(a) *For $\Delta \in \mathscr{C}$, there are $\mathbf{m}_\Delta \in \mathbb{Z}^n$ and $c_\Delta \in \Gamma$ with $f(\mathbf{u}) = \mathbf{m}_\Delta \cdot \mathbf{u} + c_\Delta$ on Δ.*

(b) *$f(\mathbf{u} + \lambda) = f(\mathbf{u}) + z_\lambda(\mathbf{u})$ $(\lambda \in \Lambda, \mathbf{u} \in \mathbb{R}^n)$.*

Let $\| \ \|_{\mathscr{L}}$ be the formal metric of L associated to $\mathscr{L}$ (see Example 3.3). Then the corresponding $f_{\mathscr{L}} : \mathbb{R}^n \to \mathbb{R}$ is uniquely determined by

$$f_{\mathscr{L}} \circ \mathrm{val} = -\log \circ (p^*\| \ \|_{\mathscr{L}}/q^*\| \ \|_{\mathscr{H}}), \tag{8}$$

where the quotient of the metrics on $q^*(H) = p^*(L^{\mathrm{an}})$ is evaluated at any non-zero local section.

Proof. Let $\mathscr{L}$ be a formal $\mathbb{K}^\circ$-model of L on A which is trivial on every $\mathscr{U}_{[V,\Delta]}$. Using the identification $q^*(H) = p^*(L^{\mathrm{an}})$, we may view $p^*\| \ \|_{\mathscr{L}}/q^*\| \ \|_{\mathscr{H}}$ as a metric on O_E. The corresponding real function is obtained by evaluating this metric at the constant section 1. Since formal metrics are continuous, the right-hand side of (8) is a continuous function on E.

Our first goal is to show that this function descends to $\mathbb{R}^n$, i.e. there is $f_{\mathscr{L}} : \mathbb{R}^n \to \mathbb{R}$ with (8). We choose a connected $V \in \mathfrak{T}$ and $\Delta \in \mathscr{C}$. By assumption, the formal affine open subset $\mathscr{V}$ with generic fibre V trivializes the formal line bundle $\mathscr{H}$, i.e. there is a nowhere-vanishing section $s_V \in \Gamma(\mathscr{V}, \mathscr{H})$. We may consider s_V as a section of $H|_V$ with $\|s_V\|_{\mathscr{H}} = 1$ on V. By assumption, $\mathscr{L}$ is trivial on $\mathscr{U}_{[V,\Delta]}$ and hence there is a nowhere-vanishing section $t_V \in \Gamma(V, L)$ with $\|t_V\|_{\mathscr{L}} = 1$ on $U_{[V,\Delta]}$. We apply Lemma 4.6 to the unit $h := q^*(s_V)/p^*(t_V)$ on $U_{V,\Delta} \cong V \times U_\Delta$; hence, there are $\mathbf{m}_\Delta \in \mathbb{Z}^n$ and $a_{V,\Delta} \in \mathscr{O}(V)^\times$ with

$$p^*\| \ \|_{\mathscr{L}}/q^*\| \ \|_{\mathscr{H}} = |h| = |a_{V,\Delta}\mathbf{x}^{\mathbf{m}_\Delta}| \tag{9}$$

on $U_{V,\Delta}$. A priori, $\mathbf{m}_\Delta$ depends also on V but, since the functions $|x_i|$ are well defined on E, it follows easily from (9) that we may select $\mathbf{m}_\Delta$ independently from $V \in \mathfrak{T}$. We conclude that $|a_{V,\Delta}| = |a_{W,\Delta}|$ on $V \cap W$ for every $V, W \in \mathfrak{T}$. If we vary $V \in \mathfrak{T}$ keeping Δ fixed, we get a formal $\mathbb{K}^\circ$-model $\mathscr{G}$ of O_B on $\mathscr{B}$, given by trivializations $a_{V,\Delta} \in \mathscr{O}(V)^\times$. Since the special fibre $\tilde{\mathscr{B}}$ is smooth, the formal metric $\| \ \|_{\mathscr{G}}$ on O_B induces a constant function $\|1\|_{\mathscr{G}}$ (see [Gub03, Proposition 7.6]). This means that $|a_{V,\Delta}|$ is constant on V and hence there is $a_\Delta \in \mathbb{K}^\times$ with $|a_{V,\Delta}| = |a_\Delta|$ for all $V \in \mathfrak{T}$. For $x \in U_{V,\Delta}$, we conclude that $|h(x)|$ in (9) depends only on $\mathrm{val}(x)$ and hence there is a unique function $f_{\mathscr{L}}$ with (8). Moreover, we have proved that (a) holds with $c_\Delta := v(a_\Delta)$. Since $\mathscr{C}$ is a polytopal complex, it is clear that continuity of $f_{\mathscr{L}}$ follows from (a).

Finally, we prove (b) for $f_{\mathscr{L}}$. Let $x \in E$ with $\mathrm{val}(x) = \mathbf{u}$ and let $\gamma \in M$ with $\mathrm{val}(\gamma) = \lambda$. Then (b) follows from

$$\begin{aligned} f_{\mathscr{L}} \circ \mathrm{val}(\gamma \cdot x) &= -\log((p^*\| \ \|_{\mathscr{L}})_{\gamma\cdot x}/(q^*\| \ \|_{\mathscr{H}})_{\gamma\cdot x}) \\ &\overset{4.3}{=} -\log((p^*\| \ \|_{\mathscr{L}})_{x}/(e^{z_\lambda(\mathbf{u})}q^*\| \ \|_{\mathscr{H}})_{x}) \\ &= f_{\mathscr{L}}(\mathbf{u}) + z_\lambda(\mathbf{u}). \end{aligned}$$

Conversely, let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying (a) and (b). We define a metric $\| \ \|'$ on $p^*(L^{\mathrm{an}}) = q^*(H)$ by

$$\| \ \|'/q^*\| \ \|_{\mathscr{H}} = e^{-f\circ \mathrm{val}}.$$

As a consequence of (b), $\| \ \|'$ descends to a metric $\| \ \|_f$ on $L^{\mathrm{an}} = p^*(L^{\mathrm{an}})/M$. It is uniquely characterized by the property

$$f \circ \mathrm{val} = -\log(p^*\| \ \|_f/q^*\| \ \|_{\mathscr{H}}). \tag{10}$$

We choose $\mathbf{m}_\Delta \in \mathbb{Z}^n$ and $c_\Delta \in \Gamma$ from (a). There is $a_\Delta \in \mathbb{K}^\times$ with $c_\Delta = v(a_\Delta)$. For $V \in \mathfrak{T}$ and $\Delta \in \mathscr{C}$, the metric $p^*\| \ \|_f$ is given on $U_{V,\Delta}$ by

$$p^*\| \ \|_f/q^*\| \ \|_{\mathscr{H}} = |a_\Delta| \cdot |\mathbf{x}^{\mathbf{m}_\Delta}| \tag{11}$$

as a consequence of (10). Using $s_V \in \Gamma(V, H)$ from above, we deduce that the nowhere-vanishing section $t_{V,\Delta} := (a_\Delta \mathbf{x}^{\mathbf{m}_\Delta})^{-1} \cdot q^*(s_V) \in \Gamma(U_{V,\Delta}, q^*(H))$ satisfies

$$p^*\|t_{V,\Delta}\|_f = 1$$

on $U_{V,\Delta}$. We may view $(t_{V,\Delta})_{V\in\mathfrak{T},\Delta\in\mathscr{C}}$ as a family of frames of L^{an} of constant $\| \ \|_f$-metric 1 and hence $\| \ \|_f$ is the metric on L associated to a unique formal $\mathbb{K}^\circ$-model $\mathscr{L}_f$ of L [Gub98, Proposition 7.5], as desired.

It remains to show that $f \mapsto \mathscr{L}_f$ is inverse to $\mathscr{L} \mapsto f_{\mathscr{L}}$. Here, the same argument as for [Gub07a, Proposition 6.6] applies. □

PROPOSITION 4.12. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $\mathbb{R}^n/\Lambda$ with associated Mumford model $\mathscr{A}$ of A over $\mathbb{K}^\circ$ obtained from the formal $\mathbb{K}^\circ$-model $\mathscr{E}$ of the Raynaud extension E as in 4.7. We assume that there is a $\mathbb{K}^\circ$-model $\mathscr{L}$ of L on $\mathscr{A}$ as in Proposition 4.11 corresponding to the affine function $f_{\mathscr{L}}$. Let $p : \mathscr{E} \to \mathscr{A}$ be the quotient map. Then $(p^*\mathscr{L})^\sim$ is relatively ample with respect to the canonical reduction $\tilde{q} : \tilde{\mathscr{E}} \to \tilde{\mathscr{B}}$ if and only if $f_{\mathscr{L}}$ is a strongly polyhedral convex function with respect to $\mathscr{C}$ (see 4.4).*

Proof. Let $\mathbf{u}$ be a vertex of $\mathscr{C}$. By Proposition 4.8, we get a corresponding irreducible component $Y'_{\mathbf{u}}$ of $\tilde{\mathscr{E}}$. Moreover, we have seen that $Y'_{\mathbf{u}}$ is a fibre bundle over $\tilde{\mathscr{B}}$ which is trivial over $\tilde{\mathscr{V}} = \tilde{V}$ for every $V \in \mathfrak{T}$ with associated formal scheme $\mathscr{V}$ over $\mathbb{K}^\circ$. The fibre $Z_{\mathbf{u}}$ of the bundle is the $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-toric variety associated to the local cones of the polytopes $\Delta \in \mathscr{C}$ with vertex $\mathbf{u}$. We claim that the restriction of $(p^*\mathscr{L})^\sim$ to the trivialization $\tilde{V} \times Z_{\mathbf{u}}$ is the pull-back of a line bundle on $Z_{\mathbf{u}}$. Indeed, the toric variety $Z_{\mathbf{u}}$ is given by pasting the family

$$(U_\Delta)^\sim := \operatorname{Spec}(\tilde{\mathbb{K}}[\tilde{\mathbf{x}}^{S_\Delta}]),$$

where Δ is ranging over the polytopes of $\mathscr{C}$ with vertex $\mathbf{u}$ and

$$\tilde{\mathbf{x}}^{S_\Delta} := \{\tilde{\mathbf{x}}^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^n,\ \mathbf{u}' \cdot \mathbf{m} \geqslant 0\ \forall \mathbf{u}' \in \Delta - \mathbf{u}\}.$$

For such a Δ, we use the presentation $f_{\mathscr{L}}(\mathbf{u}') = \mathbf{m}_\Delta \cdot \mathbf{u}' + c_\Delta$ from Proposition 4.11(a). If we change the identification $p^*(L^{\mathrm{an}}) = q^*(H)$, then $\| \ \|_{\mathscr{L}}$ is replaced by a positive multiple and hence we may assume that $f_{\mathscr{L}}(\mathbf{u}) = 0$. There is $a_\Delta \in \mathbb{K}^\times$ with $c_\Delta = v(a_\Delta)$. The functions $\tilde{a}_\Delta \tilde{\mathbf{x}}^{\mathbf{m}_\Delta}|_{(U_\Delta)^\sim}$ define a $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$-equivariant Cartier divisor D on $Z_{\mathbf{u}}$. Since $\mathscr{H}$ is trivial over $\mathscr{V}$ (see 4.10), we deduce easily from Proposition 4.11 that $\tilde{p}^*\tilde{\mathscr{L}}|_{\tilde{V} \times Z_{\mathbf{u}}}$ is isomorphic to the pull-back of $O_{Z_{\mathbf{u}}}(D)$ with respect to the second projection.

The claim follows from the fact that D is ample if and only if $f_{\mathscr{L}}$ is a strongly polyhedral convex function in the vertex $\mathbf{u}$ [Ful93, § 3.4]. □

4.13 Let X be a closed subscheme of A. Then the subset $\overline{\mathrm{val}}(X^{\mathrm{an}})$ of $\mathbb{R}^n/\Lambda$ is called the *tropical variety* associated to X.

We note the analogue to tropical algebraic geometry, where one studies the tropical variety associated to an algebraic subvariety of $\mathbb{G}_m^n$. However, the lift $p^{-1}(X^{\mathrm{an}})$ of X to the Raynaud extension E is only an analytic subvariety and hence our tropical varieties are best studied in the framework of Berkovich analytic spaces (see [Gub07a] for details about tropical analytic geometry).

Proposition 4.14. *The tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a finite union of Γ-rational polytopes in $\mathbb{R}^n/\Lambda$ of dimension at most $\dim(X)$. If X is connected, then the tropical variety is also connected.*

Proof. Let E be the Raynaud extension of A and let $\mathfrak{T}$ be an atlas of trivializations of E over B as in 4.2. We choose any Γ-rational polytope Δ of $\mathbb{R}^n$ inducing a polytope $\overline{\Delta}$ of $\mathbb{R}^n/\Lambda$ and $V \in \mathfrak{T}$. The trivialization leads to $U_{V,\Delta} \cong V \times U_\Delta$. By [Ber04, Corollary 6.2.2], $\mathrm{val}(U_{V,\Delta} \cap p^{-1}(X^{\mathrm{an}}))$ is a finite union of Γ-rational polytopes in $\mathbb{R}^n$ of dimension at most $\dim(X)$. Since A^{an} is covered by finitely many $U_{[V,\Delta]}$, we conclude easily that $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a finite union of Γ-rational polytopes in $\mathbb{R}^n/\Lambda$ of dimension at most $\dim(X)$. If X is connected, then X^{an} is also connected [Ber90, Theorem 3.4.8]. By continuity of $\overline{\mathrm{val}}$, we see that $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is also connected. □

Theorem 4.15. *Let X be a purely d-dimensional closed subscheme of A and let b be the dimension of the abelian variety B of good reduction in the Raynaud extension (3) of A. Then the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a finite union of Γ-rational polytopes in $\mathbb{R}^n/\Lambda$ of dimension at least $d-b$ and at most d.*

Proof. By Proposition 4.14, there are Γ-rational polytopes $\overline{\Delta_1}, \dots, \overline{\Delta_k} \in \mathbb{R}^n/\Lambda$ of dimension at most d with $\overline{\mathrm{val}}(X^{\mathrm{an}}) = \overline{\Delta_1} \cup \cdots \cup \overline{\Delta_k}$. We may assume that no polytope $\overline{\Delta_j}$ may be omitted in this decomposition. We have to prove that $\dim(\overline{\Delta_j}) \geqslant d-b$. For $\overline{\mathbf{u}} \in \overline{\mathrm{val}}(X^{\mathrm{an}})$, it is enough to show that the dimension of the polytopal set $\overline{\mathrm{val}}(X^{\mathrm{an}})$ in a neighborhood of $\overline{\mathbf{u}}$ is at least $d-b$.

We choose a lift $\mathbf{u}$ of $\overline{\mathbf{u}}$ to $\mathbb{R}^n$ and an n-dimensional Γ-rational polytope Δ with $\mathbf{u} \in \mathrm{relint}(\Delta)$. Using the notation of the proof of Proposition 4.14, we choose $V \in \mathfrak{T}$ such that $\mathbf{u} \in \mathrm{val}(X_{V,\Delta})$ for $X_{V,\Delta} := U_{V,\Delta} \cap p^{-1}(X^{\mathrm{an}})$. The claim follows now from the following more general result.

Let Y be any closed analytic subvariety of $U_{V,\Delta}$ of pure dimension d such that $\mathbf{u} \in \mathrm{val}(Y)$ and let N be the dimension of $\mathrm{val}(Y)$ in a neighborhood of $\mathbf{u}$. Then we have $N \geqslant d - b$.

The proof is by induction on N and follows as in [Gub07a, Proposition 5.4]. If $N = 0$, then we may assume that $\mathrm{val}(Y) = \{\mathbf{u}\}$ by passing to a smaller Δ. By our choice of V, we have the trivialization $U_{V,\Delta} \cong V \times U_\Delta$, where U_Δ is the polytopal domain in $(\mathbb{G}_m^n)^{\mathrm{an}}$ associated to Δ. Passing to the associated admissible formal affine $\mathbb{K}^\circ$-schemes, we get $\mathscr{U}_{V,\Delta} \cong \mathscr{V} \times \mathscr{U}_\Delta$. By abuse of notation, we will use the projection p_2 also on $\mathscr{U}_{V,\Delta}$. Let $\mathscr{Y}$ be the closure of Y in $\mathscr{U}_{V,\Delta}$. By [Gub07a, Proposition 4.4], the open face $\tau := \mathrm{relint}(\Delta)$ induces the stratum $Z_\tau := \pi(\mathrm{val}^{-1}(\tau))$ of $\tilde{U}_\Delta$ of dimension $n - \dim(\tau) = 0$, where $\pi : U_\Delta \to \tilde{U}_\Delta$ is the reduction map. Now we use that $\pi \circ p_2^{\mathrm{an}} = \tilde{p}_2 \circ \pi$ on Y and that the reduction π on the right-hand side is a surjective map from Y onto the special fibre $\tilde{\mathscr{Y}}$ (see 2.6). We conclude from $\mathrm{val}(Y) = \{\mathbf{u}\}$ that $\tilde{p}_2$ maps $\tilde{\mathscr{Y}}$ to the closed point Z_τ. Since Y is of pure dimension d, the same is true for the special fibre $\tilde{\mathscr{Y}}$ and hence we get

$$d \leqslant \dim(\tilde{p}_2^{-1}(Z_\tau)) \leqslant \dim(\tilde{\mathscr{V}}) = b.$$

This proves the claim for $N = 0$.

Now we prove the case $N > 0$. By [Ber04, Corollary 6.2.2], $\mathrm{val}(Y)$ is a finite union of Γ-rational polytopes. We conclude that $\mathbf{u}$ is contained in an N-dimensional Γ-rational polytope $\sigma \subset \mathrm{val}(Y)$. Note that any point $\mathbf{u}' \in \sigma$ contained in a sufficiently small neighborhood of $\mathbf{u}$ has also local dimension N. By density of $Y(\mathbb{K}) \in Y^{\mathrm{an}}$ [Ber90, Proposition 2.1.15], we find such an $\mathbf{u}'$ with $\mathbf{u}' = \mathrm{val}(y)$ for some $y \in Y(\mathbb{K})$. Moreover, we may assume that $\mathbf{u}'$ has an n-dimensional Γ-rational polytope Δ' as a neighborhood such that $\Delta' \cap \mathrm{val}(Y) = \Delta' \cap \sigma$. There are $\alpha \in \mathbb{K}$ and $\mathbf{m} \in \mathbb{Z}^n$ such that the hyperplane $H = \{\mathbf{x}^{\mathbf{m}} = \alpha\}$ passes through y and such that $\mathrm{val}(H^{\mathrm{an}}) = \{\omega \cdot \mathbf{m} = v(\alpha)\}$ intersects $\mathrm{val}(Y) \cap \Delta'$ transversally. By Krull's *Hauptidealsatz*, the closed analytic subvariety $Y' := Y \cap H^{\mathrm{an}} \cap U_{V,\Delta'}$ of $U_{V,\Delta'}$ has pure dimension $d - 1$. We deduce from

$$\mathrm{val}(Y') \subset \mathrm{val}(Y) \cap \{\omega \cdot \mathbf{m} = v(\alpha)\} \cap \Delta',$$

that $\mathrm{val}(Y')$ has dimension $N' \leqslant N - 1$ in a neighborhood of $\mathbf{u}'$. By induction applied to Y', we get $N - 1 \geqslant N' \geqslant d - 1 - b$, proving the claim. $\square$

Remark 4.16. We now assume that X is an irreducible d-dimensional closed subvariety of A. In the preprint [Gub08b] of this paper, it was claimed in Theorem 4.15 that $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is of pure dimension. As pointed out by the referee, the argument was based on a wrong application of Chevalley's theorem, which does not hold in the category of analytic spaces, and so this question remains open.

However, if A is isogeneous to $B_1 \times B_2$, where B_1 (respectively B_2) is an abelian variety with good (respectively totally degenerate) reduction at v, then $\overline{\mathrm{val}}(X^{\mathrm{an}})$ has indeed pure dimension $d - e$ for some $e \in \{0, \min(b, d)\}$.

To prove this, let $\varphi : A \to B_1 \times B_2$ be an isogeny. By [BL91, Theorem 1.2], φ lifts to an isogeny $\phi : E \to B_1^{\mathrm{an}} \times (\mathbb{G}_m^n)^{\mathrm{an}}$ between the associated uniformizations of the Raynaud extension. Obviously, $(\mathbb{G}_m^n)^{\mathrm{an}}$ is also the torus part in the Raynaud extension of A and ϕ restricts to an isogeny $(\mathbb{G}_m^n)^{\mathrm{an}} \to (\mathbb{G}_m^n)^{\mathrm{an}}$. On the other hand, an (analytic) endomorphism of $\mathbb{G}_m^n$ is given by $\phi^*(x_j) = \mathbf{x}^{\mathbf{m}_j}$ for some $\mathbf{m}_j \in \mathbb{Z}^n$, $j = 1, \ldots, n$. We conclude that the linear isomorphism ϕ_{aff}, given by the matrix $(\mathbf{m}_1, \ldots, \mathbf{m}_n)^t$, maps $\overline{\mathrm{val}}(X^{\mathrm{an}})$ onto $\overline{\mathrm{val}}(\phi(X)^{\mathrm{an}})$. Hence, we may assume that $A = B_1 \times B_2$. Let $p_2 : A \to B_2$ be the second projection. Then $Y = p_2(X)$ is an irreducible closed subvariety of B_2 of dimension $d - e$ for some $e \in \{0, \min(b, d)\}$ with $b = \dim(B_1)$. By construction, we have $\overline{\mathrm{val}}(X^{\mathrm{an}}) = \overline{\mathrm{val}}(Y^{\mathrm{an}})$ and hence the claim follows from the fact that the

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tropical variety of an irreducible d'-dimensional closed subvariety of a totally degenerate abelian variety has pure dimension d' (see Theorem 4.15 or [Gub07a, Theorem 6.9]). $\square$

5. Subdivisions of the skeleton

In this section, $\mathbb{K}$ denotes an algebraically closed field endowed with a non-trivial, non-archimedean complete absolute value $|\ |$. Let $v := -\log|\ |$ be the valuation with value group $\Gamma := v(\mathbb{K}^\times)$, valuation ring $\mathbb{K}^\circ$ and residue field $\tilde{\mathbb{K}}$.

A smooth variety X' over $\mathbb{K}$ has not always a smooth formal $\mathbb{K}^\circ$-model and hence we study a strictly semi-stable $\mathbb{K}^\circ$-model $\mathscr{X}'$. Its special fibre $\tilde{\mathscr{X}}'$ may be viewed as a divisor with normal crossings on $\mathscr{X}'$. The skeleton of $\mathscr{X}'$ is a metrized polytopal set of $(X')^{\mathrm{an}}$ closely related to the stratification of $\tilde{\mathscr{X}}'$. We will see that the skeleton has similar properties to a tropical variety.

We will study the effect of subdivisions on the models. In particular, this is interesting if X' maps to an abelian variety. The most important result of this somehow technical section is at the end, where we will compute the degree of an irreducible component of $\tilde{\mathscr{X}}'$ in this setting under a certain transversality assumption. In the next section, this result is used to compute canonical measures on abelian varieties.

5.1 Let $\mathscr{X}'$ be a *strictly semi-stable* admissible formal scheme over $\mathbb{K}^\circ$, i.e. $\mathscr{X}'$ is covered by formal open subsets $\mathscr{U}'$ with an étale morphism

$$\psi : \mathscr{U}' \longrightarrow \mathscr{S} := \mathrm{Spf}(\mathbb{K}^\circ\langle x_0', \ldots, x_d'\rangle / \langle x_0' \cdots x_r' - \pi\rangle)$$

for $r \leqslant n$ and $\pi \in \mathbb{K}^\times$ with $|\pi| < 1$. The generic fibre U' of $\mathscr{U}'$ is smooth and hence the generic fibre X' of $\mathscr{X}'$ is a smooth analytic space. For simplicity, we assume that X' is connected. Then X' is d dimensional, but r and π may depend on the choice of $\mathscr{U}'$.

Note that $\mathscr{S} = \mathrm{Spf}(\mathbb{K}^\circ\langle x_0', \ldots, x_r'\rangle / \langle x_0' \cdots x_r' - \pi\rangle) \times \mathrm{Spf}(\mathbb{K}^\circ\langle x_{r+1}', \ldots, x_d'\rangle)$. For $i = 1, 2$, we denote the ith factor by $\mathscr{S}_i$ and the corresponding projection by $p_i : \mathscr{S} \to \mathscr{S}_i$. The second factor $\mathscr{S}_2$ is the affine formal scheme associated to the closed unit ball of dimension $d - r$. The first factor $\mathscr{S}_1$ is isomorphic to the affine formal scheme over $\mathbb{K}^\circ$ associated to the polytopal domain U_Δ in $\mathbb{G}_m^r$, where Δ is the simplex $\{u_1' + \cdots + u_r' \leqslant v(\pi)\}$ in $\mathbb{R}_+^r$.

We will use the strata of the special fibre $\tilde{\mathscr{X}}$, which were introduced in § 1.2. We will always normalize the formal open covering as in the following proposition. The reason will become clear in the construction of the skeleton.

Proposition 5.2. *Any formal open covering of $\mathscr{X}'$ admits a refinement $\{\mathscr{U}'\}$ by formal open subsets $\mathscr{U}'$ as in 5.1 and which has the following properties.*

(a) *Every $\mathscr{U}'$ is a formal affine open subscheme of $\mathscr{X}'$.*

(b) *There is a distinguished stratum S of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$ such that, for any stratum T of $\tilde{\mathscr{X}}'$, we have $S \subset \overline{T}$ if and only if $\tilde{\mathscr{U}}' \cap \overline{T} \neq \emptyset$.*

(c) *$\tilde{\psi}^{-1}(\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2)$ is the stratum of $\tilde{\mathscr{U}}'$ which is equal to $\tilde{\mathscr{U}}' \cap S$ for the distinguished stratum S from (b).*

(d) *Every stratum of $\tilde{\mathscr{X}}'$ is the distinguished stratum of a suitable $\mathscr{U}'$.*

Proof. We start with the formal open covering $\{\mathscr{U}'\}$ from 5.1. We will refine it successively to get the claim. First, we may assume that the covering is a refinement of the given formal open covering of $\mathscr{X}'$. Let $\tilde{P}$ be any point of $\tilde{\mathscr{X}}'$ and let S be the stratum of $\tilde{\mathscr{X}}'$ which contains $\tilde{P}$.

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There is a $\mathscr{U}'$ with $\tilde{P} \in \tilde{\mathscr{U}}'$. We remove from $\mathscr{U}'$ the closure of all strata T of $\tilde{\mathscr{X}}'$ with $S \cap \overline{T} = \emptyset$. Note that the closure of a stratum in $\tilde{\mathscr{X}}$ is a strata subset (see [Ber99, Proposition 2.1]) and that the closures of two strata are either disjoint or one closure is contained in the other. Hence, we get from $\mathscr{U}'$ a formal open subset which contains $\tilde{P}$ and which has property (b) for our S. By passing to a formal affine open subset containing $\tilde{P}$, we get also (a). If we do this for every point $\tilde{P}$, we get a formal open subcovering with properties (a), (b) and (d). So, we may assume that the covering $\{\mathscr{U}'\}$ satisfies (a), (b) and (d). We will show that this implies (c).

By [Ber99, Lemma 2.2], the restriction of $\tilde{\psi}$ to a stratum of $\tilde{\mathscr{U}}'$ induces an étale morphism to a stratum of $\tilde{\mathscr{S}}$ and hence the preimage of a stratum of $\tilde{\mathscr{S}}$ is a stratum of $\tilde{\mathscr{U}}'$. We conclude that $\tilde{\psi}^{-1}(\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2)$ is the union of $d-r$-dimensional strata S_i of $\tilde{\mathscr{U}}'$. Let S be the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. By (b), S is contained in the closure of every S_i. Since $\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2$ is a closed stratum of $\tilde{\mathscr{S}}$, $\tilde{\psi}(S \cap \tilde{\mathscr{U}}')$ is contained in $\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2$. This proves $S = S_i$ for some i. By dimensionality reasons, we get $S = S_i$ for every i, proving (c). □

5.3 Next, we describe the *skeleton* of a strictly semi-stable formal scheme $\mathscr{X}'$ over $\mathbb{K}^\circ$. For details, we refer the reader to [Ber99, § 4], [Ber04, § 4] and [Gub07a, 9.1].

We start with the model example $\mathscr{S}$ from 5.1. Replacing x_0' by $\pi/(x_1' \cdots x_r')$, every analytic function f on $\mathscr{S}^{\mathrm{an}}$ has a unique representation as a convergent Laurent series of the form

$$f = \sum_{m_1,\ldots,m_r \in \mathbb{Z}} \sum_{m_{r+1},\ldots,m_d \in \mathbb{N}} a_{\mathbf{m}} (x_1')^{m_1} \cdots (x_d')^{m_d}.$$

For every $\mathbf{u}$ in the simplex $\Delta := \{\mathbf{u} \in \mathbb{R}_+^r \mid u_1' + \cdots + u_r' \leqslant v(\pi)\}$, we get an element $\xi_{\mathbf{u}} \in \mathscr{S}^{\mathrm{an}}$ using the bounded multiplicative semi-norm

$$|f(\xi_{\mathbf{u}})| := \max_{\mathbf{m}} |a_{\mathbf{m}}| e^{-m_1 u_1 - \cdots - m_r u_r}.$$

We define the skeleton of $\mathscr{S}$ as $\{\xi_{\mathbf{u}} \mid \mathbf{u} \in \Delta\}$. It is a closed subset of $\mathscr{S}^{\mathrm{an}}$ homeomorphic to Δ. To omit the preference of the coordinate x_0', it is better to identify the skeleton of $\mathscr{S}'$ with the simplex $\{u_0' + \cdots + u_r' = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$.

Next, we consider a formal open subset $\mathscr{U}'$ of $\mathscr{X}'$ as in Proposition 5.2. Then the skeleton $S(\mathscr{U}')$ of $\mathscr{U}'$ is defined as the preimage of the skeleton of $\mathscr{S}$ with respect to the morphism ψ^{an}. It is a closed subset of the generic fibre U' of $\mathscr{U}'$. Using Proposition 5.2(b), one can show that ψ^{an} induces a homeomorphism of $S(\mathscr{U}')$ onto the skeleton of $\mathscr{S}$. Using the above, we may identify $S(\mathscr{U}')$ again with the metrized simplex $\{u_0' + \cdots + u_r' = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$. This is independent of ψ up to reordering the coordinates $u_0', \ldots, u_r'$.

Finally, the skeleton $S(\mathscr{X}')$ of $\mathscr{X}'$ is the union of all skeletons $S(\mathscr{U}')$. Berkovich has shown that $S(\mathscr{X}')$ is a closed subset of the generic fibre X' which depends only on the formal model $\mathscr{X}'$, but neither on the choice of the formal covering $\{\mathscr{U}'\}$ nor on the choice of the étale morphisms ψ.

The skeleton $S(\mathscr{X}')$ has a canonical structure as an abstract metrized simplicial set, which reflects the incidence relations between the strata of $\tilde{\mathscr{X}}'$: for every stratum S of codimension r, there is a *canonical simplex* Δ_S in $S(\mathscr{X}')$ defined in the following way. We choose a formal affine open subset $\mathscr{U}'$ as in Proposition 5.2 such that S is the distinguished stratum associated to $\mathscr{U}'$. Then we define Δ_S as the skeleton of $\mathscr{U}'$. It is easy to see that Δ_S does not depend on the choice of $\mathscr{U}'$ and hence we may identify Δ_S with the simplex $\{u_0' + \cdots + u_r' = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$. The canonical simplices have the properties:

(a) the canonical simplices $(\Delta_S)_{S\in\mathrm{str}(\tilde{\mathscr{X}}')}$ cover $S(\mathscr{X}')$;

(b) for $S\in\mathrm{str}(\tilde{\mathscr{X}}')$, the map $T\mapsto\Delta_T$ gives a bijective order reversing correspondence between $T\in\mathrm{str}(\tilde{\mathscr{X}}')$ with $S\subset\overline{T}$ and closed faces of Δ_S;

(c) for $R,S\in\mathrm{str}(\tilde{\mathscr{X}}')$, $\Delta_R\cap\Delta_S$ is the union of all Δ_T with $T\in\mathrm{str}(\tilde{\mathscr{X}}')$ and $\overline{T}\supset R\cup S$.

There is a continuous map $\mathrm{Val}:X'\to S(\mathscr{X}')$ which restricts to the identity on $S(\mathscr{X}')$. It is enough to define it for $p\in U'$, where U' is the generic fibre of a formal affine open subset $\mathscr{U}'$ as above. Using the identification $\Delta_S=\{u_0'+\cdots+u_r'=v(\pi)\}$, we set

$$\mathrm{Val}(p):=(-\log\circ p(\psi^*(x_0')),\ldots,-\log\circ p(\psi^*(x_r')))\in\Delta_S.$$

By [Ber99, Theorem 5.2], the map Val gives a proper strong deformation retraction of X' to the skeleton $S(\mathscr{X}')$.

5.4 It would be tempting to call the family of canonical simplices a polytopal decomposition of $S(\mathscr{X}')$. However, we note that the family is not necessarily face to face; only the weaker property 5.3(c) holds instead.

In the following, we now consider a Γ-rational *polytopal subdivision* $\mathscr{D}$ of $S(\mathscr{X}')$. This means that $\mathscr{D}$ is a family of Γ-rational polytopes, each contained in a canonical simplex, such that $\mathscr{D}\cap\Delta_S:=\{\Delta\in\mathscr{D}\mid\Delta\subset\Delta_S\}$ is a polytopal decomposition of Δ_S for every $S\in\mathrm{str}(\tilde{\mathscr{X}}')$.

Proposition 5.5. *There is a coarsest formal analytic structure $\mathfrak{X}''$ on X' which refines $(\mathscr{X}')^{\mathrm{f-an}}$ in such a way that $\mathrm{Val}^{-1}(\Delta)$ is a formal open subset for every $\Delta\in\mathscr{D}$.*

Proof. Let $S\in\mathrm{str}(\tilde{\mathscr{X}}')$ and let U' be the generic fibre of a set $\mathscr{U}'$ as in Proposition 5.2. We note that such sets U', for varying S, form a formal affinoid atlas of $(\mathscr{X}')^{\mathrm{f-an}}$. To prove the claim, it is enough to show that the sets

$$(U'\cap\mathrm{Val}^{-1}(\Delta))_{\Delta\in\mathscr{D}\cap\Delta_S}\tag{12}$$

define a formal affinoid atlas on U'. The polytope $\Delta\in\mathscr{D}\cap\Delta_S$ is given by finitely many inequalities of the form $\mathbf{m}\cdot\mathbf{u}'+v(\lambda)\geqslant 0$ for some $\mathbf{m}\in\mathbb{Z}^{r+1}$ and $\lambda\in\mathbb{K}^\times$. In terms of the semi-stable coordinates $x_0',\ldots,x_r'$ of U', the subset $U'\cap\mathrm{Val}^{-1}(\Delta)$ is given by finitely many inequalities of the form $|\lambda\psi^*(\mathbf{x}')^{\mathbf{m}}|\leqslant 1$ and hence it is an affinoid subdomain of U'. This description yields easily that (12) is a formal affinoid atlas of U', proving the claim. □

Remark 5.6. Let $\mathscr{U}'$ be a formal open subset of $\mathscr{X}'$ as in Proposition 5.2 with étale morphism $\psi:\mathscr{U}'\to\mathscr{S}$. The generic fibre U' is a formal open subset of the formal analytic variety $\mathfrak{X}''$ from Proposition 5.5 and we write suggestively $U'\cap\mathfrak{X}''$ for the formal analytic structure on U' induced by $\mathfrak{X}''$.

Let S be the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. The first factor $\mathscr{S}_1$ from 5.1 is the formal scheme over $\mathbb{K}^\circ$ associated to the polytopal domain $\mathrm{val}^{-1}(\Delta_S)$ in $\{\mathbf{x}'\in\mathbb{G}_m^{r+1}\mid x_0'\cdots x_r'=v(\pi)\}$.

The polytopal decomposition $\mathscr{D}\cap\Delta_S$ of Δ_S leads to a formal analytic refinement of $\mathscr{S}_1^{\mathrm{f-an}}$ inducing an admissible formal scheme $\mathscr{S}_1'$ over $\mathbb{K}^\circ$ and a canonical morphism $\iota_1:\mathscr{S}_1'\to\mathscr{S}_1$ extending the identity from the generic fibre. By base change, we get a morphism $\iota:\mathscr{S}'\to\mathscr{S}$ with the same property. Note that $\mathscr{S}'=\mathscr{S}_1'\times\mathscr{S}_2$ has reduced special fibre (see 2.6).

Since the base change $\psi':\mathscr{U}''\to\mathscr{S}'$ of ψ with respect to ι is étale, $\mathscr{U}''$ has also reduced special fibre (see [GD67, 17.5.7]). By 2.6, $\mathscr{U}''$ is an admissible formal scheme over $\mathbb{K}^\circ$ associated to a formal analytic variety. By the proof of Proposition 5.5, the latter is $U' \cap \mathfrak{X}''$ and hence $\mathscr{U}'' = (U' \cap \mathfrak{X}'')^{\mathrm{f-sch}}$.

In the following, $\mathscr{X}''$ denotes the admissible formal $\mathbb{K}^\circ$-scheme associated to $\mathfrak{X}''$ and hence we may identify the special fibre $\tilde{\mathfrak{X}}''$ with the reduction $\tilde{\mathscr{X}}''$ (see 2.6). Since $\mathscr{D}$ is a polytopal subdivision of the skeleton, the identity is a formal analytic morphism $\mathfrak{X}'' \to (\mathscr{X}')^{\mathrm{f-an}}$ and hence we get a unique morphism $\iota' : \mathscr{X}'' \to \mathscr{X}'$ extending the identity on the generic fibre.

Recall that the order on the strata is given by inclusion of closures. Similarly, we define an order on the open faces of $\mathscr{D}$.

PROPOSITION 5.7. *Let $\mathfrak{X}''$ be the formal analytic variety associated to $\mathscr{D}$ as described in Proposition 5.5. Then there is a bijective correspondence between open faces τ of $\mathscr{D}$ and strata R of $\tilde{\mathfrak{X}}''$, given by*

$$R = \pi(\mathrm{Val}^{-1}(\tau)), \quad \tau = \mathrm{Val}(\pi^{-1}(Y)), \tag{13}$$

where $\pi : X' \to \tilde{\mathfrak{X}}''$ is the reduction map and Y is any non-empty subset of R.

Proof. Let τ be an open face of $\mathscr{D}$. We have to prove that $R := \pi(\mathrm{Val}^{-1}(\tau))$ is a stratum of $\tilde{\mathfrak{X}}''$. There is a unique $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ such that τ is contained in $\mathrm{relint}(\Delta_S)$. We choose a formal affine open subset $\mathscr{U}'$ as in Proposition 5.2 such that S is the distinguished stratum associated to $\mathscr{U}'$. Note that strata are compatible with localization and hence we may assume that $\mathscr{X}' = \mathscr{U}'$. By Remark 5.6, we have a cartesian diagram of admissible formal schemes over $\mathbb{K}^\circ$:

$$\begin{tikzcd}
\mathscr{X}'' \arrow[r, "\psi'"] \arrow[d, "\iota'"] & \mathscr{S}' \arrow[r, "p_1'"] \arrow[d, "\iota"] & \mathscr{S}_1' \arrow[d, "\iota_1"] \\
\mathscr{X}' \arrow[r, "\psi"] & \mathscr{S} \arrow[r, "p_1"] & \mathscr{S}_1
\end{tikzcd}$$

with ψ and ψ' étale. Let $\psi_1 := p_1 \circ \psi$ and $\psi_1' := p_1' \circ \psi'$.

The idea of the proof is to use ψ_1 to reduce the claim to the corresponding statement for the polytopal domain $\mathscr{S}_1$ in $\mathbb{G}_m^r$. We describe this result here in terms of the torus $\mathbb{G}_m^{r+1}$ and in terms of the valuation map

$$\mathrm{val} : (\mathbb{G}_m^{r+1})^{\mathrm{an}} \to \mathbb{R}^{r+1}, \quad p \mapsto (-\log \circ\, p(x_0'), \ldots, -\log \circ\, p(x_r'))$$

to omit the preference of the first coordinate. By [Gub07a, Propositions 4.4 and 4.7], there is a bijective correspondence between open faces σ of $\mathscr{D}$ (which is a polytopal decomposition of $\Delta_S = \{\mathbf{u} \in \mathbb{R}^{r+1} \mid u_0' + \cdots + u_r' = v(\pi)\}$ by the assumption $\mathscr{U}' = \mathscr{X}'$) and strata T_1' of $\tilde{\mathscr{S}}_1'$, given by

$$T_1' = \pi(\mathrm{val}^{-1}(\sigma) \cap \mathscr{S}_1^{\mathrm{an}}), \quad \sigma = \mathrm{val}(\pi^{-1}(T_1')), \tag{14}$$

where $\pi : \mathscr{S}_1^{\mathrm{an}} = (\mathscr{S}_1')^{\mathrm{an}} \to \tilde{\mathscr{S}}_1'$ denotes the reduction map. In fact, one can replace T_1' in the second formula of (14) by any non-empty subset of T_1'. To see this, note that the formal affinoid subtorus $D = \{|x_0| = \cdots = |x_r| = 1,\ x_0 \cdots x_r = 1\}$ of $\mathbb{G}_m^{r+1}$ acts on $\mathscr{S}_1^{\mathrm{an}}$ and this extends to an action of the formal torus on $\mathscr{S}_1'$. The strata of $\mathscr{S}_1'$ are the same as the torus orbits. We conclude that D acts transitively on the set $\{\pi^{-1}(\tilde{P}) \mid \tilde{P} \in T_1'(\tilde{\mathbb{K}})\}$. Since the map val is invariant under the D-action, we conclude that $\mathrm{val}(\pi^{-1}(T_1')) = \mathrm{val}(\pi^{-1}(\tilde{P}))$ for any $\tilde{\mathbb{K}}$-rational point $\tilde{P}$ of T_1'. Note that we may use base extension to make any non-closed point rational; therefore, since the map val is invariant under base extension, we conclude that $\mathrm{val}(\pi^{-1}(T_1')) = \mathrm{val}(\pi^{-1}(\tilde{P}))$ holds for *any* (i.e. not necessarily closed) point $\tilde{P}$ of T_1'. This proves the second formula in (14) with T_1' replaced by any non-empty subset.

Now let T'_1 be the stratum of $\tilde{\mathscr{S}}'_1$ corresponding to the given open face τ. Obviously, $T' := (\tilde{p}'_1)^{-1}(T'_1) = T'_1 \times \tilde{\mathscr{S}}_2$ is a stratum of $\tilde{\mathscr{S}}' = \tilde{\mathscr{S}}'_1 \times \tilde{\mathscr{S}}_2$. We would like to prove that the preimage of T' with respect to $\tilde{\psi}'$ is equal to R. Using (14), we first note that

$$(\tilde{\psi}')^{-1}(T') = (\tilde{\psi}'_1)^{-1}(\pi(\mathrm{val}^{-1}(\tau) \cap \mathscr{S}_1^{\mathrm{an}})). \tag{15}$$

Next, we prove the following formula:

$$(\tilde{\psi}'_1)^{-1}(\pi(\mathrm{val}^{-1}(\tau) \cap \mathscr{S}_1^{\mathrm{an}})) = \pi((\psi_1^{\mathrm{an}})^{-1}(\mathrm{val}^{-1}(\tau))). \tag{16}$$

The inclusion '$\supset$' follows immediately from $\pi \circ \psi_1^{\mathrm{an}} = \tilde{\psi}'_1 \circ \pi$. Here, we have used that $\psi_1^{\mathrm{an}} = (\psi'_1)^{\mathrm{an}}$. To prove the reverse inclusion, let us choose a point $\tilde{x}' \in (\tilde{\psi}'_1)^{-1}(\pi(\mathrm{val}^{-1}(\tau) \cap \mathscr{S}_1^{\mathrm{an}}))$. The reduction map is surjective; hence, there is $x' \in X'$ with $\pi(x') = \tilde{x}'$. By assumption, there is $z \in \mathrm{val}^{-1}(\tau) \cap \mathscr{S}_1^{\mathrm{an}}$ with

$$\pi(z) = \tilde{\psi}'_1(\tilde{x}') = \tilde{\psi}'_1(\pi(x')) = \pi \circ \psi_1^{\mathrm{an}}(x').$$

By (14), we get $\pi \circ \psi_1^{\mathrm{an}}(x') \in T'_1$. An application of (14) shows that $\mathrm{val}(\psi_1^{\mathrm{an}}(x')) \in \tau$. We conclude that $\tilde{x}' = \pi(x') \in \pi((\psi_1^{\mathrm{an}})^{-1}(\mathrm{val}^{-1}(\tau)))$, proving (16).

Using (15) and (16), we get finally the desired relation between R and T':

$$(\tilde{\psi}')^{-1}(T') = \pi((\psi_1^{\mathrm{an}})^{-1}(\mathrm{val}^{-1}(\tau))) = \pi(\mathrm{Val}^{-1}(\tau)) = R. \tag{17}$$

By [Ber99, Lemma 2.2], the preimage of the stratum T' with respect to the étale morphism $\tilde{\psi}'$ is the union of strata of the same dimension. This argument was already used in the proof of Proposition 5.2. To prove that R is a stratum, it is enough to show that R is irreducible. Note that $\tilde{\iota}_1(T'_1) = \{\tilde{\mathbf{0}}\}$ in $\tilde{\mathscr{S}}_1$ and hence

$$(\tilde{\psi}')^{-1}(T') = T' \times_{\tilde{\mathscr{S}}} \tilde{\mathscr{X}}' = T' \times_{\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2} \tilde{\psi}^{-1}(\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2) \cong \mathbb{G}_m^{r-t} \times S, \tag{18}$$

where $t := \dim(\tau)$. Here, we have used that T'_1 is an $(r-t)$-dimensional torus orbit and that $S = \tilde{\psi}^{-1}(\{\tilde{\mathbf{0}}\} \times \tilde{\mathscr{S}}_2)$ (see Proposition 5.2 and [Gub07a, Proposition 4.4]). We conclude that $R = (\tilde{\psi}')^{-1}(T')$ is irreducible, proving that $R \in \mathrm{str}(\tilde{\mathfrak{X}}'')$.

Since the open faces of $\mathscr{D}$ form a covering of the skeleton $S(\mathscr{X}')$, we conclude that every $R \in \mathrm{str}(\tilde{\mathfrak{X}}'')$ has the form $R = \pi(\mathrm{Val}^{-1}(\tau))$ for an open face τ of $\mathscr{D}'$.

It remains to show that τ may be reconstructed from R by the second formula in (13). By the same argument as used in the paragraph after (14), it is enough to prove this for $Y = \{\tilde{y}\}$ for any $\tilde{\mathbb{K}}$-rational point $\tilde{y}$ of R. Since ψ' is étale, the formal fibre $X'_+(\tilde{y}) := \pi^{-1}(\tilde{y})$ is isomorphic to the formal fibre over $\tilde{z} := \tilde{\psi}'(\tilde{y})$ in $\mathscr{S}'$ (see [Gub07a, Proposition 2.9]). For $\tilde{z}_1 := \tilde{p}_1(\tilde{z})$, we get the following isomorphism of formal fibres:

$$X'_+(\tilde{y}) \cong (\mathscr{S}'_1)^{\mathrm{an}}_+(\tilde{z}_1) \times (\mathscr{S}_2)^{\mathrm{an}}_+(\tilde{\mathbf{0}}). \tag{19}$$

The $(d-r)$-dimensional ball $\mathscr{S}_2^{\mathrm{an}}$ does not contribute to Val. Using the analogue of the claim for the polytopal domain $\mathscr{S}_1^{\mathrm{an}}$ deduced after (14), we get

$$\mathrm{Val}(X'_+(\tilde{y})) = \mathrm{val}((\mathscr{S}'_1)^{\mathrm{an}}_+(\tilde{z}_1)) = \mathrm{val}(\pi^{-1}(T'_1)) = \tau.$$

This proves the second formula in (13). □

Remark 5.8. Let $\mathscr{U}'$ be a formal affine open subset of $\mathscr{X}'$ as in Proposition 5.2 and let $\psi : \mathscr{U}' \to \mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$ be the étale morphism from 5.1. Let us consider the composition $\psi_1 : \mathscr{U}' \to \mathscr{S}_1$ of the first projection with ψ and let $\psi'_1 : \mathscr{U}'' \to \mathscr{S}'_1$ be the base change of ψ_1 induced by the polytopal decomposition $\mathscr{D}$. We have seen in the above proof that the preimage of any stratum of $\tilde{\mathscr{S}}'_1$ with respect to $\tilde{\psi}'_1$ is a stratum of $\tilde{\mathscr{U}}''$.

Recall that $\mathscr{X}'' = (\mathfrak{X}'')^{\mathrm{f-sch}}$ and that we have a canonical morphism $\iota' : \mathscr{X}'' \to \mathscr{X}'$ extending the identity on the generic fibre.

COROLLARY 5.9. *Let $R \in \mathrm{str}(\tilde{\mathfrak{X}}'')$ with corresponding open face τ of $\mathscr{D}$.*

(a) $\dim(\tau) = \mathrm{codim}(R, \tilde{\mathfrak{X}}'')$.

(b) $S := \tilde{\iota}'(R) \in \mathrm{str}(\tilde{\mathscr{X}}')$.

(c) *$R \xrightarrow{\tilde{\iota}'} S$ is a fibre bundle with fibre $(\mathbb{G}_m)_{\tilde{\mathbb{K}}}^{\dim(R)-\dim(S)}$.*

(d) *Every stratum of $\tilde{\mathfrak{X}}''$ is smooth.*

(e) *The closure $\overline{R}$ is the union of the strata of $\tilde{\mathfrak{X}}''$ corresponding to the open faces σ of $\mathscr{D}$ with $\tau \subset \overline{\sigma}$.*

(f) *For open faces τ_1, τ_2 of $\mathscr{D}$ with corresponding strata R_1, R_2 of $\tilde{\mathfrak{X}}''$, we have $\overline{\tau_1} \subset \overline{\tau_2}$ if and only if $\overline{R_1} \supset \overline{R_2}$.*

(g) *For an irreducible component Y of $\tilde{\mathfrak{X}}''$, let ξ_Y be the unique point of X' with reduction equal to the generic point of Y. Then $Y \mapsto \xi_Y$ is a bijection between the irreducible components of $\tilde{\mathfrak{X}}''$ and the vertices of $\mathscr{D}$.*

Proof. We use the proof of Proposition 5.7. We have $\dim(\tau) = \mathrm{codim}(T_1', \tilde{\mathscr{S}}_1')$ by [Gub07a, Proposition 4.4]. Using that R is locally equal to $(\tilde{\psi}_1')^{-1}(T_1')$ and the smoothness of $\tilde{\psi}_1'$, we get (a). Let $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ with $\tau \subset \mathrm{relint}(\Delta_S)$. By (a) and (18), we deduce (b) and (c). Since S is smooth by Proposition 5.2(c), we get (d) from (c).

Since strata are compatible with localization, it is enough to prove (e) in case of $\mathscr{X}' = \mathscr{U}'$ for a formal affine $\mathscr{U}'$ as in Proposition 5.2 such that S is the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. Since $\tilde{\psi}_1'$ is flat, we get $\overline{R} = (\tilde{\psi}_1')^{-1}(\overline{T_1'})$. By [Gub07a, Remark 4.8], $\overline{T_1'}$ is the union of all strata of $\tilde{\mathscr{S}}_1'$ corresponding to the open faces σ of $\mathscr{D}$ with $\overline{\sigma} \supset \overline{\tau}$. If we take preimages of this decomposition, we get (e). Note that (f) is a consequence of (e).

By using the unique dense stratum of an irreducible component, it follows from (a) and Proposition 5.7 that the map $Y \mapsto \mathrm{Val}(\xi_Y)$ is a bijection between the irreducible components of $\tilde{\mathfrak{X}}''$ and the vertices of $\mathscr{D}$. To prove (g), it remains to see that $\xi_Y \in S(\mathscr{X}')$. There are a formal affine open subset $\mathscr{U}'$ of $\mathscr{X}'$ as in Proposition 5.2 with $Y \cap \mathscr{U}' \neq \emptyset$ and an étale morphism $\psi : \mathscr{U}' \to \mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$. By Remark 5.8, there is an irreducible component Z of $\tilde{\mathscr{S}}'$ such that $\psi(\xi_Y) = \xi_Z$. Since ψ is étale, it is enough to prove that $\xi_Z \in S(\mathscr{S})$ (see [Ber04, Corollary 4.3.2]). Since $Z = Z_{\mathbf{u}} \times \tilde{\mathscr{S}}_2$ for the irreducible component $Z_{\mathbf{u}}$ of $\tilde{\mathscr{S}}_1$ corresponding to the vertex $\mathbf{u} = \mathrm{Val}(\xi_Y)$ of $\mathscr{D}$ (see [Gub03, Proposition 4.7]), it is easy to see that the point $\xi_{\mathbf{u}}$ from 5.3 reduces to the generic point of Z and hence we get $\xi_{\mathbf{u}} = \xi_Z$, proving the claim. □

5.10 For the remaining part of this section, we fix the following situation: let A be an abelian variety over $\mathbb{K}$ with uniformization E such that $A^{\mathrm{an}} = E/M$ as in 4.1. We recall that M is a discrete subgroup of $E(\mathbb{K})$ such that $\mathrm{val} : E \to \mathbb{R}^n$ maps M isomorphically onto a complete lattice Λ of $\mathbb{R}^n$.

We assume that we have a morphism $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$, where $\mathscr{X}'$ is still a strictly semi-stable scheme over $\mathbb{K}^\circ$ and $\mathscr{A}_0$ is the Mumford model of A associated to a Γ-rational polytopal decomposition $\overline{\mathscr{C}_0}$ of $\mathbb{R}^n/\Lambda$. Let $f : X' \to A$ be the generic fibre of φ_0.

PROPOSITION 5.11. *There is a unique map $\overline{f}_{\mathrm{aff}} : S(\mathscr{X}') \to \mathbb{R}^n/\Lambda$ with $\overline{f}_{\mathrm{aff}} \circ \mathrm{Val} = \overline{\mathrm{val}} \circ f$ on X'. The map $\overline{f}_{\mathrm{aff}}$ is continuous. For every $S \in \mathrm{str}(\tilde{\mathscr{X}}')$, the restriction of $\overline{f}_{\mathrm{aff}}$ to the canonical simplex Δ_S is an affine map and there is a unique $\overline{\Delta} \in \overline{\mathscr{C}_0}$ with $\overline{f}_{\mathrm{aff}}(\mathrm{relint}(\Delta_S)) \subset \mathrm{relint}(\overline{\Delta})$.*

Proof. We recall the construction of $\mathscr{A}_0$ given in 4.7. Let $\mathscr{V}$ be a formal affine open subset of the formal abelian scheme $\mathscr{B}$ which trivializes the Raynaud extension (3) of A. For the generic fibre V of $\mathscr{V}$ and $\Delta \in \mathscr{C}_0$, we get a formal affinoid subdomain $U_{V,\Delta}$ of E with associated affine formal schemes $\mathscr{U}_{V,\Delta}$. With varying V and Δ, we get a formal affinoid atlas on E with associated $\mathbb{K}^\circ$-model $\mathscr{E}_0$ of E which is covered by the formal open subsets $\mathscr{U}_{V,\Delta}$. By passing to the quotient by M, we get $\mathscr{A}_0 = \mathscr{E}_0/M$ and the quotient morphism maps $\mathscr{U}_{V,\Delta}$ isomorphically onto the formal open chart $\mathscr{U}_{[V,\Delta]}$ of $\mathscr{A}_0$.

There is a formal open covering $\{\mathscr{U}'\}$ of $\mathscr{X}'$ as in Proposition 5.2 such that for any $\mathscr{U}'$ of the covering, there are V, Δ as above with $\mathscr{U}' \subset \varphi_0^{-1}(\mathscr{U}_{[V,\Delta]})$. We denote the generic fibre of $\mathscr{U}'$ by U'. By construction, there is a unique lift $F : U' \to U_{V,\Delta}$ of f. Now we use the coordinates $x_1, \ldots, x_n$ of the torus T from the Raynaud extension (4) of A. They are defined on $U_{V,\Delta}$ by using the trivialization $U_{V,\Delta} \cong V \times U_\Delta$ from 4.2 for the polytopal domain U_Δ of T. Note that $F^*(x_i)$ is a unit of U'. By [Gub07a, Proposition 2.11], there are $u_i \in \mathscr{O}(\mathscr{U}')^\times$, $\mathbf{m}_i \in \mathbb{Z}^{r+1}$ and $\lambda_i \in \mathbb{K}^\times$ with

$$F^*(x_i) = \lambda_i u_i \psi^*(\mathbf{x}')^{\mathbf{m}_i} \tag{20}$$

for $i = 1, \ldots, n$, where $\psi : \mathscr{U}' \to \mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$ is again the étale morphism and $\mathbf{x}' = (x'_0, \ldots, x'_r)$ are the semi-stable coordinates from 5.1. Let $S \in \operatorname{str}(\tilde{\mathscr{X}}')$ be the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. Then the canonical simplex Δ_S may be identified with the simplex $\{u'_0 + \cdots + u'_r = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$ and we define $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ by

$$f_{\mathrm{aff}}(u'_0, \ldots, u'_r) := (\mathbf{m}_i \cdot \mathbf{u}' + v(\lambda_i))_{i=1,\ldots,n}. \tag{21}$$

We note that this definition depends only on $|\lambda_i|$, $|x'_i|$ and $\mathbf{m}_i$; hence, it is independent of the trivialization of $U_{V,\Delta}$. If we change V and Δ, then the new lift is obtained from F by an M-translation. We deduce easily that the locally defined maps f_{aff} induce a well-defined map $\overline{f}_{\mathrm{aff}} : S(\mathscr{X}') \to \mathbb{R}^n/\Lambda$. All the claimed properties follow from the construction and uniqueness is clear from surjectivity of Val. □

Remark 5.12. More generally, Berkovich [Ber04, Corollary 6.1.2] has shown that a morphism between strongly non-degenerate pluristable formal schemes over $\mathbb{K}^\circ$ induces a piecewise linear map between the skeletons. Now Proposition 5.11 describes precisely the domain of affineness and we will see in Remark 5.19 that this holds also if $\mathscr{X}'$ is a strongly non-degenerate strictly pluristable formal scheme over $\mathbb{K}^\circ$.

By Propositions 4.8 and 5.7, we conclude easily that every stratum of $\tilde{\mathscr{X}}'$ is mapped into a stratum of $\tilde{\mathscr{A}}_0$. This will be proved in a more general context in Lemma 5.15. The preserving of strata is a key fact which will allow us to describe canonical measures on $X = f(X')$ in terms of the skeleton of $\mathscr{X}'$.

Proposition 5.13. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal subdivision of $\overline{\mathscr{C}_0}$ with associated Mumford model $\mathscr{A}$ of A. Then $(\mathscr{X}' \times_{\mathscr{A}_0} \mathscr{A})^{\mathrm{f-an}}$ is the formal analytic variety $\mathfrak{X}''$ from Proposition 5.5 associated to the Γ-rational subdivision $\mathscr{D}$ of $S(\mathscr{X}')$ given by*

$$\mathscr{D} := \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \operatorname{str}(\tilde{\mathscr{X}}'), \overline{\sigma} \in \overline{\mathscr{C}}\}.$$

Proof. We will use the notation from the proof of Proposition 5.11. Let $\sigma \in \mathscr{C}$ be contained in $\Delta \in \mathscr{C}_0$, let V be the generic fibre of a formal affine open subset $\mathscr{V}$ of $\mathscr{B}$ and let $\mathscr{U}'$ be a formal affine open subset of $\mathscr{X}'$ as in Proposition 5.2 with $\varphi_0(\mathscr{U}') \subset \mathscr{U}_{[V,\Delta]}$. Then the sets

$$(\mathscr{U}' \times_{\mathscr{U}_{[V,\Delta]}} \mathscr{U}_{[V,\sigma]})^{\mathrm{an}} = U' \times_{A^{\mathrm{an}}} U_{[V,\sigma]}$$

form a formal affinoid atlas of $(\mathscr{X}' \times_{\mathscr{A}_0} \mathscr{A})^{\mathrm{f-an}}$. We have $U_{[V,\sigma]} = U_{[V,\Delta]} \cap \overline{\mathrm{val}}^{-1}(\overline{\sigma})$ and hence Proposition 5.11 yields

$$U' \times_{A^{\mathrm{an}}} U_{[V,\sigma]} = U' \cap f^{-1}(\overline{\mathrm{val}}^{-1}(\overline{\sigma})) = U' \cap \mathrm{Val}^{-1}(\overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma})).$$

Let S be the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. Therefore, we have $\mathrm{Val}(U') = \Delta_S$ and we deduce that

$$(\mathscr{U}' \times_{\mathscr{U}_{[V,\Delta]}} \mathscr{U}_{[V,\sigma]})^{\mathrm{an}} = U' \cap \mathrm{Val}^{-1}(\sigma')$$

for the polytope $\sigma' := \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \cap \Delta_S \in \mathscr{D}$. These sets form the formal affinoid atlas (12) for $\mathfrak{X}''$, proving $\mathfrak{X}'' = (\mathscr{X}' \times_{\mathscr{A}_0} \mathscr{A})^{\mathrm{f-an}}$. □

Proposition 5.14. *We keep the above assumptions and notation. Then $\mathfrak{X}''$ is the coarsest formal analytic variety on X' such that $f : X' \to A^{\mathrm{an}}$ induces a formal analytic morphism $\phi : \mathfrak{X}'' \to \mathscr{A}^{\mathrm{f-an}}$. If R is the stratum of $\tilde{\mathfrak{X}}''$ corresponding to the open face τ of $\mathscr{D}$, then $\tilde{\phi}(R)$ is contained in the stratum of $\tilde{\mathscr{A}}$ corresponding to the unique open face $\overline{\sigma}$ of $\overline{\mathscr{C}}$ with $\overline{f}_{\mathrm{aff}}(\tau) \subset \overline{\sigma}$.*

Proof. The first claim is clear by construction. By definition of $\mathscr{D}$, there is an open face $\overline{\sigma}$ of $\overline{\mathscr{C}}$ with $\overline{f}_{\mathrm{aff}}(\tau) \subset \overline{\sigma}$. If π denotes the reduction map, then we get

$$\tilde{\phi}(R) = \tilde{\phi}(\pi(\mathrm{Val}^{-1}(\tau))) = \pi(f(\mathrm{Val}^{-1}(\tau))) \overset{5.11}{\subset} \pi(\overline{\mathrm{val}}^{-1}(\overline{\sigma})).$$

By Proposition 4.8, we deduce that $\tilde{\phi}(R)$ is contained in the stratum of $\tilde{\mathscr{A}}$ corresponding to $\overline{\sigma}$. □

Again, let $\mathscr{X}''$ be the admissible formal $\mathbb{K}^\circ$-scheme associated to the formal analytic variety $\mathfrak{X}''$ from Propositions 5.13 and 5.5. The following commutative diagram gives an overview of the occurring canonical morphisms of admissible formal schemes, where $\mathscr{E}_0$ (respectively $\mathscr{E}$) is the $\mathbb{K}^\circ$-model of the uniformization E associated to $\mathscr{C}_0$ (respectively $\mathscr{C}$) as in 4.7 and where the vertical maps extend the identity on the generic fibre.

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "\varphi"] \arrow[d, "\iota'"] & \mathscr{A} \arrow[d, "\iota_0"] & \mathscr{E} \arrow[l, "p"'] \arrow[d, "\iota"] \\ \mathscr{X}' \arrow[r, "\varphi_0"] & \mathscr{A}_0 & \mathscr{E}_0 \arrow[l, "p_0"'] \end{tikzcd} \tag{22}$$

Lemma 5.15. *Let $R \in \mathrm{str}(\tilde{\mathscr{X}}'')$. Then $S := \tilde{\iota}'(R)$ is a stratum of $\tilde{\mathscr{X}}'$. The restricted morphism $\tilde{\varphi}_0 : \overline{S} \to \tilde{\mathscr{A}_0} = \tilde{\mathscr{E}_0}/M$ has a lift $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}_0}$, unique up to the M-action on $\tilde{\mathscr{E}_0}$. Moreover, there is a unique lift $\tilde{\Phi} : \overline{R} \to \tilde{\mathscr{E}}$ of $\tilde{\varphi} : \overline{R} \to \tilde{\mathscr{A}} = \tilde{\mathscr{E}}/M$ with $\tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{\iota} \circ \tilde{\Phi}$ on $\overline{R}$.*

Proof. The first claim was proved in Corollary 5.9. The proof of the remaining claims follows standard arguments from the theory of coverings (applied to the quotient maps $\tilde{p}_0 : \tilde{\mathscr{E}_0} \to \tilde{\mathscr{A}_0} = \tilde{\mathscr{E}_0}/M$ and $\tilde{p} : \tilde{\mathscr{E}} \to \tilde{\mathscr{A}} = \tilde{\mathscr{E}}/M$).

Let Y be an irreducible component of $\tilde{\mathscr{A}_0}$ with $\tilde{\varphi}_0(\overline{S}) \subset Y$. By Proposition 4.8, Y corresponds to a vertex $\overline{\mathbf{u}}$ of $\overline{\mathscr{C}_0}$ and $\tilde{p}_0^{-1}(Y)$ is the disjoint union of the irreducible components $Y_{\mathbf{u}}$ of $\tilde{\mathscr{E}_0}$ associated to the vertices $\mathbf{u}$ of $\mathscr{C}_0$ with residue class $\overline{\mathbf{u}} \in \mathbb{R}^n/\Lambda$. Moreover, $Y_{\mathbf{u}}$ is mapped isomorphically onto Y by $\tilde{p}_0$. Using composition with the inverse $Y \to Y_{\mathbf{u}}$ of this isomorphism, we get the desired lift $\tilde{\Phi}_0$ of the restriction of $\tilde{\varphi}_0$ to $\overline{S}$. Uniqueness up to the M-action on $\tilde{\mathscr{E}_0}$ is obvious. Similarly, we get a lift $\tilde{\Phi}$ of the restriction of $\tilde{\varphi}$ to $\overline{R}$ by working with $\overline{\mathscr{C}}$ instead of $\overline{\mathscr{C}_0}$. The lift $\tilde{\Phi}$ is also unique up to the M-action on $\tilde{\mathscr{E}}$.

It remains to prove that $\tilde{\Phi}_0$ determines $\tilde{\Phi}$ uniquely by the condition $\tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{\iota} \circ \tilde{\Phi}$. Let us choose $\tilde{x}' \in R$ and let $\tilde{x} := \tilde{\iota}'(\tilde{x}')$. Note that the lift $\tilde{\Phi}_0$ is uniquely determined by choosing an

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element $\tilde{y} \in \tilde{p}_0^{-1}(\tilde{\varphi}_0(\tilde{x}))$ if we require that $\tilde{\Phi}_0(\tilde{x}) = \tilde{y}$. Similarly, $\tilde{\Phi}$ is determined by $\tilde{\Phi}(\tilde{x}') = \tilde{y}'$ for some $\tilde{y}' \in \tilde{p}^{-1}(\tilde{\varphi}(\tilde{x}))$. Since M acts faithfully and transitively on $\tilde{p}_0^{-1}(\tilde{\varphi}_0(\tilde{x}))$ (respectively on $\tilde{p}^{-1}(\tilde{\varphi}(\tilde{x}))$), there is a unique $\tilde{y}'$ with $\tilde{\iota}(\tilde{y}') = \tilde{y}$. Note that $\tilde{p}_0$ and $\tilde{p}$ are local isomorphisms. Since $\tilde{\varphi}_0 \circ \tilde{\iota}' = \tilde{\iota}_0 \circ \tilde{\varphi}$, this lifts to the identity $\tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{\iota} \circ \tilde{\Phi}$ on $\overline{R}$ for a unique $\tilde{\Phi}$. □

Remark 5.16. We may use the same techniques to construct a lift of the morphism $f : X' \to A^{\mathrm{an}} = E/M$ to the uniformization E of A. In general, such a lift does not exist globally on X'. By [BL91, Theorem 1.2], such a lift exists if $H^1(X', \mathbb{Z}) = 0$. Let us consider the formal open subset $U_S := \mathrm{Val}^{-1}(\Delta_S)$ of X' for the canonical simplex Δ_S associated to a stratum S of $\tilde{\mathscr{X}}'$. Then U_S is the generic fibre of a formal open subset $\mathscr{U}_S$ of $\mathscr{X}'$. Obviously, $\mathscr{U}_S$ is strictly semi-stable with skeleton Δ_S. Since the skeleton is a proper deformation retraction of the generic fibre, we get $H^1(U_S, \mathbb{Z}) = 0$ and hence we may apply the above results to get the desired lift $F : U_S \to E$ of $f|_{U_S}$.

Note that it is not necessary to appeal to such sophisticated results. We may just use Proposition 5.11 to conclude that $\overline{f}_{\mathrm{aff}}(\Delta_S)$ is contained in a polytope $\overline{\Delta} \in \overline{\mathscr{C}_0}$ and hence $f(U_S)$ is contained in the formal open subset $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$ of A^{an}. The preimage of $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$ in E is the disjoint union of the formal open subsets $\mathrm{val}^{-1}(\Delta)$, where Δ ranges over all polytopes of $\mathscr{C}_0$ mapping (bijectively) onto $\overline{\Delta}$ with respect to the residue map $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$. Obviously, M acts faithfully and transitively on the set of all such $\mathrm{val}^{-1}(\Delta)$. Since the quotient morphism $p : E \to A^{\mathrm{an}} = E/M$ maps $\mathrm{val}^{-1}(\Delta)$ isomorphically onto $\overline{\mathrm{val}}^{-1}(\overline{\Delta})$, we get a lift $F : U_S \to E$ of the restriction of f to U_S, unique up to the M-action. Note that this construction was partially used in the proof of Proposition 5.11.

By Proposition 5.11, we get a unique map $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ such that $f_{\mathrm{aff}} \circ \mathrm{Val} = \mathrm{val} \circ F$ on U_S. Moreover, f_{aff} is affine on Δ_S. Conversely, every lift of $\overline{f}_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n/\Lambda$ to $\mathbb{R}^n$ is an affine map $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ and there is a unique lift $F : U_S \to E$ of the restriction of f to U_S such that $f_{\mathrm{aff}} \circ \mathrm{Val} = \mathrm{val} \circ F$ on U_S. This follows from the fact that the lift of $\overline{f}_{\mathrm{aff}}$ is unique up to $\Lambda = \mathrm{val}(M)$-translation.

Finally, we note that we may use such lifts F to construct the lifts $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}}_0$ and $\tilde{\Phi} : \overline{R} \to \tilde{\mathscr{E}}$ from Lemma 5.15. We will give the construction for $\tilde{\Phi}_0$, but everything works similarly for $\tilde{\Phi}$. The map f is the generic fibre of the formal morphism $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$ and $\mathscr{U}_S$ is a formal open subset of $\mathscr{X}'$; hence, the lift F is the generic fibre of a formal lift $\Phi_S : \mathscr{U}_S \to \mathscr{E}_0$ of φ_0. We conclude that the reduction $\tilde{\Phi}_S$ agrees with a lift $\tilde{\Phi}_0$ from Lemma 5.15 on the dense stratum S of $\overline{S}$. Similarly, we could argue for every other stratum $T \subset \overline{S}$ to describe the restriction of $\tilde{\Phi}_0$ to T as the reduction of a formal lift of φ_0. However, it is not always possible to describe $\tilde{\Phi}_0$ by the reduction of a single formal lift defined on a formal open subset of $\mathscr{X}'$. The problem arises if there are two strata T_1, T_2 in $\overline{S}$ such that $f_{\mathrm{aff}}(\Delta_{T_1}) \cup f_{\mathrm{aff}}(\Delta_{T_2})$ does not map bijectively onto $\overline{f}_{\mathrm{aff}}(\Delta_{T_1}) \cup \overline{f}_{\mathrm{aff}}(\Delta_{T_2})$. In this case, the lift F will be multivalued on $U_{T_1} \cup U_{T_2}$ and the above covering argument breaks down. This problem can be omitted if we start with a sufficiently fine polytopal decomposition $\overline{\mathscr{C}_0}$ of $\mathbb{R}^n/\Lambda$ and then $\tilde{\Phi}_0$ is indeed the reduction of a single formal lift.

5.17 Our goal is to compute the degree of an irreducible component Y of $\tilde{\mathfrak{X}}''$ with respect to a line bundle $\mathscr{L}$ on $\mathscr{A}$. This can be done in terms of convex geometry under the following hypotheses fulfilled in our applications.

We still have our abelian variety A over $\mathbb{K}$ with uniformization E and the morphism $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$, where $\mathscr{A}_0$ is the Mumford model of A associated to the Γ-rational polytopal decomposition $\overline{\mathscr{C}_0}$ of $\mathbb{R}^n/\Lambda$ and where $\mathscr{X}'$ is a strictly semi-stable formal scheme over $\mathbb{K}^\circ$ with connected generic fibre X'. We assume that the generic fibre $f : X' \to A^{\mathrm{an}}$ of φ_0 is proper and hence the special fibre $\tilde{\varphi}_0$ is also proper (see [Gub98, Remark 3.14]). Let $\overline{\mathscr{C}_1}$ be a Γ-rational polytopal decomposition of $\mathbb{R}^n/\Lambda$ with associated Mumford model $\mathscr{A}_1$ of A. We now choose $\overline{\mathscr{C}} := \overline{\mathscr{C}_0} \cap \overline{\mathscr{C}_1} := \{\overline{\Delta_0} \cap \overline{\Delta_1} \mid \overline{\Delta_0} \in \overline{\mathscr{C}_0}\,, \overline{\Delta_1} \in \overline{\mathscr{C}_1}\}$. Let $\mathscr{A}$ be the Mumford model of A associated to $\overline{\mathscr{C}}$. We apply Propositions 5.13 and 5.14 to this setup. By (22), we get the following commutative diagram of canonical morphisms of admissible formal schemes over $\mathbb{K}^\circ$.

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "\varphi"] \arrow[d, "\iota'"] & \mathscr{A} \arrow[r, "\iota_1"] \arrow[d, "\iota_0"] & \mathscr{A}_1 \\ \mathscr{X}' \arrow[r, "\varphi_0"] & \mathscr{A}_0 & \end{tikzcd} \tag{23}$$

Recall that all admissible formal schemes in (23) are associated to formal analytic varieties and that the morphism φ is determined by the fact that the rectangle is cartesian on the level of formal analytic varieties.

By Corollary 5.9, the irreducible component Y of $\tilde{\mathscr{X}}''$ corresponds to the vertex $\mathbf{u}' = \xi_Y$ of the Γ-rational subdivision

$$\mathscr{D} = \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}'),\ \overline{\sigma} \in \overline{\mathscr{C}}\}$$

of $S(\mathscr{X}')$. There is a unique $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ such that $\mathbf{u}' \in \mathrm{relint}(\Delta_S)$. If S' is the dense stratum in Y, then Corollary 5.9 yields $S = \tilde{\iota}'(S')$. We choose a lift $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ of $\overline{f}_{\mathrm{aff}}$. By Lemma 5.15, there is a lift $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}}_0$ (respectively $\tilde{\Phi} : Y \to \tilde{\mathscr{E}}$) of $\tilde{\varphi}_0$ (respectively $\tilde{\varphi}$) to the special fibre of the formal $\mathbb{K}^\circ$-model $\mathscr{E}_0$ (respectively $\mathscr{E}$) of E associated to $\mathscr{C}_0$ (respectively $\mathscr{C}$) with $\tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{\iota} \circ \tilde{\Phi}$.

Let L be a line bundle on A. The role of $\mathscr{A}_1$ now becomes clear as we assume that L has a formal $\mathbb{K}^\circ$-model $\mathscr{L}$ of L on $\mathscr{A}_1$ corresponding to a continuous piecewise affine function $f_{\mathscr{L}}$ as in Proposition 4.11 (applied to $\overline{\mathscr{C}_1}$). We assume that $g := f_{\mathscr{L}} \circ f_{\mathrm{aff}}$ is convex in a neighborhood of $\mathbf{u}'$. In the light of Proposition 4.12, this is a natural positivity assumption for $\mathscr{L}$. We have seen in 5.3 that we may identify Δ_S with the simplex $\{w_0' + \cdots + w_r' = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$. In the following, it is more convenient to identify Δ_S with the simplex $\{w_1' + \cdots + w_r' \leqslant v(\pi)\}$ in $\mathbb{R}_+^r$ obtained by omitting the coordinate w_0'. Then we define a polytope $\{\mathbf{u}'\}^g$ in $\mathbb{R}^r$ by

$$\{\mathbf{u}'\}^g := \{\omega \in \mathbb{R}^r \mid \mathbf{w}' \in \Delta \in \mathrm{star}_r(\mathbf{u}') \Rightarrow \omega \cdot (\mathbf{w}' - \mathbf{u}') \leqslant g(\mathbf{w}') - g(\mathbf{u}')\},$$

where $\mathrm{star}_r(\mathbf{u}')$ is the set of r-dimensional polytopes in $\mathscr{D}$ with vertex $\mathbf{u}'$. The volume of $\{\mathbf{u}'\}^g$ with respect to the Lebesgue measure on $\mathbb{R}^r$ will be denoted by $\mathrm{vol}(\{\mathbf{u}'\}^g)$. By 4.3, there is a line bundle $\mathscr{H}$ on the formal abelian scheme $\mathscr{B}$ from the Raynaud extension (4) such that $p^*(L^{\mathrm{an}}) = q^*(H)$ for the generic fibre H of $\mathscr{H}$ and the canonical morphisms $p : E \to A^{\mathrm{an}} = E/M$, $q : E \to B = \mathscr{B}^{\mathrm{an}}$. We now have the following commutative diagram of varieties over $\tilde{\mathbb{K}}$.

$$\begin{tikzcd} Y \arrow[r, "\tilde{\Phi}"] \arrow[d, "\tilde{\iota}'"] & \tilde{\mathscr{E}} \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{\iota}_0"] & \tilde{\mathscr{B}} \arrow[d, "\mathrm{id}"] \\ \overline{S} \arrow[r, "\tilde{\Phi}_0"] & \tilde{\mathscr{E}}_0 \arrow[r, "\tilde{q}_0"] & \tilde{\mathscr{B}} \end{tikzcd} \tag{24}$$

For simplicity of notation, we will write $\deg_{\mathscr{L}}(Y)$ for the degree of Y with respect to the line bundle $(\tilde{\iota}_1 \circ \tilde{\varphi})^*(\tilde{\mathscr{L}})$ and similarly for other degrees. It is always understood that we use the pull-backs of the line bundles with respect to the canonical morphisms from (23) or (24).

By Proposition 5.11, it is easy to deduce that

$$\mathscr{D} = \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}'),\ \overline{\sigma} \in \overline{\mathscr{C}_1}\}. \tag{25}$$

There is a unique $\overline{\Delta_1} \in \overline{\mathscr{C}_1}$ with $\overline{\mathbf{u}} := \overline{f}_{\mathrm{aff}}(\mathbf{u}') \in \mathrm{relint}(\overline{\Delta_1})$. Since the vertex $\mathbf{u}'$ of $\mathscr{D}$ is contained in $\mathrm{relint}(\Delta_S)$, it follows from (25) that

$$\{\mathbf{u}'\} = \overline{f}_{\mathrm{aff}}^{-1}(\overline{\Delta_1}) \cap \Delta_S, \quad \{\overline{\mathbf{u}}\} = \overline{\Delta_1} \cap \overline{f}_{\mathrm{aff}}(\Delta_S).$$

The first equality yields that the affine map $\overline{f}_{\mathrm{aff}}|_{\Delta_S}$ is injective and hence $\overline{f}_{\mathrm{aff}}(\Delta_S)$ is a $(d-e)$-dimensional simplex in $\mathbb{R}^n/\Lambda$, where $d := \dim(X')$ and $e := \dim(S) = d - \dim(\Delta_S)$. We now make the *transversality assumption*

$$d - e = \mathrm{codim}(\Delta_1, \mathbb{R}^n). \tag{26}$$

Proposition 5.18. *Using the assumptions from 5.17, we have*

$$\deg_{\mathscr{L}}(Y) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}'\}^g).$$

Proof. Let $\mathscr{E}_1$ be the $\mathbb{K}^\circ$-model of E associated to $\overline{\mathscr{C}_1}$. In the following, we will always use the canonical morphisms $p_1 : \mathscr{E}_1 \to \mathscr{A}_1$, $q_1 : \mathscr{E}_1 \to \mathscr{B}$, $i_1 : \mathscr{E} \to \mathscr{E}_1$ and $\tilde{\Phi}_1 := \tilde{\imath}_1 \circ \tilde{\Phi}$ to compute degrees. Using $p^*(L^{\mathrm{an}}) = q^*(H)$, we have the decomposition

$$p_1^*(\mathscr{L}) = q_1^*(\mathscr{H}) \otimes \mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}) \tag{27}$$

for a formal $\mathbb{K}^\circ$-model $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})$ of O_E on $\mathscr{E}_1$. The reason behind the notation is that the formal metric on the trivial bundle O_E associated to the formal model $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})$ (see Example 3.3) satisfies

$$-\log \|s\|_{\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})} = f_{\mathscr{L}} \circ \mathrm{val}, \tag{28}$$

where s is the unique meromorphic section of $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})$ extending the canonical section 1 of O_E. This follows immediately from the definition of $f_{\mathscr{L}}$ in (8). In the decomposition (27), $q_1^*(\mathscr{H})$ reflects the contribution of the abelian part $\mathscr{B}$ to $\mathscr{L}$ and $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})$ measures the combinatorial contribution from the polytopal decomposition $\mathscr{C}_1$ and from the piecewise affine function $f_{\mathscr{L}}$. We deduce that

$$\deg_{\mathscr{L}}(Y) = \sum_{\ell=0}^{d} \binom{d}{\ell} d_\ell(Y), \tag{29}$$

from (27), where $d_\ell(Y) := \deg_{\underbrace{\mathscr{H}, \ldots, \mathscr{H}}_{\ell}, \underbrace{\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}), \ldots, \mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})}_{d-\ell}}(Y)$.

Our goal is now to prove that $d_\ell(Y) = 0$ for $\ell \neq e$ and to compute $d_e(Y)$ using the projection formula with respect to $\tilde{\imath}' : Y \to \overline{S}$ and

$$\tilde{\Phi}^*(\tilde{q}^*(\tilde{\mathscr{H}})) = (\tilde{\imath}')^*(\tilde{\Phi}_0^*(\tilde{q}_0^*(\tilde{\mathscr{H}}))) \tag{30}$$

obtained from (24).

Step 1. The cycle class $c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y$ in $CH(Y)$ is algebraically equivalent to a strata cycle of Y.

It is always understood that $c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))$ operates by pull-back with respect to $\tilde{\Phi}_1$ on Y. Again, we denote by s the unique meromorphic section of $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})$ extending the canonical section 1 of O_E. It is enough to show that $\mathrm{div}(s).\overline{S''}$ is algebraically equivalent to a strata cycle of Y for every $S'' \in \mathrm{str}(\tilde{\mathscr{X}}'')$ with $S'' \subset Y$. We have seen in Proposition 5.14 that $\tilde{\varphi} = \tilde{\phi}$ maps strata into strata. By Proposition 4.8, we easily deduce the same property for $\tilde{\imath}_1$. Passing to the lift $\tilde{\Phi}_1$, we see that $\tilde{\Phi}_1(S'')$ is contained in a stratum Z of $\tilde{\mathscr{E}}_1$. By Proposition 4.8, Z corresponds

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to $\mathrm{relint}(\sigma)$ for a unique $\sigma \in \mathscr{C}_1$. Using

$$f_{\mathscr{L}}(\mathbf{u}) = \mathbf{m}_\sigma \cdot \mathbf{u} + v(a_\sigma)$$

on σ with $\mathbf{m}_\sigma \in \mathbb{Z}^n$ and $a_\sigma \in \mathbb{K}^\times$, we deduce from (28) that the Cartier divisor $\mathrm{div}(s)$ is given on $\mathrm{val}^{-1}(\sigma)$ by $a_\sigma \cdot \mathbf{x}^{\mathbf{m}_\sigma}$. Here, we consider $\chi = \mathbf{x}^{\mathbf{m}_\sigma}$ as a meromorphic section of $q_1^*(\mathscr{O}_\chi)$ which restricts to a nowhere-vanishing global section on the generic fibre $q^*(O_\chi)$ (see 4.2). We consider the Cartier divisor $D := \mathrm{div}(s/(a_\sigma \cdot \mathbf{x}^{\mathbf{m}_\sigma}))$ on $\mathscr{E}_1$. It has a well-defined reduction $\tilde{D}$ on a neighborhood of $\overline{Z}$ which is trivial on $(\mathrm{val}^{-1}(\sigma))^\sim$ and hence $\tilde{\Phi}_1^*(\tilde{D})$ is a Cartier divisor on Y which is trivial on S''. Since $\overline{S''}$ is a strata subset, $\tilde{\Phi}_1^*(\tilde{D}).\overline{S''}$ is a strata cycle in Y. We have seen in 4.2 that $\tilde{\mathscr{O}}_\chi$ is algebraically equivalent to 0 and hence $\mathscr{O}(\tilde{D})|_{\overline{Z}}$ is algebraically equivalent to $\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}})^\sim|_{\overline{Z}}$. By construction, $\mathrm{div}(s).\overline{S''}$ is algebraically equivalent to $\tilde{\Phi}_1^*(\tilde{D}).\overline{S''}$, proving the first step.

We need an explicit description of the Cartier divisor $\tilde{\Phi}_1^*(\tilde{D})$ on Y from Step 1. Let $\mathscr{U}'$ be a formal affine open subset of $\tilde{\mathscr{X}}'$ as in Proposition 5.2 with étale morphism $\psi : \mathscr{U}' \to \mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$ such that S is the distinguished stratum of $\tilde{\mathscr{X}}'$ associated to $\mathscr{U}'$. Passing to a formal open refinement, we may assume that $\varphi_0(\mathscr{U}')$ is contained in a formal trivialization of the Raynaud extension (3) and hence the torus coordinates $x_1, \ldots, x_n$ make sense on $\varphi_0(\mathscr{U}')$.

We denote by $\mathscr{S}_1'$ the $\mathbb{K}^\circ$-model of the polytopal domain $\mathscr{S}_1^{\mathrm{an}}$ in $(\mathbb{G}_m^r)^{\mathrm{an}}$ associated to the refinement $\mathscr{D} \cap \Delta_S$ and let $\mathscr{U}'' := (\iota')^{-1}(\mathscr{U}')$. We have seen in Remark 5.6 that $\psi' : \mathscr{U}'' \to \mathscr{S}' = \mathscr{S}_1' \times \mathscr{S}_2$ is the base change of ψ to $\mathscr{S}'$ and hence ψ' is étale. Let $\psi_1 : \mathscr{U}' \to \mathscr{S}_1'$ be the composition of the first projection with ψ and let ψ_1' be the base change of ψ_1 to $\mathscr{S}_1'$. Then $\tilde{\psi}_1'$ is a smooth morphism such that the preimage of a stratum of $\tilde{\mathscr{S}}_1'$ is a stratum of $\tilde{\mathscr{U}}''$ (see Remark 5.8). We conclude that $\tilde{\psi}_1'(Y \cap \tilde{\mathscr{U}}'')$ is dense in an irreducible component of $\tilde{\mathscr{S}}_1'$, which we denote by $Y_{\mathbf{u}'}$. This notation is justified by the fact that the irreducible components of $\mathscr{S}_1'$ are in bijective correspondence with the vertices of $\mathscr{D} \cap \Delta_S$ (see [Gub07a, Proposition 4.7]).

Step 2. There is a Cartier divisor $\tilde{D}_1$ on $Y_{\mathbf{u}'}$ with $(\tilde{\psi}_1')^*(\tilde{D}_1) = \tilde{\Phi}_1^*(\tilde{D})|_{Y \cap \tilde{\mathscr{U}}''}$.

By Remark 5.16, the lift $\tilde{\Phi}_1 : Y \cap \tilde{\mathscr{U}}'' \to \tilde{\mathscr{E}}_1$ is equal to the reduction of a suitable lift $F : U' \to E$ of f. Moreover, F induces a lift $\Delta_S \to \mathbb{R}^n$ of $\overline{f}_{\mathrm{aff}}$. We may assume that the lift is equal to f_{aff} from 5.17. Indeed, f_{aff} is determined up to Λ-translation and hence the polytope $\{\mathbf{u}'\}^g$ is also determined up to translation, which does not affect the volume in Proposition 5.18.

We consider the polytopes ν of $\mathscr{C}_1$ with closed face Δ_1 from 5.17. We have seen in the first step that the Cartier divisor D is given on $\mathrm{val}^{-1}(\nu)$ by $(a_\nu/a_\sigma) \cdot \mathbf{x}^{\mathbf{m}_\nu - \mathbf{m}_\sigma}$. It follows easily from the definitions that the polytopes $\mu := (f_{\mathrm{aff}})^{-1}(\nu) \cap \Delta_S$ are just the polytopes of $\mathscr{D} \cap \Delta_S$ with vertex $\mathbf{u}'$. By Proposition 5.7 and Corollary 5.9, the reductions of the formal open subsets $\mathrm{Val}^{-1}(\mu)$ cover Y. By construction, F induces a formal morphism $\Phi_1 : \mathscr{U}'' \to \mathscr{A}_1$ with reduction $\tilde{\Phi}_1$ and hence $\tilde{\Phi}_1^*(\tilde{D}) = (\Phi_1^*(D))^\sim$ on $Y \cap \tilde{\mathscr{U}}''$. By [Gub07a, Proposition 2.11], there is an $n \times r$ matrix M with entries in $\mathbb{Z}$, $\gamma_i \in \mathscr{O}(\mathscr{U}')^\times$ and $\lambda \in (\mathbb{K}^\times)^n$ with

$$F^*(x_i) = \lambda_i \cdot \gamma_i \cdot (\psi^{\mathrm{an}})^*(x_1')^{M_{i1}} \cdots (\psi^{\mathrm{an}})^*(x_r')^{M_{ir}}$$

on the generic fibre U' of $\mathscr{U}'$ for $i = 1, \ldots, n$. We have seen in (21) that

$$f_{\mathrm{aff}}(\mathbf{w}') = M\mathbf{w}' + \lambda$$

for $\mathbf{w}' = (w_1', \ldots, w_r') \in \Delta_S$. Note that we use here the identification of Δ_S with the simplex $\Sigma_S := \{w_1' + \cdots + w_r' \leqslant v(\pi)\}$ in $\mathbb{R}_+^r$, which is different from the one used in (21).

Let $\mathbf{y} := (x_1', \dots, x_r')$. We conclude that $\Phi_1^*(D)$ is given on $\mathrm{Val}^{-1}(\mu) \cap U'$ by

$$\frac{a_\nu}{a_\sigma} \cdot F^*(\mathbf{x}^{\mathbf{m}_\nu - \mathbf{m}_\sigma}) = \frac{a_\nu}{a_\sigma} \cdot \lambda^{\mathbf{m}_\nu - \mathbf{m}_\sigma} \cdot \gamma \cdot (\psi^{\mathrm{an}})^*(\mathbf{y})^{(\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M}$$

for some $\gamma \in \mathscr{O}(\mathscr{U}')^\times$. For a Cartier divisor, such a unit γ can be omitted. Let $\alpha_\mu \in \mathbb{K}^\times$ with $v(\alpha_\mu) + ((\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M) \cdot \mathbf{u}' = 0$. Then $\Phi_1^*(D)$ is given by $\alpha_\mu \cdot (\psi^{\mathrm{an}})^*(\mathbf{y})^{(\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M}$ on $\mathrm{Val}^{-1}(\mu) \cap U'$. These functions are also defined on the formal open subsets $\mathrm{val}^{-1}(\mu)$ of $(\mathscr{S}_1')^{\mathrm{f-an}}$. Let $\mathscr{U}_\mu$ be the formal affine open subset of $\mathscr{S}_1'$ associated to $\mathrm{val}^{-1}(\mu)$ and let $D_1 := \{\mathscr{U}_\mu, \alpha_\mu \cdot \mathbf{y}^{(\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M}\}$ with μ ranging over the polytopes of $\mathscr{D} \cap \Delta_S$ with vertex $\mathbf{u}'$. It is easy to see that D_1 is a Cartier divisor on the open subset $\bigcup_\mu \mathscr{U}_\mu$ of $\mathscr{S}_1'$ containing $Y_{\mathbf{u}'}$. We conclude that $(\tilde{\psi}_1')^*(\tilde{D}_1) = \tilde{\Phi}_1^*(\tilde{D})|_{Y \cap \tilde{\mathscr{U}}''}$, proving the second step.

We note that the Cartier divisor D_1 depends on the choice of the stratum S'', but the linear equivalence class of the Cartier divisor D_1 is independent of S'' and hence the same is true for the linear equivalence class of $\tilde{D}_1$ on $Y_{\mathbf{u}'}$.

Step 3. $d_\ell(Y) = 0$ for $\ell \neq e$.

If $\ell > e$, then $c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y$ has dimension $\ell > e = \dim(S)$ and hence the projection formula with respect to $\tilde{\iota}' : Y \to \overline{S}$ and (30) prove that

$$\begin{aligned} d_\ell(Y) &= \deg(c_1(\mathscr{H})^\ell . c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y) \\ &= \deg(c_1(\mathscr{H})^\ell . \tilde{\iota}'_*(c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y)) = 0. \end{aligned}$$

It remains to consider $\ell < e$. We will use the first step for the dense stratum S' in Y (instead of S''). We conclude that $\tilde{\Phi}_1(S')$ is contained in the stratum Z of $\tilde{\mathscr{E}}_1$ corresponding to $\mathrm{relint}(\Delta_1)$. By Proposition 4.8, we have $\dim(Z) = \dim(A) - \dim(\Delta_1)$. By the construction in the first step, the cycle class

$$\alpha := (\tilde{\Phi}_1)_*(c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y) = c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.(\tilde{\Phi}_1)_*(Y)$$

is algebraically equivalent to a cycle supported in $\overline{Z_1}$ for a strata subset Z_1 of codimension $\geqslant d - \ell$ in $\overline{Z}$. We have

$$e = d - \mathrm{codim}(\Delta_1, \mathbb{R}^n) = d + \dim(B) - \dim(Z)$$

by our transversality assumption. Using $\ell < e$, we get

$$\dim(Z_1) \leqslant \dim(Z) - (d - \ell) = \dim(B) + \ell - e < \dim(B).$$

By Proposition 4.8, all strata of $\tilde{\mathscr{E}}_1$ have dimension $\geqslant \dim(B)$ and hence α is algebraically equivalent to 0. The projection formula now shows that

$$d_\ell(Y) = \deg(c_1(\mathscr{H})^\ell . (\tilde{\Phi}_1)_*(c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^{d-\ell}.Y)) = \deg(c_1(\mathscr{H})^\ell.\alpha) = 0.$$

Step 4. $d_e(Y) = (d-e)! \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}'\}^g)$.

Recall that $r = d - e$. By the first step, we know that $c_1(\mathscr{O}_{\mathscr{E}_1}(f_{\mathscr{L}}))^r.Y$ is algebraically equivalent to an e-dimensional strata cycle W of $\tilde{\mathscr{X}}''$. Since W has support in Y, its components have the form $\overline{S_i}$, where the $S_i \in \mathrm{str}(\tilde{\mathscr{X}}'')$ correspond to an open face of $\mathscr{D}_S \cap \Delta_S$ with vertex $\mathbf{u}'$. This follows from Corollary 5.9 as well as the fact that $\tilde{\iota}'$ maps S_i isomorphically onto S.

Now we will use the formal open subset $\mathscr{U}'' := (\iota')^{-1}(\mathscr{U}')$ of $\mathscr{X}''$ from Step 2. Since $\tilde{\iota}'(S_i) = S$, we have $S_i \cap \mathscr{U}'' \neq \emptyset$. The same holds for every stratum relevant in the intersection process for W described in Step 1. We conclude that we may compute W on $\mathscr{U}''$. We note that $\tilde{\psi}_1' : \tilde{\mathscr{U}}'' \to \tilde{\mathscr{S}}_1'$ is a smooth morphism and that the stratification of $\tilde{\mathscr{U}}''$ is obtained by the preimages of the strata

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of $\tilde{\mathscr{S}}'_1$ (see Remark 5.8). By the second step and the compatibility of flat pull-back with the intersection operations (see [Ful84, Proposition 2.3]), the intersection process on $Y \cap \mathscr{U}''$ leading to $W \cap \mathscr{U}''$ may be first performed on $Y_{\mathbf{u}'}$ giving a cycle W' and then $W \cap \mathscr{U}'' = (\tilde{\psi}'_1)^*(W')$. To obtain W', we have just to replace Y by $Y_{\mathbf{u}'}$ and the Cartier divisors $\tilde{\Phi}_1^*(\tilde{D})$ by $\tilde{D}_1$. It is clear that $W' = \sum n_i S'_i$, where the zero-dimensional strata S'_i of $\tilde{\mathscr{S}}'_1$ correspond to the same open faces as the S_i. We deduce that $W = \sum n_i \overline{S_i}$ and $\sum n_i = \deg_{D_1}(Y_{\mathbf{u}'})$. Using the projection formula with respect to $\tilde{\iota}' : Y \to \overline{S}$ and (30), we get

$$d_e(Y) = \deg(c_1(\mathscr{H})^e.W) = \deg(c_1(\mathscr{H})^e.\tilde{\iota}'_*(W)).$$

We have noticed that $S_i \cong S$ and hence we get

$$d_e(Y) = \deg_{\mathscr{H}}(\overline{S}) \sum_i n_i[\overline{S_i} : \overline{S}] = \deg_{\mathscr{H}}(\overline{S}) \deg_{D_1}(Y_{\mathbf{u}'}). \tag{31}$$

To compute the degree of $Y_{\mathbf{u}'}$, we will use the theory of toric varieties. The projection $\mathbb{G}_m^{r+1} \to \mathbb{G}_m^r$, given by $(x'_0, \ldots, x'_r) \mapsto (x'_1, \ldots, x'_r)$, leads to an isomorphism of $\mathscr{S}_1^{\mathrm{f-an}}$ with the polytopal domain $\mathrm{val}^{-1}(\Sigma_S)$ for the simplex $\Sigma_S := \{w'_1 + \cdots + w'_r \leqslant v(\pi)\}$ in $\mathbb{R}_+^r$. We recall from 5.17 that we identify Σ_S with Δ_S and hence $Y_{\mathbf{u}'}$ is equal to the $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$-toric variety associated to the vertex $\mathbf{u}'$ of $\mathscr{D} \cap \Sigma_S$ (see [Gub07a, Proposition 4.7]). As in the second step, let $\nu \in \mathscr{C}_1$ with closed face Δ_1 and let $\mu := f_{\mathrm{aff}}^{-1}(\nu) \cap \Delta_S$. Then the polytopes μ are just the polytopes of $\mathscr{D} \cap \Sigma_S$ with vertex $\mathbf{u}'$. We have seen in the second step that the Cartier divisor D_1 is given on the formal open subset $\mathrm{val}^{-1}(\mu)$ of $(\mathscr{S}'_1)^{\mathrm{an}}$ by $\alpha_\mu \cdot \mathbf{y}^{(\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M}$. In the theory of toric varieties, the Cartier divisor $\tilde{D}_1|_{Y_{\mathbf{u}'}}$ induces a polyhedron P as the set of all $\omega \in \mathbb{R}^r$ with

$$\forall \mathbf{w}' \in \mu \in \mathrm{star}_r(\mathbf{u}') \Rightarrow \omega \cdot (\mathbf{w}' - \mathbf{u}') \leqslant (\mathbf{m}_\nu - \mathbf{m}_\sigma)^t \cdot M \cdot (\mathbf{w}' - \mathbf{u}').$$

It is easy to see that P is a translate of our polytope $\{\mathbf{u}'\}^g$. By [Ful93, § 5.3], Corollary on p. 111, we get

$$\deg_{D_1}(Y_{\mathbf{u}'}) = r! \cdot \mathrm{vol}(P) = r! \cdot \mathrm{vol}(\{\mathbf{u}'\}^g).$$

Together with (31), this proves the fourth step. Finally, the proposition is a consequence of (29), Steps 3 and 4. □

Remark 5.19. Berkovich has defined the skeleton $S(\mathscr{X}')$ more generally for a non-degenerate pluristable formal scheme $\mathscr{X}'$ over $\mathbb{K}^\circ$ and he has shown that $S(\mathscr{X}')$ has a canonical piecewise linear structure (see [Ber04]). If $\mathscr{X}'$ is strongly non-degenerate, then there is a well-defined proper strong deformation retraction from the generic fibre X' to $S(\mathscr{X}')$ which generalizes the map Val.

All the results of § 5 can be generalized to a strongly non-degenerate *strictly* pluristable $\mathscr{X}'$. This is based on the following facts proved in the appendix: the linear pieces of $S(\mathscr{X}')$ are given by canonical plurisimplices Δ_S corresponding to the strata S of $\tilde{\mathscr{X}}'$. Moreover, Δ_S is a polytope with associated polytopal domain U_{Δ_S} (see 4.5). In analogy to Proposition 5.2, $\mathscr{X}'$ consists locally of open building blocks $\mathscr{U}'$ such that $S(\mathscr{U}')$ is a canonical plurisimplex Δ_S of $S(\mathscr{X}')$ and there is an étale morphism $\psi : \mathscr{U}' \to U_{\Delta_S}^{\mathrm{f-sch}}$.

Similarly as in the strictly semi-stable case, this allows us to prove the results of this section by using well-known results for polytopal domains. Moreover, we could replace strictly semi-stable formal schemes in § 6 by strongly non-degenerate strictly pluristable formal schemes. This is straightforward and we leave the details to the reader.

6. Canonical measures

In this section, K is a field with a discrete valuation v. We denote by $\mathbb{K}$ the completion of the algebraic closure of K. Note that $\mathbb{K}$ is algebraically closed [BGR84, Proposition 3.4.1/3] and the value group Γ is equal to $\mathbb{Q}$.

We consider a geometrically integral d-dimensional closed subvariety X of A over K. In § 3, we have defined canonical measures on X. Now we will compute them explicitly in terms of convex geometry. The main idea is to choose a Mumford model of A and a semi-stable alteration of X to apply the results from §§ 4 and 5. Note that the restriction to geometrically integral varieties is not a serious restriction. In general, we may perform a finite base change and then we can proceed by linearity in the components.

6.1 For our computations, Proposition 5.18 will be crucial. To fulfill its transversality assumption (26), we shall choose the polytopal decomposition of the Mumford model 'completely irrational'. We choose an infinite-dimensional $\mathbb{Q}$-subspace Γ' of $\mathbb{R}$ containing $\mathbb{Q}$. By [Bou64, ch. VI, $n°$ 10, Proposition 1], there is an algebraically closed field $\mathbb{K}'$, complete with respect to a valuation v' extending v, such that the value group $v'((\mathbb{K}')^\times)$ is Γ'.

6.2 We denote the analytic space over $\mathbb{K}$ associated to X by X^{an}. Let $\overline{\mathscr{C}_0}$ be a rational polytopal decomposition of $\mathbb{R}^n/\Lambda$ with associated Mumford model $\mathscr{A}_0$ of A^{an} over $\mathbb{K}^\circ$. We denote the closure of X^{an} in $\mathscr{A}_0$ by $\mathscr{X}_0$, which is a formal $\mathbb{K}^\circ$-model of X^{an} (see [Gub98, Proposition 3.3]). By de Jong's alteration theorem [deJ96, Theorem 6.5], applied to a projective $\mathbb{K}^\circ$-model of X^{an} dominating $\mathscr{X}_0$ (see [Gub03, Proposition 10.5]), there is always a *semi-stable alteration* $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$, which means that the generic fibre $f : X' \to X^{\mathrm{an}}$ of φ is a proper surjective morphism and X' is an irreducible d-dimensional analytic space which is the generic fibre of a strictly semi-stable admissible formal scheme $\mathscr{X}'$ over $\mathbb{K}^\circ$. It follows from [Gub98, Remark 3.14] that $\tilde{\varphi}$ is a proper surjective morphism between the special fibres.

6.3 We will use the notation from the previous sections. Let E be the uniformization of A, i.e. $A^{\mathrm{an}} = E/M$ for a discrete subgroup M in E with complete lattice $\Lambda = \mathrm{val}(M)$ in $\mathbb{R}^n$. Let $\mathscr{E}_0$ be the $\mathbb{K}^\circ$-model of E associated to the polytopal decomposition $\mathscr{C}_0$ of $\mathbb{R}^n$ (see 4.7).

Let $S \in \mathrm{str}(\tilde{\mathscr{X}'})$ with canonical simplex Δ_S in the skeleton $S(\mathscr{X}')$. By Lemma 5.15, there is a lift $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}_0}$ of $\tilde{\varphi}_0 : \overline{S} \to \tilde{\mathscr{A}_0}$, unique up to the M-action on $\tilde{\mathscr{E}_0}$. If $q_0 : \mathscr{E}_0 \to \mathscr{B}$ denotes the unique morphism extending $q : E \to B = \mathscr{B}^{\mathrm{an}}$ from the Raynaud extension (4), then $\tilde{q}_0 \circ \tilde{\Phi}_0$ is unique up to $q(M)$-translation on the abelian variety $\tilde{\mathscr{B}}$ over $\tilde{\mathbb{K}}$.

A canonical simplex Δ_S is called *non-degenerate* with respect to the morphism f if the conditions $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S)) = \dim(\Delta_S)$ and $\dim(\tilde{q}_0 \circ \tilde{\Phi}_0(S)) = \dim(S)$ are fulfilled. This definition does not depend on the choice of the lift $\tilde{\Phi}_0$. Moreover, it depends only on $\mathscr{X}'$ and f, but not on the choice of $\mathscr{C}_0$. This means that if we have a second rational polytopal decomposition $\overline{\mathscr{C}_0'}$ of $\mathbb{R}^n/\Lambda$ with associated Mumford model $\mathscr{A}_0'$ and with a semi-stable alteration $\varphi_0' : \mathscr{X}' \to \mathscr{A}_0'$ such that the generic fibre is again f, then the definitions of non-degenerate canonical simplices agree. Indeed, the independence of the first condition is obvious and the invariance of the second condition follows from an easy diagram chase involving Lemma 5.15 by passing to the common refinement $\mathscr{C}_0 \cap \mathscr{C}_0'$.

6.4 Let Σ be a Λ-periodic set of polytopes such that $\overline{\Sigma} := \{\overline{\sigma} \subset \mathbb{R}^n/\Lambda \mid \sigma \in \Sigma\}$ is a finite set. If σ is a polytope in Σ, then we assume that all closed faces of σ are also in Σ. Let $\mathbb{A}_\sigma$ be the affine space in $\mathbb{R}^n$ generated by the polytope σ.

The polytopal decomposition $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$ is said to be $\overline{\Sigma}$-*generic* if the following conditions hold for every $\sigma \in \Sigma$, $\Delta \in \mathscr{C}$:

(a) $\dim(\mathbb{A}_\sigma \cap \mathbb{A}_\Delta) = D$ if $D := \dim(\sigma) + \dim(\Delta) - n \geqslant 0$;

(b) $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta = \emptyset$ if $D < 0$.

By [Gub07a, Proposition 8.2], every $\overline{\Sigma}$-generic $\overline{\mathscr{C}}$ is $\overline{\Sigma}$-*transversal*, which means that $\Delta \cap \sigma$ is either empty or of dimension $\dim(\Delta) + \dim(\sigma) - n$ for all $\Delta \in \mathscr{C}$, $\sigma \in \Sigma$.

LEMMA 6.5. *Let L be an ample line bundle on A. Then there is a Γ'-rational polytopal decomposition $\overline{\mathscr{C}_1}$ of $\mathbb{R}^n/\Lambda$ with the following properties.*

(a) *$\overline{(1/m)\mathscr{C}_1}$ is $\overline{\Sigma}$-generic and hence $\overline{\Sigma}$-transversal for all $m \in \mathbb{N}\backslash\{0\}$.*

(b) *If $\mathscr{A}_1$ denotes the formal $(\mathbb{K}')^\circ$-model of $A_{\mathbb{K}'}$ associated to $\overline{\mathscr{C}_1}$, then there are $N \in \mathbb{N}\backslash\{0\}$ and a formal $(\mathbb{K}')^\circ$-model $\mathscr{L}$ of $L^{\otimes N}$ on $\mathscr{A}_1$ corresponding to a function $f_{\mathscr{L}}$ as in Proposition 4.11 which is a strongly polyhedral convex function with respect to $\mathscr{C}_1$.*

Proof. In [Gub07a, Lemma 8.4], this was proved for a totally degenerate abelian variety A. Using Proposition 4.11, the same proof applies here. □

6.6 We keep the assumptions from 6.2 and we consider an ample line bundle $\overline{L}$ on A endowed with a canonical metric.

Let Δ_S be a canonical simplex of the skeleton $S(\mathscr{X}')$ which is non-degenerate with respect to f. By 5.3, we may identify Δ_S with the simplex $\{u_0' + \cdots + u_r' = v(\pi)\}$ in $\mathbb{R}_+^{r+1}$. In the following, it is more convenient to identify Δ_S with the simplex $\Sigma_S := \{\mathbf{u}' \in \mathbb{R}_+^r \mid u_1' + \cdots + u_r' \leqslant v(\pi)\}$ by omitting the coordinate u_0'. Let us choose an affine lift $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ of the map $\overline{f}_{\mathrm{aff}}$ from Proposition 5.11. Using the identification $\Delta_S = \Sigma_S$, there is a unique injective linear map $\ell_S^{(0)} : \mathbb{R}^r \to \mathbb{R}^n$ extending $f_{\mathrm{aff}} - f_{\mathrm{aff}}(\mathbf{0})$. By (21), $\ell_S^{(0)}$ is defined over $\mathbb{Z}$ and hence $\Lambda_S := (\ell_S^{(0)})^{-1}(\Lambda)$ is a complete rational lattice in $\mathbb{R}^r$. The positive definite bilinear form b associated to L (see 4.3) induces a complete lattice

$$\Lambda_S^L := \{b(\ell_S^{(0)}(\cdot), \lambda) \mid \lambda \in \Lambda\}$$

on $(\mathbb{R}^r)^* = \mathbb{R}^r$. We denote by vol the volume with respect to the Lebesgue measure on $\mathbb{R}^r$.

There is an ample line bundle $\mathscr{H}$ on the abelian scheme $\mathscr{B}$ from the Raynaud extension (4) of A with generic fibre H such that $p^*(L) = q^*(H)$ on E (see 4.3). As in 5.17, we define the degree $\deg_{\mathscr{H}}(\overline{S})$ of $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ by using the lift $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}}_0$ and $\tilde{q}_0 : \tilde{\mathscr{E}}_0 \to \tilde{\mathscr{B}}$.

THEOREM 6.7. *Under the hypothesis in 6.6, the support of the positive measure $\mu := c_1(f^*(\overline{L}))^{\wedge d}$ is equal to the union of the canonical simplices of $S(\mathscr{X}')$ which are non-degenerate with respect to f. For a measurable subset Ω contained in the relative interior of such a simplex Δ_S and $r := \dim(\Delta_S)$, we have*

$$\mu(\Omega) = \frac{d!}{(d-r)!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \frac{\mathrm{vol}(\Lambda_S^L)}{\mathrm{vol}(\Lambda_S)} \cdot \mathrm{vol}(\Omega). \tag{32}$$

Remark 6.8. The theorem generalizes easily to several canonically metrized ample line bundles $\overline{L_1}, \ldots, \overline{L_d}$ on A. Let $\mu := c_1(f^*(\overline{L_1})) \wedge \cdots \wedge c_1(f^*(\overline{L_d}))$ and let $\mathscr{H}_j$ be an ample line bundle on $\mathscr{B}$ with $p^*(L_j) = q^*(H_j)$. Then the support of μ will be again equal to the union of all canonical simplices Δ_S which are non-degenerate with respect to f.

We are going to describe the canonical measure $\mu(\Omega)$ for any measurable subset Ω of a canonical simplex Δ_S. For $r := \dim(\Delta_S)$, let $\mathrm{vol}(\Lambda_S^{L_1}, \dots, \Lambda_S^{L_r})$ be the mixed volume in $\mathbb{R}^r$ of the corresponding fundamental lattices. This is a positive number which agrees with $\mathrm{vol}(\Lambda')$ if all lattices $\Lambda_S^{L_i}$ are equal to a single lattice Λ'. Moreover, the mixed volume is symmetric and multilinear with respect to the Minkowski sum of fundamental lattices. We conclude that $\mathrm{vol}(\Lambda_S^{L_1}, \dots, \Lambda_S^{L_r})$ is multilinear and symmetric with respect to the line bundles $L_1, \dots, L_r$. For more details, we refer the reader to [Gub07a, A6].

The generalization of Theorem 6.7 can now be stated as

$$\mu(\Omega) = r! \sum_{\mathbf{i}} \deg_{\mathscr{H}_{j_1}, \dots, \mathscr{H}_{j_s}}(\overline{S}) \cdot \frac{\mathrm{vol}(\Lambda_S^{L_{i_1}}, \dots, \Lambda_S^{L_{i_r}})}{\mathrm{vol}(\Lambda_S)} \cdot \mathrm{vol}(\Omega), \tag{33}$$

where $\mathbf{i}$ ranges over $\{1, \dots, d\}^r$ with $i_1 < i_2 < \cdots < i_r$ and where $j_1 < \cdots < j_s$ is the complement of $\mathbf{i}$ in $\{1, \dots, d\}$. Both sides of (33) are multilinear and symmetric with respect to $\overline{L_1}, \dots, \overline{L_d}$. Since symmetric real-valued multilinear forms are determined by the restriction to the diagonal, (33) follows from (32).

COROLLARY 6.9. *If $\overline{L_1}, \dots, \overline{L_d}$ are arbitrary line bundles on A endowed with canonical metrics, then $\mu := c_1(f^*(\overline{L_1})) \wedge \cdots \wedge c_1(f^*(\overline{L_d}))$ is supported in the union of canonical simplices of $S(\mathscr{X}')$ which are non-degenerate with respect to f and the restriction of μ to such a simplex is a multiple of the Lebesgue measure.*

Proof. This follows from (33) and multilinearity. □

Remark 6.10. Theorem 6.7 is well known in the two extreme cases of abelian varieties. If A is an abelian variety of potentially good reduction, then (32) shows that $\mu = \sum_Y \deg_{\mathscr{H}}(Y)\delta_{\xi_Y}$ with Y ranging over the irreducible components of $\tilde{\mathscr{X}}'$. This is a special case of Proposition 3.8(d) and was first proved by Chambert-Loir [Cha06].

If A is an abelian variety which is totally degenerate at v, then Theorem 6.7 shows that the support of μ is equal to the union of all d-dimensional canonical simplices Δ_S of $S(\mathscr{X}')$ with $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S)) = d$ and we have $\mu(\Omega) = d! \cdot \mathrm{vol}(\Lambda_S^L) \cdot \mathrm{vol}(\Omega)/\mathrm{vol}(\Lambda_S)$ for any measurable subset Ω of Δ_S. This was proved in [Gub07a, Theorem 9.6].

Proof of Theorem 6.7. By [Gub07a, Remark 3.14], the measure μ is independent of the odd part L_- of L. Moreover, L_- does not influence the bilinear form b of L and hence we may assume that L is a symmetric ample line bundle. It will be crucial for the proof to choose a Mumford model $\mathscr{A}$ of A as 'generic' as possible. Let $\overline{\Sigma}$ be the set of simplices $\{\overline{f}_{\mathrm{aff}}(\Delta_S) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}')\}$ together with all their closed faces. Then we will use the Γ'-rational polytopal decomposition $\overline{\mathscr{C}_1}$ of $\mathbb{R}^n/\Lambda$ from Lemma 6.5 with associated Mumford model $\mathscr{A}_1$. By multilinearity, we may assume that the strongly polyhedral convex function $f_{\mathscr{L}}$ from Lemma 6.5 induces a model $\mathscr{L}$ of L on $\mathscr{A}_1$. For $m \geqslant 1$, let $\mathscr{A}_m'$ be the Mumford model of A associated to the Γ'-rational polytopal decomposition $\overline{\mathscr{C}_m'} := \overline{\mathscr{C}_0} \cap \overline{(1/m)\mathscr{C}_1}$ (see 5.17). Note that $\mathscr{A}_m'$, $\mathscr{L}$ and $\mathscr{A}$ are only defined over the valuation ring of the 'large' field extension $\mathbb{K}'$. Since μ is invariant under base change [Gub07a, Remark 3.10], we may perform analytic calculations for μ over $\mathbb{K}'$.

We fix a rigidification on L such that the given canonical metric $\| \ \|_{\mathrm{can}}$ on L is given by (2) in Example 3.7. Let $\mathscr{X}_m$ be the closure of X in $\mathscr{A}_m'$. If we apply Propositions 5.13 and 5.14 to the polytopal decomposition $\mathscr{C}_m'$ instead of $\mathscr{C}$, then we get a minimal formal analytic structure $\mathfrak{X}_m''$ on X' which refines $(\mathscr{X}')^{\mathrm{f-an}}$ such that our given morphism $f : X' \to A^{\mathrm{an}}$ extends to a morphism $\phi_m : \mathfrak{X}''_m \to (\mathscr{A}'_m)^{\mathrm{f-an}}$. Let $\varphi_m : \mathscr{X}''_m \to \mathscr{A}'_m$ be the associated morphism of admissible formal schemes over $\mathbb{K}^\circ$.

Step 1. μ is the weak limit of discrete measures μ_m on X' which are supported in the preimages of the generic points of the irreducible components of $\tilde{\mathscr{X}}''_m$ with respect to the reduction map.

This will be a consequence of Tate's limit argument (see (2) in Example 3.7). We may assume that the metric $\|\ \|$ in (2) is equal to the formal metric $\|\ \|_{\mathscr{L}}$. We note that multiplication by m extends uniquely to a morphism $\psi_m : \mathscr{A}'_m \to \mathscr{A}_1$ (see [Gub07a, Proposition 6.4]). Then $\mathscr{L}_m := \psi_m^*(\mathscr{L})$ is a $(\mathbb{K}')^\circ$-model of $[m]^*(L)$ on $\mathscr{A}'_m$ with associated formal metric $[m]^*\|\ \|$ and hence we have $f^*([m]^*\|\ \|) = \|\ \|_{\mathscr{L}''_m}$ for the formal $(\mathbb{K}')^\circ$-model $\mathscr{L}''_m := \varphi_m^*(\mathscr{L}_m)$ of $f^*([m]^*(L)) = f^*(L)^{\otimes m^2}$. By (2), we get

$$f^*\|\ \|_{\mathrm{can}} = \lim_{m\to\infty} \|\ \|_{\mathscr{L}''_m}^{1/m^2}.$$

If we use this uniform limit together with Proposition 3.8, then we get

$$\mu = \lim_{m\to\infty} m^{-2d} \sum_Z \deg_{\tilde{\mathscr{L}}''_m}(Z)\, \delta_{\xi_Z}, \tag{34}$$

where Z ranges over all irreducible components of $\tilde{\mathscr{X}}''_m$.

Step 2. A first determination of $\mathrm{supp}(\mu)$.

By Corollary 5.9(g) and Proposition 5.13, the points ξ_Z are the vertices of the subdivision $\mathscr{D}_m := \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}'),\ \overline{\sigma} \in \overline{\mathscr{C}'_m}\}$ of $S(\mathscr{X}')$. As we have seen in 5.17, a vertex may only occur in the interior of a canonical simplex Δ_S with $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S)) = \dim(\Delta_S)$. By (34), we conclude that the support of μ is contained in the union of such Δ_S.

Step 3. Transformation of the limit in (34) into a multiple of $\mathrm{vol}(\Omega)$.

To prove (32), we may assume that Ω is a polytope contained in the interior of a canonical simplex Δ_S with $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S)) = \dim(\Delta_S)$. Using the identification $\Delta_S = \Sigma_S$, the lift $f_{\mathrm{aff}} : \Delta_S \to \mathbb{R}^n$ extends to an affine map $f_0 : \mathbb{R}^r \to \mathbb{R}^n$ which is also one-to-one and the polytopal decomposition $\mathscr{D} := \{f_0^{-1}(\Delta) \mid \Delta \in \mathscr{C}_1\}$ is periodic with respect to the lattice Λ_S from 6.6. Similarly as in (25), we have $\mathscr{D}_m = \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}'),\ \overline{\sigma} \in \overline{(1/m)\mathscr{C}_1}\}$. We conclude that there is a bijective correspondence between the irreducible components Z of $\tilde{\mathscr{X}}''_m$ with $\xi_Z \in \Omega$ and the vertices $\mathbf{u}'$ of $(1/m)\mathscr{D}$ contained in Ω. We note that our situation matches 5.17. By our above choice of $\overline{\Sigma}$, the transversality assumption (26) in the vertex $f_0(\mathbf{u}')$ follows easily from $\overline{\Sigma}$-transversality in Lemma 6.5. From Proposition 5.18, we get

$$\deg_{\tilde{\mathscr{L}}''_m}(Z) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}^{\otimes m^2}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}'\}^{g_m}),$$

where $e := \dim(S) = d - r$ and $g_m := f_{\mathscr{L}_m} \circ f_0$. We deduce that

$$\deg_{\tilde{\mathscr{L}}''_m}(Z) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}'\}^{g_m}) \cdot m^{2e}. \tag{35}$$

We define the dual polytope of the vertex $\mathbf{u} := m\mathbf{u}'$ of $\mathscr{D}$ with respect to the convex function $g := f_{\mathscr{L}} \circ f_0 : \mathbb{R}^r \to \mathbb{R}$ by

$$\{\mathbf{u}\}^g := \{\omega \in \mathbb{R}^r \mid \omega \cdot (\mathbf{w} - \mathbf{u}) \leqslant g(\mathbf{w}) - g(\mathbf{u})\ \forall \mathbf{w} \in U\},$$

where U is a sufficiently small neighborhood of $\mathbf{u}$ in $\mathbb{R}^r$. Since $\{\mathbf{u}'\}^{g_m} = m\{\mathbf{u}\}^g$, (35) yields

$$\deg_{\tilde{\mathscr{L}}''_m}(Z) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}\}^g) \cdot m^{2e+r}. \tag{36}$$

Let F be the fundamental domain of the lattice Λ_S in $\mathbb{R}^r$. For $m \gg 0$, the number of $(1/m)\Lambda_S$-translates of $(1/m)F$ contained in Ω (respectively intersecting $\partial\Omega$) is $m^r \mathrm{vol}(\Omega)/\mathrm{vol}(F) + O(m^{r-1})$ (respectively $O(m^{r-1})$). By (34) and (36), we deduce that

$$\mu(\Omega) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \sum_{\mathbf{u}} \mathrm{vol}(\{\mathbf{u}\}^g) \cdot \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\Lambda_S)}, \tag{37}$$

where $\mathbf{u}$ ranges over all vertices of $\mathscr{D}$ modulo Λ_S. The set $\{\{\mathbf{u}\}^g \mid \mathbf{u} \text{ vertex of } \mathscr{D}\}$ is invariant under Λ_S-translation. By [McM94, Theorem 3.1], this set is a Λ_S^L-periodic tiling of $\mathbb{R}^r$, which means that $\mathbb{R}^r$ is covered by these r-dimensional polytopes and they meet face-to-face. Together with (37), this proves (32). Since $\tilde{\mathscr{H}}$ is ample on $\tilde{\mathscr{B}}$, we have $\deg_{\mathscr{H}}(\overline{S}) \neq 0$ if and only if Δ_S is non-degenerate with respect to f. By Step 2, we get also the claim about the support. □

Remark 6.11. By the projection formula (b) in Proposition 3.8, Theorem 6.7 gives also an explicit description for the canonical measure

$$c_1(\overline{L}|_X)^{\wedge d} = f_*(c_1(f^*\overline{L})^{\wedge d})$$

on X. We conclude that the support of such a canonical measure is equal to the union of all $f(\Delta_S)$, where Δ_S ranges over all canonical simplices of $S(\mathscr{X}')$ which are non-degenerate with respect to f. Note that this set is independent of the choice of L. We call it the *canonical subset* of X^{an}.

The referee has suggested that the canonical subset is a piecewise linear space. In the following, a piecewise linear space means always a piecewise $R_{\mathbb{Z}_+}$-linear space for $R := \mathbb{Q} \cap (0, 1]$ in the sense of [Ber04, ch. 1]. We will always skip $R_{\mathbb{Z}_+}$ for brevity.

Theorem 6.12. *The canonical subset of X^{an} has a unique structure as a piecewise linear space T such that for any semi-stable alteration $\varphi_0 : \mathscr{X}' \to \mathscr{A}_0$ as in 6.2 with generic fibre $f : X' \to A^{\mathrm{an}}$, the restriction of f to the union of all canonical simplices which are non-degenerate with respect to f induces a piecewise linear map to T with finite fibres.*

Proof. Let $\mathscr{X}_0$ be the closure of X in a Mumford model $\mathscr{A}_0$ of A over $\mathbb{K}^\circ$ associated to the rational polytopal decomposition $\overline{\mathscr{C}_0}$ of $\mathbb{R}^n/\Lambda$. By a result of de Jong, there is a finite group G acting on a strongly non-degenerate pluristable formal scheme $\mathscr{Y}$ over $\mathbb{K}^\circ$ with the following properties (see [Ber99, Lemma 9.2]).

(a) We endow $\mathscr{X}_0$ with the trivial G-action. Then there is a dominant G-equivariant morphism $\gamma : \mathscr{Y} \to \mathscr{X}_0$.

(b) The generic fibre Y of $\mathscr{Y}$ is the analytic space associated to an irreducible smooth projective variety over $\mathbb{K}$.

(c) The generic fibre $g : Y \to X^{\mathrm{an}}$ of γ is a generically finite proper morphism.

(d) The fixed field $\mathbb{K}(Y)^G$ is a purely inseparable extension of the field of rational functions $\mathbb{K}(X)$.

Now we choose a semi-stable alteration $\eta : \mathscr{X}' \to \mathscr{Y}$ with generic fibre $h : X' \to Y$. Then $\varphi_0 := \gamma \circ \eta$ plays the role of the semi-stable alteration in 6.2 and $f := g \circ h$ is its generic fibre.

Let Δ_S be a canonical simplex of $S(\mathscr{X}')$ which is non-degenerate with respect to f. Since $\overline{f}_{\mathrm{aff}} \circ \mathrm{Val} = \overline{\mathrm{val}} \circ g \circ h$, it is clear that h is one-to-one on Δ_S. We claim that $h(\Delta_S)$ is contained in the skeleton $S(\mathscr{Y})$ of $\mathscr{Y}$.

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By continuity, it is enough to prove that $h(\mathbf{u}') \in S(\mathscr{Y})$ for every $\mathbf{u}' \in \mathrm{relint}(\Delta_S)$ with rational coordinates. We choose a rational polytopal decomposition $\overline{\mathscr{C}_1}$ of $\mathbb{R}^n/\Lambda$ with associated Mumford model $\mathscr{A}_1$ such that $\mathbf{u}'$ is a vertex of the subdivision $\mathscr{D} := \{\Delta_R \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid R \in \mathrm{str}(\tilde{\mathscr{X}}'), \overline{\sigma} \in \overline{\mathscr{C}_1}\}$ satisfying the transversality condition (26) and such that $g := f_{\mathscr{L}} \circ f_{\mathrm{aff}}$ strongly polyhedral convex function in $\mathbf{u}'$ for a symmetric ample line bundle L on A with a formal $\mathbb{K}^\circ$-model $\mathscr{L}$ on $\mathscr{A}_1$ associated to a piecewise affine function $f_{\mathscr{L}}$ as in Proposition 4.10. This is much easier to construct than the simultaneous transversality conditions in Lemma 6.5 and does not require a base change.

Let $\mathscr{A}_1'$ be the Mumford model associated to $\overline{\mathscr{C}_0} \cap \overline{\mathscr{C}_1}$. We get a commutative diagram of admissible formal schemes over $\mathbb{K}^\circ$ with reduced special fibres

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "\eta_1"] \arrow[d, "\iota_1'"] & \mathscr{Y}_1 \arrow[r, "\gamma_1"] \arrow[d, "j_1"] & \mathscr{A}_1' \arrow[r, "\psi_1"] \arrow[d, "\iota_1"] & \mathscr{A}_1 \\ \mathscr{X}' \arrow[r, "\eta"] & \mathscr{Y} \arrow[r, "\gamma"] & \mathscr{A}_0 & \end{tikzcd} \tag{38}$$

by assuming that the rectangles are cartesian on the level of formal analytic varieties. The vertical maps and ψ_1 are the identity on the generic fibre.

By Corollary 5.9, there is a unique irreducible component Z of $\tilde{\mathscr{X}}''$ with $\mathbf{u}' = \xi_Z$. Since the assumptions of 5.17 are satisfied, Proposition 5.18 yields

$$\deg_{\mathscr{L}}(Z) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}(\{\mathbf{u}'\}^g).$$

Since $\mathscr{H}$ is ample (see 4.3) and Δ_S is non-degenerate with respect to f, we have $\deg_{\mathscr{H}}(\overline{S}) > 0$. By strict convexity of g in $\mathbf{u}'$, we get also $\mathrm{vol}(\{\mathbf{u}'\}^g) > 0$ and hence $\deg_{\mathscr{L}}(Z) > 0$. By the projection formula, we have $\deg_{\mathscr{L}}(Z) = \deg_{\mathscr{L}}(\tilde{\beta}_*(Z))$ for $\beta := \psi_1 \circ \gamma_1 \circ \eta_1$. Note that $\dim(\tilde{\beta}(Z)) = \dim(Z) = d$ is necessary for the positivity of the degree. Now (38) yields that $\tilde{\eta}_1(Z)$ is also d dimensional. Since Y is d dimensional, we conclude that $\tilde{\eta}_1(Z)$ is an irreducible component W of $\tilde{\mathscr{Y}_1}$ and hence $h(\xi_Z) = \xi_W$. By the generalization of Corollary 5.9(g) to $\mathscr{Y}$ (see Lemma 6.13 below), we know that ξ_W is a vertex of a subdivision $\mathscr{D}_1$ of $S(\mathscr{Y})$ and hence $h(\mathbf{u}') = \xi_W \in S(\mathscr{Y})$, proving $h(\Delta_S) \subset S(\mathscr{Y})$.

By Remark 6.11 and the above, h maps the support of $\mu := c_1(f^*\overline{L}^{\wedge d})$ into $S(\mathscr{Y})$. By [Ber04, Corollary 6.1.3], h restricts to a piecewise linear map from the piecewise linear subspace $\mathrm{supp}(\mu)$ to $S(\mathscr{Y})$. Moreover, the skeleton $S(\mathscr{Y})$ is invariant under G and the G-transformations induce piecewise linear automorphisms of the skeleton [Ber04, Corollary 6.1.2].

There is a Zariski dense open subset U of X^{an} such that $g: V \to U$ is finite for $V := g^{-1}(U)$ (see (c)). By [Ber99, Corollary 8.6], the quotient V/G exists. By [Ber99, Corollary 8.4], we have $S(\mathscr{Y}) \subset V$. We note that the compact subset $S(\mathscr{Y})/G$ of V/G has a canonical structure as a piecewise linear space. Indeed, the skeleton $S(\mathscr{Y})$ is a piecewise linear space because it is the geometric realization of a polysimplicial set D (see [Ber04, Theorem 5.1.1]). As $S(\mathscr{Y})/G$ is the geometric realization of the polysimplicial set D/G, we deduce that $S(\mathscr{Y})/G$ is also a piecewise linear space (see [Ber04, Proposition 3.5.3]). We conclude that $h(\mathrm{supp}(\mu))$ is a piecewise linear subspace of $S(\mathscr{Y})$ which maps onto a piecewise linear subspace of $S(\mathscr{Y})/G$. By shrinking U and using (d), we may assume that the canonical morphism $V/G \to U$ is radicial. In particular, it is a homeomorphism of the underlying topological spaces (see [Ber07, Remark 2.2.2]). As a consequence, we get a piecewise linear structure on $f(\mathrm{supp}(\mu)) = g(h(\mathrm{supp}(\mu)))$. By Remark 6.11, this is the canonical subset T of X^{an}.

The domains of linearity for the piecewise linear map $f:\mathrm{supp}(\mu)\to T$ are subsets of the canonical simplices of $S(\mathscr{X}')$ which are non-degenerate with respect to f. By Proposition 5.5, they induce a finer formal analytic structure on X' and we may apply de Jong's alteration theorem also to the associated formal scheme over $\mathbb{K}^\circ$. Replacing the alteration η by the composition of the two alterations, we may assume that the domains of linearity are really equal to the canonical simplices of $S(\mathscr{X}')$ which are non-degenerate with respect to f. Then a linear atlas of T is given by the charts $f(\Delta_S)$, where Δ_S ranges over all such canonical simplices and f is a linear isomorphism from Δ_S onto $f(\Delta_S)$.

Let us now consider any semi-stable alteration $\varphi_0':\mathscr{Z}'\to\mathscr{A}_0'$ as in 6.2 with generic fibre $f':Z'\to A^{\mathrm{an}}$. Then there is a semi-stable alteration which factors through φ_0 and φ_0'. Using the above atlas, it follows easily that f' induces a piecewise linear map $S(\mathscr{Z}')\to T$. Uniqueness is obvious. $\square$

We now sketch how the results of the first part of § 5 generalize to the strongly non-degenerate pluristable formal scheme $\mathscr{Y}$ from the above proof. By definition, there are a strongly non-degenerate strictly pluristable formal scheme $\mathscr{Y}'$ and a surjective étale morphism $\rho:\mathscr{Y}'\to\mathscr{Y}$. By [Ber04, Theorem 5.1.1], the skeleton $S(\mathscr{Y})$ has a piecewise linear structure which is a cokern of the piecewise linear structure on $S(\mathscr{Y}')$ described in Remark 5.19 and in the appendix. Moreover, there is a 'polytopal' subdivision of $S(\mathscr{Y})$ given by canonical 'plurisimplices' Δ_S, which are in bijective correspondence to the strata S of $\tilde{\mathscr{Y}}$. We use quotation marks because Δ_S is only a quotient of a canonical plurisimplex $\Delta_{S'}$ of $S(\mathscr{Y}')$ for any stratum S' with $\tilde\rho(S')\subset S$. To construct Δ_S, we have to identify closed faces Δ_P and Δ_Q in the boundary of $\Delta_{S'}$ if and only if the strata P and Q map into the same stratum of $\tilde{\mathscr{Y}}$. By [Ber04], there are well-defined proper strong deformation retractions $\mathrm{Val}:Y\to S(\mathscr{Y})$ and $\mathrm{Val}':Y'\to S(\mathscr{Y}')$, where Y and Y' are the generic fibres of $\mathscr{Y}$ and $\mathscr{Y}'$.

Lemma 6.13. *There is a unique map $\overline{g}_{\mathrm{aff}}:S(\mathscr{Y})\to\mathbb{R}^n/\Lambda$ with $\overline{g}_{\mathrm{aff}}\circ\mathrm{Val}=\overline{\mathrm{val}}\circ g$ on Y. We get a 'polytopal' subdivision $\mathscr{D}_1:=\{\Delta_S\cap\overline{g}_{\mathrm{aff}}^{-1}(\overline{\sigma})\mid S\in\mathrm{str}(\tilde{\mathscr{Y}}),\overline{\sigma}\in\overline{\mathscr{C}_1}\}$ of $S(\mathscr{Y})$ defining a formal analytic structure $\mathfrak{Y}_1$ on Y as in Proposition 5.5 with $\mathfrak{Y}_1=\mathscr{Y}_1^{\mathrm{f-an}}$. Moreover, Proposition 5.7 and Corollary 5.9*(a), (b), (e), (f) *and* (g) *hold for $\mathfrak{Y}_1$ (instead of $\mathfrak{X}''$).*

Proof. By Remark 5.19, the lemma holds for the strictly pluristable $\mathscr{Y}'$. The idea is now to use the étale covering $\rho:\mathscr{Y}'\to\mathscr{Y}$ to deduce the claim for $\mathscr{Y}$. It is necessary to define $\overline{g}_{\mathrm{aff}}:=\overline{\mathrm{val}}\circ g$. Using that ρ^{an} is surjective [Ber94, Lemma 2.2] and $\rho^{\mathrm{an}}\circ\mathrm{Val}'=\mathrm{Val}\circ\rho^{\mathrm{an}}$, it is easy to prove that $\overline{g}_{\mathrm{aff}}\circ\mathrm{Val}=\overline{\mathrm{val}}\circ g$ from the corresponding property for $\overline{g}_{\mathrm{aff}}':S(\mathscr{Y}')\to\mathbb{R}^n/\Lambda$.

In Appendix A, we have studied building blocks for strongly non-degenerate strictly pluristable formal schemes over $\mathbb{K}^\circ$. We define a *building block* for $\mathscr{Y}$ as a formal affine open subscheme $\mathscr{U}$ of $\mathscr{Y}$ such that a building block $\mathscr{U}'$ of $\mathscr{Y}'$ exists with $\rho(\mathscr{U}')=\mathscr{U}$. Since an étale map is open, the building blocks cover $\mathscr{Y}$. By definition, $\tilde{\mathscr{U}}'$ has a smallest stratum S' and hence $\tilde{\mathscr{U}}$ has the smallest stratum $S:=\tilde\rho(S')$. We set $U:=\mathscr{U}^{\mathrm{an}}$. If $\mathscr{U}$ varies over all building blocks and Δ over $\mathscr{D}_1$, then $U\cap\mathrm{Val}^{-1}(\Delta)$ is a formal affinoid atlas for Y. Here, we use that the 'polytopal subdivision' $\mathscr{D}_1$ is induced by the valuation of units, i.e. Δ is given by inequalities $v(b_{\mathbf{m}})+\mathbf{m}\cdot\mathbf{u}\geqslant 0$ induced by the units $g^*(b_{\mathbf{m}}\mathbf{x}^{\mathbf{m}})$ on U coming from the inequalities of a corresponding polytope of $\overline{\mathscr{C}_1}$. It now follows analogously to the proofs of Propositions 5.5 and 5.13 that the formal affinoid atlas induces a formal analytic variety $\mathfrak{Y}_1$ isomorphic to $\mathscr{Y}_1^{\mathrm{f-an}}$.

A similar construction applies to $\mathscr{Y}'$, leading to an admissible formal scheme $\mathscr{Y}_1'$ over $\mathbb{K}^\circ$ with reduced special fibre. Since ρ is étale and surjective, the natural base change ρ_1 of ρ to $\mathscr{Y}_1$ is étale and surjective. We deduce from 2.6 that ρ_1 is the canonical map $\mathscr{Y}_1' \to \mathscr{Y}_1$.

Let R' be a stratum of $\tilde{\mathscr{Y}}_1'$. By [Ber99, Lemma 2.2], $\tilde{\rho}_1(R')$ is an open dense set of a stratum of $\tilde{\mathscr{Y}}_1$. Using the claim for $\mathscr{Y}'$, we know that $R' = \pi((\mathrm{Val}')^{-1}(\tau'))$ for a unique open face τ' of $\mathscr{D}_1'$. Note that ρ^{an} maps τ' isomorphically onto an open face τ of $\mathscr{D}_1$ and $\tilde{\rho}_1(R') \subset \pi(\mathrm{Val}^{-1}(\tau))$.

If τ varies over all open faces of $\mathscr{D}_1$, then the definition of $\mathfrak{Y}_1 = \mathscr{Y}_1^{\mathrm{f-an}}$ yields that $\pi(\mathrm{Val}^{-1}(\tau))$ is a partition of $\tilde{\mathfrak{Y}}_1 = \tilde{\mathscr{Y}}_1$. The surjectivity of $\tilde{\rho}_1$ and a partition argument show that $R := \pi(\mathrm{Val}^{-1}(\tau))$ is a strata subset equal to $\bigcup \tilde{\rho}_1(\pi((\mathrm{Val}')^{-1}(\tau')))$, where τ' ranges over all open faces of $\mathscr{D}_1'$ with $\rho^{\mathrm{an}}(\tau') = \tau$.

We claim that R is a stratum of $\tilde{\mathscr{Y}}_1$. By localizing, we may assume that $\mathscr{Y}$ and $\mathscr{Y}'$ are building blocks and that $\tau \subset \mathrm{relint}(\Delta_S)$ for the canonical 'plurisimplex' $\Delta_S = S(\mathscr{Y})$. Since ρ^{an} maps the interior of the plurisimplex $\Delta_{S'} = S(\mathscr{Y}')$ isomorphically onto $\mathrm{relint}(\Delta_S)$, there is a unique open face τ' of $\mathscr{D}_1'$ with $\rho^{\mathrm{an}}(\tau') = \tau$. We conclude that $R = \tilde{\rho}_1(\pi((\mathrm{Val}')^{-1}(\tau'))) \in \mathrm{str}(\tilde{\mathscr{Y}}_1)$.

The remaining claims are easily deduced from the corresponding claims for $\mathscr{Y}'$ using that ρ_1 is étale and surjective. □

7. Proof of the main theorem and examples

First, we will give the proof of Theorem 1.1. Then we will describe the canonical measure in two relevant examples.

Proof of Theorem 1.1. By a finite base change and using linearity in the components, we may assume that X is a d-dimensional geometrically integral closed subvariety of the abelian variety A. The argument will be based on the description of canonical measures in the previous section; hence, we will use the notation from there. By multilinearity, it is enough to consider ample line bundles in Theorem 1.1(c). We choose a semi-stable alteration $\varphi_0 : \mathscr{X}' \to \mathscr{X}_0$ as in 6.2 with generic fibre $f : X' \to X^{\mathrm{an}}$. Then we have the explicit description (33) of $\mu := c_1(f^*(\overline{L_1})) \wedge \cdots \wedge c_1(f^*(\overline{L_d}))$. Since μ is supported in $S(\mathscr{X}')$, we get

$$\overline{\mathrm{val}}_*(c_1(\overline{L_1}|_{X^{\mathrm{an}}}) \wedge \cdots \wedge c_1(\overline{L_d}|_{X^{\mathrm{an}}})) = \deg(f) \cdot (\overline{f}_{\mathrm{aff}})_*(\mu) \tag{39}$$

by Propositions 3.8 and 5.11. More precisely, Remark 6.8 shows that μ is supported in the union of the canonical simplices Δ_S which are non-degenerate with respect to f. Since

$$d - \dim(\Delta_S) = \dim(S) = \dim(\tilde{q}_0 \circ \tilde{\Phi}_0(S)) \leqslant \dim(\tilde{\mathscr{B}}) = b,$$

we get $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S) = \dim(\Delta_S) \geqslant d - b$. By Theorem 4.15, the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a finite union of rational polytopes of dimension at most d and at least $d - \min\{b, d\}\}$. Hence, we may list the simplices $\overline{f}_{\mathrm{aff}}(\Delta_S)$ as in (a), where Δ_S ranges over the canonical simplices which are non-degenerate with respect to f. Then (c) and (d) follow from (33) and (39). Finally, (b) follows from the next lemma. □

Lemma 7.1. *In the above notation, let us consider $\overline{\mathbf{u}} \in \overline{\mathrm{val}}(X^{\mathrm{an}})$ and let $d - e$ be the dimension of the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ in a neighborhood of $\overline{\mathbf{u}}$. Then there is a $(d-e)$-dimensional canonical simplex Δ_S of $S(\mathscr{X}')$ which is non-degenerate with respect to f such that $\overline{\mathbf{u}} \in \overline{f}_{\mathrm{aff}}(\Delta_S)$.*

Proof. By Proposition 5.11 and the surjectivity of f and $\overline{\mathrm{val}}$, it is clear that the simplices $\overline{f}_{\mathrm{aff}}(\Delta_T)$, $T \in \mathrm{str}(\tilde{\mathscr{X}}')$, cover $\overline{\mathrm{val}}(X^{\mathrm{an}})$. Since $\overline{f}_{\mathrm{aff}}$ is locally an affine map defined over $\mathbb{Q}$, we may assume that $\overline{\mathbf{u}}$ is an element of $\overline{\mathrm{val}}(X^{\mathrm{an}})$ with coordinates in $\mathbb{Q}$. Moreover, by density of the $\mathbb{Q}$-rational

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points in $\overline{\mathrm{val}}(X^{\mathrm{an}})$, we may assume that $\overline{\mathbf{u}}$ is not contained in any $\overline{f}_{\mathrm{aff}}(\Delta_T)$ of dimension $< d-e$. We have to prove that there is a $(d-e)$-dimensional canonical simplex Δ_S with $\overline{\mathbf{u}} \in \overline{f}_{\mathrm{aff}}(\Delta_S)$ and

$$\dim(\tilde{q}_0 \circ \tilde{\Phi}_0(\overline{S})) = e, \tag{40}$$

which yields that Δ_S is non-degenerate with respect to f. We choose a rational simplex $\overline{\Delta}$ in $\mathbb{R}^n/\Lambda$ of codimension $d-e$ such that $\overline{\mathbf{u}} \in \mathrm{relint}(\overline{\Delta})$ and

$$\overline{\Delta} \cap \overline{\mathrm{val}}(X^{\mathrm{an}}) = \{\overline{\mathbf{u}}\}. \tag{41}$$

We extend $\overline{\Delta}$ to a rational polytopal subdivision $\overline{\mathscr{C}}_1$ of $\mathbb{R}^n/\Lambda$ (which is not assumed to have the properties of Lemma 6.5). We denote the associated Mumford model of A by $\mathscr{A}_1$.

Let $\overline{\mathscr{C}} := \overline{\mathscr{C}_0} \cap \overline{\mathscr{C}_1}$ be the minimal polytopal decomposition of $\mathbb{R}^n/\Lambda$ containing $\overline{\mathscr{C}_0}$ and $\overline{\mathscr{C}_1}$. Let $\mathscr{A}$ be the Mumford model of A associated to $\overline{\mathscr{C}}$ and let $\phi : \mathfrak{X}'' \to \mathscr{A}^{\mathrm{f-an}}$ be the morphism obtained from φ_0 by base change as in Propositions 5.13 and 5.14. Since $\overline{\mathscr{C}}$ is a polytopal subdivision of $\overline{\mathscr{C}}_1$, we get a canonical formal analytic morphism $\phi_1 : \mathfrak{X}'' \to \mathscr{A}_1^{\mathrm{f-an}}$. Passing to the associated admissible formal schemes over $\mathbb{K}^\circ$, this induces a morphism $\varphi_1 : \mathscr{X}'' \to \mathscr{A}_1$ (see 2.6).

Note that $U := \overline{\mathrm{val}}^{-1}(\overline{\Delta})$ is a formal open subset of $\mathscr{A}_1^{\mathrm{f-an}}$. By (41), we get $X^{\mathrm{an}} \cap U \neq \emptyset$. Let $\mathscr{X}_1$ be the closure of X^{an} in $\mathscr{A}_1$. Then the special fibre $\tilde{\mathscr{X}_1}$ has an irreducible component Y with $Y \cap \tilde{U} \neq \emptyset$. We use here that the reduction of $\mathscr{A}_1^{\mathrm{f-an}}$ is equal to the special fibre of $\mathscr{A}_1$ (see 2.6). Since $\tilde{\varphi}_1$ maps $\tilde{\mathscr{X}}''$ onto $\tilde{\mathscr{X}}_1$, there is an irreducible component $Y_{\mathbf{u}'}$ of $\tilde{\mathscr{X}}''$ mapping onto Y. As the notation already indicates, $Y_{\mathbf{u}'}$ is the irreducible component associated to a vertex $\mathbf{u}'$ of the rational subdivision $\mathscr{D} = \{\Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid S \in \mathrm{str}(\tilde{\mathscr{X}}'),\ \overline{\sigma} \in \overline{\mathscr{C}}\}$ of $S(\mathscr{X}')$ (see Corollary 5.9). More precisely, $\mathbf{u}' = \mathrm{Val}(\xi)$ for the unique point ξ of X', which reduces to the generic point of this irreducible component $Y_{\mathbf{u}'}$. From $\tilde{\varphi}_1(Y_{\mathbf{u}'}) = Y$, we deduce that the reduction of $f(\xi)$ to the special fibre $\tilde{\mathscr{A}_1}$ is equal to the generic point of Y. We conclude that $f(\xi) \in U$ and hence $\overline{f}_{\mathrm{aff}}(\mathbf{u}') = \overline{\mathrm{val}}(f(\xi)) \in \overline{\Delta}$ by Proposition 5.11 and by the definition of U. Since $f(\xi) \in X^{\mathrm{an}}$, we get $\overline{f}_{\mathrm{aff}}(\mathbf{u}') = \overline{\mathbf{u}}$ from (41).

Let S be the unique stratum of the chosen strictly semi-stable $\mathbb{K}^\circ$-model $\mathscr{X}'$ of X' with $\mathbf{u}' \in \mathrm{relint}(\Delta_S)$. We note first that $\overline{\mathbf{u}} = \overline{f}_{\mathrm{aff}}(\mathbf{u}') \in \overline{f}_{\mathrm{aff}}(\Delta_S)$. As we have remarked at the end of 5.17, the fact that $\mathrm{relint}(\Delta_S)$ contains a vertex of $\mathscr{D}$ implies that $\dim(\overline{f}_{\mathrm{aff}}(\Delta_S)) = \dim(\Delta_S)$. From the non-degeneracy assumption on $\overline{\mathbf{u}}$, we deduce that $\dim(\Delta_S) = d-e$ and hence S is an e-dimensional stratum.

It remains to prove (40). By Corollary 5.9, the canonical morphism $\tilde{\mathscr{X}}'' \to \tilde{\mathscr{X}}'$ maps $Y_{\mathbf{u}'}$ onto $\overline{S}$. By Lemma 5.15, we have lifts $\tilde{\Phi}_0 : \overline{S} \to \tilde{\mathscr{E}}_0$ and $\tilde{\Phi} : Y_{\mathbf{u}'} \to \tilde{\mathscr{E}}$ of $\tilde{\varphi}_0$ and $\tilde{\varphi}$, where $\mathscr{E}_0$ and $\mathscr{E}$ are the $\mathbb{K}^\circ$-models of the uniformization E of A associated to the polytopal decompositions $\mathscr{C}_0$ and $\mathscr{C}$. Using that $\mathscr{C}$ is a polytopal subdivision of $\mathscr{C}_1$, the map $\tilde{\Phi}$ induces a canonical morphism $\tilde{\Phi}_1 : Y_{\mathbf{u}'} \to \tilde{\mathscr{E}}_1$, which is a lift of the restriction of $\tilde{\varphi}_1$ to $Y_{\overline{\mathbf{u}}'}$. This lift may be also constructed by the fact that $\mathscr{E}_1$ and $\mathscr{A}_1 = \mathscr{E}_1/M$ are locally isomorphic. Since $\tilde{\varphi}_1(Y_{\mathbf{u}'}) = Y$, Proposition 4.8(d) yields that $Y' := \tilde{\Phi}_1(Y_{\mathbf{u}'})$ is isomorphic to Y. Lemma 5.15 and an easy diagram chase involving the canonical morphisms $q_i : \mathscr{E}_i \to \mathscr{B}$ to the formal abelian scheme $\mathscr{B}$ of the Raynaud extension show that

$$\tilde{q}_0 \circ \tilde{\Phi}_0(\overline{S}) = \tilde{q} \circ \tilde{\Phi}(Y_{\mathbf{u}'}) = \tilde{q}_1 \circ \tilde{\Phi}_1(Y_{\mathbf{u}'}).$$

We conclude that the dimension of

$$\tilde{q}_0 \circ \tilde{\Phi}_0(\overline{S}) = \tilde{q}_1(Y') \tag{42}$$

is at most e. To show equality, we consider a basic formal affinoid subdomain $U_{V,\Delta} \cong V \times U_\Delta$ from the construction of $\mathscr{E}_1^{\mathrm{f-an}}$ (see 4.7). Here, V is the generic fibre of a formal affine open

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subset of $\mathscr{B}$ which trivializes the Raynaud extension (3) of A and Δ is a simplex in $\mathbb{R}^n$ lifting the simplex $\overline{\Delta}$ considered at the beginning of the proof. We may choose V such that $\tilde{U}_{V,\Delta} \cap Y' \neq \emptyset$. Recall that U_Δ is the polytopal subdomain $\mathrm{val}^{-1}(\Delta)$ of $(\mathbb{G}_m^n)^{\mathrm{an}}$. By the choice of V, we have $\tilde{U}_{V,\Delta} \cong \tilde{V} \times \tilde{U}_\Delta$.

We claim that the second projection $\tilde{p}_2$ maps the generic point of Y' into the torus orbit Z of $\tilde{U}_\Delta$ corresponding to $\mathrm{relint}(\Delta)$. Recall from [Gub07a, Proposition 4.4] that $Z = \pi(\mathrm{val}^{-1}(\Delta))$, where π is the reduction map. We have seen above that $f(\xi)$ reduces to the generic point of Y and that $\overline{\mathrm{val}}(f(\xi)) = \overline{\mathbf{u}}$. Let ξ' be the unique lift of $f(\xi)$ to E whose reduction $\pi(\xi')$ is the generic point of Y'. We conclude that $\mathrm{val}(p_2(\xi')) = \mathrm{val}(\xi')$ is the unique point $\mathbf{u} \in \mathrm{relint}(\Delta)$ which lifts $\overline{\mathbf{u}}$. Therefore, we have $\pi(p_2(\xi')) \in Z$. Since $\pi(p_2(\xi')) = \tilde{p}_2(\pi(\xi'))$, we get the above claim. Since Z is the closed orbit of $\tilde{U}_\Delta$, we conclude that $\tilde{p}_2(Y') \subset Z$.

By [Gub07a, Proposition 4.4], the dimension of the torus orbit Z is $\mathrm{codim}(\Delta) = d - e$. The above claim shows that $Y' \cap \tilde{U}_{V,\Delta}$ is contained in the closed subset $(\tilde{q}_1(Y') \cap \tilde{V}) \times Z$ of $\tilde{U}_{V,\Delta} \cong \tilde{V} \times \tilde{U}_\Delta$. Since Y' is d dimensional and the product is at most d dimensional, we get $Y' \cong (\tilde{q}_1(Y') \cap \tilde{V}) \times Z$. Moreover, we deduce that $\dim(\tilde{q}_1(Y')) = e$. By (42), we get (40) and the lemma. □

Remark to the proof of Lemma 7.1. The argument shows that the irreducible component Y' of $\tilde{\mathscr{E}}_1$ is a fibre bundle over $\tilde{q}_1(Y')$ with fibre isomorphic to the toric variety Y_Δ associated to $\mathrm{star}(\Delta) = \{\sigma \in \mathscr{C}_1 \mid \Delta \subset \sigma\}$ (see [Gub07a, Remark 4.8]). Indeed, the choice of V as a trivialization yields that $\tilde{q}_1^{-1}(\tilde{V}) \cong \tilde{V} \times \tilde{\mathscr{X}}$, where $\mathscr{X}$ is the formal $\mathbb{K}^\circ$-model of $(\mathbb{G}_m^n)^{\mathrm{an}}$ associated to the polytopal decomposition $\mathscr{C}_1$. Now the proof of the lemma shows that $Y' \cap \tilde{q}_1^{-1}(\tilde{V}) \cong (\tilde{q}_1(Y') \cap \tilde{V}) \times Y_\Delta$, proving the claim.

Example 7.2. Let us consider the special case $X = A$ in Theorem 6.7. For every $\overline{\mathbf{u}} \in \mathbb{R}^n/\Lambda$, there is a canonical point $\xi_{\overline{\mathbf{u}}} \in A^{\mathrm{an}}$ which we describe as follows: let V be the generic fibre of a non-empty formal affine open subset of the abelian scheme $\mathscr{B}$ which trivializes the Raynaud extension (3) of A. Then $U_{V,\{\mathbf{u}\}} = \mathrm{val}^{-1}(\mathbf{u}) \cap q^{-1}(V) \cong V \times U_{\{\mathbf{u}\}}$ is an affinoid subdomain of the uniformization E. Using $A^{\mathrm{an}} \cong E/M$, it is obvious that $U_{V,\{\mathbf{u}\}}$ is isomorphic to an affinoid subdomain $U_{[V,\mathbf{u}]}$ of A^{an}. By Lemma 4.6, we may write every analytic function h on $U_{[V,\mathbf{u}]}$ as a strictly convergent Laurent series

$$h = \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}} x_1^{m_1} \cdots x_n^{m_n}$$

in the torus coordinates $x_1, \ldots, x_n$ on the polytopal domain $U_{\{\mathbf{u}\}}$ in $(\mathbb{G}_m^n)^{\mathrm{an}}$, where the $a_{\mathbf{m}} \in \mathscr{O}(V)$ are uniquely determined by h. Then we define $\xi_{\overline{\mathbf{u}}} \in U_{[V,\mathbf{u}]}$ by

$$|h(\xi_{\overline{\mathbf{u}}})| = \sup_{\mathbf{m} \in \mathbb{Z}^n} |a_{\mathbf{m}}|_{\sup} \cdot e^{-\mathbf{m} \cdot \mathbf{u}}.$$

It is easy to see that $\xi_{\overline{\mathbf{u}}}$ does not depend on the choice of V and the representative $\mathbf{u}$. The subset $S(A) := \{\xi_{\overline{\mathbf{u}}} \mid \overline{\mathbf{u}} \in \mathbb{R}^n/\Lambda\}$ of A^{an} is called the *skeleton* of A (see [Ber90, § 6.5]).

By a combinatorial result of Knudsen and Mumford [KKMS73, ch. III], there is a rational triangulation $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$ (even refining any given rational polytopal decomposition) and $m_{\mathscr{C}} \in \mathbb{N}\backslash\{0\}$ such that for every maximal $\Delta \in \mathscr{C}$, the simplex $m_{\mathscr{C}}\Delta$ is $\mathrm{GL}(n, \mathbb{Z})$-isomorphic to a $\mathbb{Z}^n$-translate of the standard simplex $\{\mathbf{u} \in \mathbb{R}_+^n \mid u_1 + \cdots + u_n \leqslant 1\}$. Then the Mumford model $\mathscr{A}$ of A associated to $\overline{\mathscr{C}}$ is strictly semi-stable. Künnemann used this to construct projective strictly semi-stable $\mathbb{K}^\circ$-models for abelian varieties (see [Kün98] and also the erratum in [Kün01, 5.8]).

Let $\Delta \in \mathscr{C}$ with $\mathbf{u} \in \Delta$. A similar application of Lemma 4.6 as above shows that $\xi_{\overline{\mathbf{u}}}$ is contained in the affinoid chart $U_{[V,\Delta]} \cong V \times U_\Delta$ of A^{an} and that $|h(\xi_{\overline{\mathbf{u}}})| \geqslant |h(x)|$ for all $h \in \mathscr{O}(U_{[V,\Delta]})$ and all $x \in U_{[V,\Delta]}$ with $\overline{\mathrm{val}}(x) = \overline{\mathbf{u}}$. By [Ber99, Theorem 5.2], this maximality implies that $\xi_{\overline{\mathbf{u}}}$ is contained in the skeleton $S(\mathscr{A})$ of $\mathscr{A}$. We conclude that $S(A) = S(\mathscr{A})$ and $\overline{\mathrm{val}} = \mathrm{Val}$ maps the skeleton homeomorphically onto $\mathbb{R}^n/\Lambda$.

We apply Theorem 6.7 with $X' = X = A$ and $\mathscr{X}' = \mathscr{A}$. The canonical simplices of $S(\mathscr{A})$ are just the elements of $\overline{\mathscr{C}}$. By Proposition 4.8 and its proof, a stratum S of $\tilde{\mathscr{A}}$ has locally the form $S \cap \tilde{U}_{[V,\Delta_S]} \cong \tilde{V} \times Z_\tau$ for V as above and the open face $\tau = \mathrm{relint}(\Delta_S)$ with corresponding stratum $Z_\tau := \pi(\mathrm{val}^{-1}(\tau))$ in $\tilde{U}_{\Delta_S}$. Hence, Δ_S is a non-degenerate simplex of $S(\mathscr{A})$ (with respect to $f = \mathrm{id}$) if and only if $\dim(Z_\tau) = 0$. We conclude that the non-degenerate canonical simplices of $S(\mathscr{A})$ are just the n-dimensional simplices of $\overline{\mathscr{C}}$. The lattice Λ_S^L in 6.6 does not depend on the choice of such a simplex $\Delta = \Delta_S$. By Proposition 3.8 and Theorem 6.7, we conclude that $c_1(\overline{L})^{\wedge d}$ is supported in $S(A)$ and corresponds to the unique Haar measure ν on $\mathbb{R}^n/\Lambda$ with $\nu(A) = \deg_L(A)$. Using multilinearity for non-ample line bundles and Remark 6.8, we deduce easily the following corollary.

Corollary 7.3. *Let $\overline{L_1}, \ldots, \overline{L_d}$ be canonically metrized line bundles on the abelian variety A over K of dimension d. Then $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$ is supported in the skeleton $S(A)$ and corresponds to the Haar measure on $\mathbb{R}^n/\Lambda$ with total measure $\deg_{L_1,\ldots,L_d}(A)$.*

Example 7.4. We will show that the whole spectrum of values $\{d-b, \ldots, d-e\}$ in Theorem 1.1 may occur for a single canonical measure, where $d-e$ denotes the dimension of the tropical variety. We assume that K is the function field $k(C)$ for an irreducible regular projective curve C over an algebraically closed field k of characteristic 0. Let v be the discrete valuation on K defined by the order in a given closed point $P \in C$. It is easy to use the construction below to give similar examples for other fields.

We consider a product $A = B_1 \times B_2$ of abelian varieties over K, where B_1 has good reduction at v and where B_2 is totally degenerate at v. As usual, let $\mathbb{K}$ be a minimal algebraically closed field containing K which is complete with respect to a valuation extending v. The analytic considerations will be performed over $\mathbb{K}$. 'Totally degenerate at v' means that the Raynaud extension of B_2 is an analytic torus and hence $B_2^{\mathrm{an}} \cong (\mathbb{G}_m^n)^{\mathrm{an}}/M$ for a discrete subgroup M with $\Lambda = \mathrm{val}(M)$ a complete lattice in $\mathbb{R}^n$. Then $E \cong B_1^{\mathrm{an}} \times (\mathbb{G}_m^n)^{\mathrm{an}}$ is the Raynaud extension of A and we have $A^{\mathrm{an}} \cong E/M$.

By assumption, B_1 is the generic fibre of an abelian scheme $\mathcal{B}_1$ over the discrete valuation ring K°. The associated admissible formal scheme $\mathscr{B}_1 := \hat{\mathcal{B}}_1$ over $\mathbb{K}^\circ$ (see 2.7) is just the formal abelian scheme $\mathscr{B}$ over $\mathbb{K}^\circ$ in the Raynaud extension (3) for A. To get a Mumford model $\mathscr{B}_2$ for B_2^{an}, we will use a similar polytopal decomposition $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$ as in Example 7.2. There is a rational triangulation $\overline{\mathscr{C}}$ of $\mathbb{R}^n/\Lambda$ such that the strictly semi-stable $\mathbb{K}^\circ$-model $\mathscr{B}_2$ is projective [Kün98, §§ 3 and 4]. Künnemann's proofs show that $\overline{\mathscr{C}}$ can be chosen as a refinement of any given rational polytopal decomposition of $\mathbb{R}^n/\Lambda$ (see also [Kün01, 5.5]). We get a strictly semi-stable formal $\mathbb{K}^\circ$-model $\mathscr{A} := \mathscr{B}_1 \times \mathscr{B}_2$ of A^{an}.

By Künnemann's construction, $\mathscr{B}_2$ is defined algebraically over the valuation ring F° of a finite extension F of the completion K_v, i.e. we have a strictly semi-stable algebraic F°-model $\mathcal{B}_2$ of B_2 with associated admissible formal scheme $\mathscr{B}_2 = \hat{\mathcal{B}}_2$. We choose ample line bundles $\mathcal{L}_1$ on $\mathcal{B}_1$ and $\mathcal{L}_2$ on $\mathcal{B}_2$. Then $\mathcal{L} := p_1^*(\mathcal{L}_1) \otimes p_2^*(\mathcal{L}_2)$ is an ample line bundle on $\mathcal{A} := \mathcal{B}_1 \times_{K^\circ} \mathcal{B}_2$. By passing to a suitable tensor power of $\mathcal{L}$, we may assume that $\mathcal{L}$ is very ample and that $H^0(\mathcal{A}, \mathcal{L}) \to H^0(\tilde{\mathcal{A}}, \tilde{\mathcal{L}})$ is surjective for the reduction $\tilde{\mathcal{L}}$ of $\mathcal{L}$ to the special fibre $\tilde{\mathcal{A}}$.

Let $b := \dim(B_1)$ and let us fix $m \in \{0, \ldots, \min(b, n)\}$. Let us choose generic global sections $\tilde{s}_1, \ldots, \tilde{s}_m \in H^0(\tilde{\mathcal{A}}, \tilde{\mathcal{L}})$. By assumption, they are the reductions of global sections $s_1, \ldots, s_m$ of $\mathcal{L}$. The generic choice of the sections leads to a closed subscheme

$$\mathcal{X} := \operatorname{div}(s_1) \cap \cdots \cap \operatorname{div}(s_m)$$

of codimension m in $\mathcal{A}$ which is flat over F°. By Bertini's theorem, the generic fibre X of $\mathcal{X}$ is an irreducible smooth variety over F of dimension $d := b + n - m$. The same argument shows that the irreducible components Y_i of the special fibre $\tilde{\mathcal{X}}$ are Cartier divisors and $\bigcap_{i \in I} Y_i$ is a smooth variety over $\tilde{F}$ of pure dimension $\dim(X) - |I| + 1$ for any non-empty subset I. By a criterion of Hartl and Lütkebohmert [HL00, Proposition 1.3], $\mathcal{X}$ is strictly semi-stable. Since $m \leqslant b$, the fibre of X over any point of B_2 is non-empty and hence

$$\overline{\mathrm{val}}(X^{\mathrm{an}}) = \mathbb{R}^n/\Lambda.$$

We conclude that the excess e of the tropical variety $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is given by

$$e := \dim(X) - \dim(\overline{\mathrm{val}}(X^{\mathrm{an}})) = b - m.$$

Now we switch from the algebraic point of view to the analytic and formal category. Then $\mathcal{X}$ has an associated admissible formal scheme $\mathscr{X} := \hat{\mathcal{X}}$ over $\mathbb{K}^\circ$ which is a strictly semi-stable formal $\mathbb{K}^\circ$-model of X^{an} and a closed formal subscheme of the Mumford model $\mathscr{A} = \mathscr{B}_1 \times \mathscr{B}_2$ of A^{an}. Let $\mathscr{L}, \mathscr{L}_1, \mathscr{L}_2$ be the formal line bundles on $\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2$ induced by $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$.

If $S \in \operatorname{str}(\tilde{\mathscr{A}})$, then $S = \tilde{\mathscr{B}}_1 \times S_2$ for $S_2 \in \operatorname{str}(\tilde{\mathscr{B}}_2)$ corresponding to an open face $\overline{\tau} := \operatorname{relint}(\overline{\Delta})$ for a unique $\overline{\Delta} \in \overline{\mathscr{C}}$ (see Proposition 4.8). We note that

$$\dim(S) = \operatorname{codim}(\Delta, \mathbb{R}^n) + b \geqslant b \geqslant m.$$

We consider first the case $\dim(S) > m$. Using the generic choice of $s_1, \ldots, s_m$ again, Bertini's theorem yields that

$$S' := \operatorname{div}(s_1) \cap \cdots \cap \operatorname{div}(s_m) \cap S \tag{43}$$

is a stratum of $\tilde{\mathscr{X}}$ with $\dim(S') = \dim(S) - m$. If $\dim(S) = m = b$, then S' is a strata subset of $\tilde{\mathscr{X}}$ consisting of $\deg_{\tilde{\mathscr{L}}_1}(\tilde{\mathscr{B}}_1) = \deg_{L_1}(B_1)$ points. Therefore, the skeleton $S(\mathscr{X})$ may be identified with the triangulation $\overline{\mathscr{C}}$ by using the map Val except in the case $m = b$, where we have to count the n-dimensional simplices of $\overline{\mathscr{C}}$ with multiplicity $\deg_{L_1}(B_1)$. By construction, S' is non-degenerate (with respect to $f = \mathrm{id}$ in the sense of 6.3) if and only if $\dim(S_2) \leqslant m$. If we endow the generic fibre L of $\mathcal{L}$ with a canonical metric, then Theorem 6.7 shows that

$$\nu := \overline{\mathrm{val}}_*(c_1(\overline{L}|_X)^{\wedge d}) = \sum_{\overline{\Delta}} \lambda_{\overline{\Delta}} \cdot \delta_{\overline{\Delta}}, \tag{44}$$

where $\overline{\Delta}$ ranges over all simplices of $\overline{\mathscr{C}}$ with $\dim(\overline{\Delta}) \geqslant n - m = d - b$ and where $\lambda_{\overline{\Delta}} > 0$. Similarly as in Example 7.2, we deduce that the contribution of the n-dimensional simplices of $\overline{\mathscr{C}}$ to ν is equal to a strictly positive Haar measure ν_n on $\mathbb{R}^n/\Lambda$. Note however that for $m > 0$, we have

$$\nu_n(\mathbb{R}^n/\Lambda) < \nu(\mathbb{R}^n/\Lambda) = \deg_L(X).$$

Finally, we show that the multiplicities $\lambda_{\overline{\Delta}}$ are given completely in terms of convex geometry. The simplex $\overline{\Delta} \in \overline{\mathscr{C}}$ of dimension $r \geqslant n - m$ corresponds to a stratum S' of $\tilde{\mathscr{X}}$ as above. Let $\mathbf{m}_\Delta \in \mathbb{Z}^n$ and $c_\Delta \in \mathbb{Q}$ such that $f_{\mathscr{L}}(\mathbf{u}) = \mathbf{m}_\Delta \cdot \mathbf{u} + c_\Delta$ for all $\mathbf{u} \in \Delta$. Then the dual polytope Δ^g of Δ with respect to $g := f_{\mathscr{L}}$ is given by the face

$$\Delta^g := \{\mathbf{u}\}^g \cap (\mathbf{m}_\Delta + \Delta^\perp)$$

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of the dual polytope $\{\mathbf{u}\}^g$ of the vertex $\mathbf{u}$ of Δ (see 5.17), where $\Delta^\perp$ is the orthogonal complement of Δ in $\mathbb{R}^n$ (see [Gub07a, Appendix A]). Let $\mathbb{L}_\Delta$ be the linear space such that $\mathbf{u}+\mathbb{L}_\Delta$ is the affine space spanned by Δ and let $\Lambda_\Delta := \Lambda \cap \mathbb{L}_\Delta$ be the complete lattice in $\mathbb{L}_\Delta$ induced by Λ.

Recall that $m_{\mathscr{C}}$ is the natural number such that a suitable translate of $m_{\mathscr{C}}\sigma$ is $\mathrm{GL}(n,\mathbb{Z})$-isomorphic to the standard simplex $\{\mathbf{u} \in \mathbb{R}^n_+ \mid u_1 + \cdots + u_n \leqslant 1\}$ for every $\sigma \in \mathscr{C}$. Let vol_Δ be the Haar measure on $\mathbb{L}_\Delta$ such that $\mathrm{vol}_\Delta(\Delta - \mathbf{u}) = 1/(r! m_{\mathscr{C}})$. On $\mathbb{L}_\Delta^*$, we will use the dual measure also denoted by vol_Δ. These are the volumes from Theorem 6.7. On the other hand, we have the relative Lebesgue measure on $\mathbb{L}_\Delta \subset \mathbb{R}^n$ which is used for the Dirac measure $\delta_{\overline{\Delta}}$ and which we now denote by $\mathrm{vol}_{\mathbb{R}^n}$. Formula (32) in Theorem 6.7 yields

$$\lambda_{\overline{\Delta}} = \frac{d!}{(d-r)!} \cdot \deg_{\mathscr{L}_1}(\overline{S'}) \cdot \frac{\mathrm{vol}_\Delta((\Lambda_\Delta)^L)}{\mathrm{vol}_{\mathbb{R}^n}(\Lambda_\Delta)}.$$

Here, we have used the complete lattice $(\Lambda_\Delta)^L := \{b(\cdot,\lambda) \mid \lambda \in \Lambda_\Delta\}$ in $(\mathbb{L}_\Delta)^*$ defined by the bilinear form b associated to L (see 4.3). By using (43), we get

$$\dim(S') = d - r, \quad \deg_{\mathscr{L}_1}(\overline{S'}) = \binom{m}{n-r} \cdot \deg_{\mathscr{L}_1}(\tilde{\mathscr{B}}_1) \cdot \deg_{\mathscr{L}_2}(\overline{S_2}).$$

By the theory of toric varieties, the degree of the toric variety $\overline{S_2}$ with respect to $\tilde{\mathscr{L}}_2$ is given in terms of $\mathrm{vol}(\Delta^g)$. As in the formula in [Gub07a, Equation (36)], we get

$$\deg_{\mathscr{L}_2}(\overline{S_2}) = (n-r)! \cdot \mathrm{vol}_{\mathbb{R}^n}(\Delta^g) \cdot \mathrm{vol}_{\mathbb{R}^n}(\mathbb{Z}^n \cap \Delta^\perp)^{-1}$$

and hence $\deg_{\mathscr{L}_1}(\tilde{\mathscr{B}}_1) = \deg_{L_1}(B_1)$ yields

$$\lambda_{\overline{\Delta}} = \frac{d! \cdot m! \cdot \mathrm{vol}_{\mathbb{R}^n}(\Delta^g) \cdot \mathrm{vol}_\Delta((\Lambda_\Delta)^L) \cdot \deg_{L_1}(B_1)}{(d-r)! \cdot (m+r-n)! \cdot \mathrm{vol}_{\mathbb{R}^n}(\mathbb{Z}^n \cap \Delta^\perp) \cdot \mathrm{vol}_{\mathbb{R}^n}(\Lambda_\Delta)}.$$

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Appendix A. Building blocks

Let $\mathbb{K}$ be an algebraically closed field with a non-trivial, non-archimedean complete absolute value $|\ |$ and valuation $v := -\log|\ |$. In the appendix, we will study building blocks of strongly non-degenerate strictly pluristable formal schemes of length $l \in \mathbb{N}$ over the valuation ring $\mathbb{K}^\circ$.

A.1 Such a *building block* $\mathscr{U}_l$ is recursively defined by $\mathscr{U}_0 := \mathrm{Spf}(\mathbb{K}^\circ)$ and the following conditions:

(a) $\mathscr{U}_l$ is an affine formal scheme over $\mathbb{K}^\circ$ with generic fibre U_l;

(b) there is an étale morphism $\psi_l : \mathscr{U}_l \to \mathscr{U}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)})$ over $\mathbb{K}^\circ$ for a building block $\mathscr{U}_{l-1}$ of length $l-1$ and $\mathbf{n}^{(l)} \in (\mathbb{N}\backslash\{0\})^{p_l}$, $\mathbf{a}^{(l)} \in \mathscr{O}(\mathscr{U}_{l-1})^{p_l}$;

(c) the entries of $\mathbf{a}^{(l)} = (a_1^{(l)}, \ldots, a_{p_l}^{(l)})$ are units in $\mathscr{O}(U_{l-1})$;

(d) $\tilde{\mathscr{U}}_l$ has a smallest stratum which maps into the smallest stratum of $\tilde{\mathscr{U}}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)})$.

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Let $\mathscr{D} = \mathrm{Spf}(\mathbb{K}^\circ\langle x\rangle)$ be the formal unit disk. For $k = 1, \ldots, l$, we recall that $\mathscr{U}_{k-1}(\mathbf{n}^{(k)}, \mathbf{a}^{(k)})$ is the closed formal subscheme of

$$\mathscr{U}_{k-1} \times \mathscr{D}^{n_1^{(k)}+1} \times \cdots \times \mathscr{D}^{n_{p_k}^{(k)}+1}$$

given by the following equations:

$$x_{i0}^{(k)} \cdots x_{in_i^{(k)}}^{(k)} = a_i^{(k)} \quad (i = 1, \ldots, p_k). \tag{45}$$

Recursively, we know that $\tilde{\mathscr{U}}_{l-1}$ has a smallest stratum. It follows from [Ber99, Lemma 2.3] that $\tilde{\mathscr{U}}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)})$ has a smallest stratum and hence (d) makes sense. By [Ber99, Lemma 2.2], the smallest stratum of $\tilde{\mathscr{U}}_l$ maps onto an open dense subset of the smallest stratum of $\tilde{\mathscr{U}}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)})$. By [Ber99, Lemma 2.10], we have an isomorphism from $\mathrm{str}(\tilde{\mathscr{U}}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)}))$ onto $\mathrm{str}(\tilde{\mathscr{U}}_l)$ given by taking preimages with respect to $\tilde{\psi}_l$. It is easy to see that every strongly non-degenerate strictly pluristable formal scheme is covered by open building blocks.

For $i = 1, \ldots, p_l$ and $j = 0, \ldots, n_i^{(l)}$, let $z_{ij}^{(l)} = \psi_l^*(x_{ij}^{(l)})$ and let $\mathbf{z}^{(l)}$ be the resulting vector. Recursively, we define $\mathbf{z} = (\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(l)})$, where we use the natural pull-backs of the coordinates $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(l)})$ from the definition of the building blocks $\mathscr{U}_1, \ldots, \mathscr{U}_l$ to $\mathscr{U}_l$. It will be convenient to skip all entries with index $j = 0$, i.e. let

$$\hat{\mathbf{x}} := (x_{ij}^{(k)})_{k=1,\ldots,l;i=1,\ldots,p_k;j=1,\ldots,n_i^{(k)}}.$$

Similarly, we define $\mathbf{n} = (\mathbf{n}^{(1)}, \ldots, \mathbf{n}^{(l)})$ and $\hat{\mathbf{n}} = (\hat{\mathbf{n}}^{(1)}, \ldots, \hat{\mathbf{n}}^{(l)})$. We set

$$\mathrm{Val} : U_l \longrightarrow \mathbb{R}^{|\hat{\mathbf{n}}|}, \quad p \mapsto -\log \circ\, p(\hat{\mathbf{x}})$$

and $\Delta_l := \mathrm{Val}(U_l)$. By [Ber04, §§ 4 and 5], the map Val restricts to a homeomorphism from the skeleton $S(\mathscr{U}_l)$ onto Δ_l. It gives $S(\mathscr{U}_l)$ a canonical piecewise linear structure and induces a canonical proper strong deformation retraction $U_l \to S(\mathscr{U}_l)$.

These constructions can be globalized for any strongly non-degenerate strictly pluristable formal scheme $\mathscr{X}$ over $\mathbb{K}^\circ$. We will show in the next proposition that the building blocks induce the linear pieces of the skeleton $S(\mathscr{X})$.

PROPOSITION A.2. *Let $\mathscr{U}_l$ be a strongly non-degenerate strictly pluristable building block of length l with generic fibre U_l and let $\Delta_l := \mathrm{Val}(U_l) \subset \mathbb{R}^{|\hat{\mathbf{n}}|}$ as above. Then the following properties hold.*

(a) *Δ_l is a polytope in $\mathbb{R}^{|\hat{\mathbf{n}}|}$ defining a polytopal domain $U_{\Delta_l} := \mathrm{val}^{-1}(\Delta_l)$ in $(\mathbb{G}_m^{|\hat{\mathbf{n}}|})^{\mathrm{an}}$ and $\mathscr{U}_{\Delta_l} := U_{\Delta_l}^{\mathrm{f-sch}}$ (see 4.5).*

(b) *The pull-backs $\hat{\mathbf{z}}$ of the coordinates $\hat{\mathbf{x}}$ define an étale morphism $\mathscr{U}_l \xrightarrow{\psi} \mathscr{U}_{\Delta_l}$.*

(c) *There is a bijective order reversing correspondence between strata S of $\tilde{\mathscr{U}}_l$ and open faces τ of Δ_l. It is given by*

$$\tau = \mathrm{Val}(\pi^{-1}(T)), \quad S = \pi(\mathrm{Val}^{-1}(\tau)),$$

where $\pi : \mathscr{U}_l \to \tilde{\mathscr{U}}_l$ is the reduction map and T is any non-empty subset of S. We have $\dim(\tau) = \mathrm{codim}(S, \tilde{\mathscr{U}}_l)$.

(d) *Let Y be an irreducible component of $\tilde{\mathscr{U}}_l$. Then there is a unique $\xi_Y \in U_l$ with $\pi(\xi_Y)$ dense in Y. Moreover, we have $\xi_Y \in S(\mathscr{U}_l)$ and $\mathrm{Val}(\xi_Y)$ is the vertex of Δ_l corresponding to the dense open stratum of Y by (c).*

(e) *If $f \in \mathscr{O}(U_l)^\times$, then there are $\lambda \in \mathbb{K}^\times$, $\mu \in \mathscr{O}(\mathscr{U}_l)^\times$ and $\mathbf{m} \in \mathbb{Z}^{|\hat{\mathbf{n}}|}$ with $f = \lambda\mu\hat{\mathbf{z}}^{\mathbf{m}}$. There is a unique affine function F on Δ_l with $v \circ f = F \circ \mathrm{Val}$.*

Proof. The proof is by induction on l. Let $\mathbf{u} = (\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(l)})$ be the coordinates on $\mathbb{R}^{|\hat{\mathbf{n}}|} = \mathbb{R}^{|\hat{\mathbf{n}}^{(1)}|} \times \cdots \times \mathbb{R}^{|\hat{\mathbf{n}}^{(l)}|}$. By the induction hypothesis, $a_i^{(k)}$ induces an affine function $A_i^{(k)} = A_i^{(k)}(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k-1)})$ on the polytope $\Delta_{k-1} = \mathrm{Val}(U_{k-1})$ in $\mathbb{R}_+^{|\hat{\mathbf{n}}^{(1)}|} \times \cdots \times \mathbb{R}_+^{|\hat{\mathbf{n}}^{(k-1)}|}$ for $k = 2, \ldots, l$. Since $a_i^{(k)} \in \mathscr{O}(\mathscr{U}_{k-1})$, the values of $A_i^{(k)}$ are in $\mathbb{R}_+$. It follows from (45) that Δ_l is given as a subset of $\mathbb{R}_+^{|\hat{\mathbf{n}}|}$ by

$$u_{i1}^{(k)} + \cdots + u_{in_i^{(k)}}^{(k)} \leqslant A_i^{(k)} \quad (k = 1, \ldots, l; i = 1, \ldots, p_k), \tag{46}$$

proving (a). By induction again, we have $a_i^{(l)} = \lambda_i \mu_i \hat{\mathbf{y}}^{\mathbf{m}_i}$, where $\lambda_i \in \mathbb{K}^\times$, $\mu_i \in \mathscr{O}(\mathscr{U}_{l-1})^\times$ and $\hat{\mathbf{y}}$ is the pull-back of the coordinates $(\hat{\mathbf{x}}^{(1)}, \ldots, \hat{\mathbf{x}}^{(l-1)})$ to $\mathscr{U}_{l-1}$. For $b_i^{(l)} := \lambda_i \hat{\mathbf{y}}^{\mathbf{m}_i}$ and $c_i^{(l)} := \lambda_i \hat{\mathbf{x}}^{\mathbf{m}_i}$, we get $\mathscr{U}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)}) \cong \mathscr{U}_{l-1}(\mathbf{n}^{(l)}, \mathbf{b}^{(l)})$ and $\mathscr{U}_{\Delta_l} \cong \mathscr{U}_{\Delta_{l-1}}(\mathbf{n}^{(l)}, \mathbf{c}^{(l)})$. For the latter, we use [Ber99, Proposition 1.4]. Therefore, the canonical diagram

$$\begin{tikzcd}
\mathscr{U}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)}) \arrow[r] \arrow[d] & \mathscr{U}_{\Delta_l} \arrow[d] \\
\mathscr{U}_{l-1} \arrow[r] & \mathscr{U}_{\Delta_{l-1}}
\end{tikzcd}$$

is cartesian. The bottom line is given by $\hat{\mathbf{y}}$ and the induction hypothesis yields that this map is étale. We conclude that the upper line is étale, proving (b).

Note that (c) holds for any polytopal domain (see [Gub07a, Proposition 4.4]). It follows from [Ber99, § 2] that

$$\mathrm{str}(\tilde{\mathscr{U}}_{l-1}(\mathbf{n}^{(l)}, \mathbf{a}^{(l)})) \longrightarrow \mathrm{str}(\tilde{\mathscr{U}}_l), \quad S' \mapsto \tilde{\psi}_l^{-1}(S')$$

is a bijective order preserving map. This proves easily (c).

By [Ber99, Proposition 1.4], we have $\mathscr{O}(U_l)^\circ = \mathscr{O}(\mathscr{U}_l)$ and hence we may apply the theory of formal affinoid varieties to deduce the existence and uniqueness of ξ_Y (see § 2). Since Val maps $S(\mathscr{U}_l)$ bijectively onto Δ_l, there is $\xi \in S(\mathscr{U}_l)$ with $\mathrm{Val}(\xi)$ equal to the vertex $\mathrm{Val}(\xi_Y)$ of Δ_l given by the correspondence in (c). By the first paragraph of [Ber04, p. 332], $\pi(\xi)$ is dense in Y and hence $\xi = \xi_Y$, proving (d).

Let $\tilde{P}$ be a $\tilde{\mathbb{K}}$-rational point in the smallest stratum of $\tilde{\mathscr{U}}_l$. By [Gub07a, Proposition 2.9], $\tilde{\psi}$ induces an isomorphism $\pi^{-1}(\tilde{P}) \to \pi^{-1}(\tilde{\psi}(\tilde{P}))$ between formal fibres. This allows us to apply results for polytopal domains to the formal fibre $\pi^{-1}(\tilde{P})$. By Lemma 4.6, we have a convergent Laurent expansion

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^{|\hat{\mathbf{n}}|}} a_{\mathbf{m}} \hat{\mathbf{z}}^{\mathbf{m}}$$

on $\pi^{-1}(\tilde{P})$ and there is a dominant term $t := a_\nu \hat{\mathbf{z}}^\nu$ in the expansion, i.e.

$$|t(x)| > |a_{\mathbf{m}} \hat{\mathbf{z}}^{\mathbf{m}}(x)|$$

for all $x \in \pi^{-1}(\tilde{P})$ and $\mathbf{m} \in \mathbb{Z}^{|\hat{\mathbf{n}}|} \backslash \{\nu\}$. Let Y be an irreducible component of $\tilde{\mathscr{U}}_l$. Applying (c) with $T = \{\tilde{P}\}$, we deduce that there is a sequence $x_n \in \pi^{-1}(\tilde{P})$ with $\mathrm{Val}(x_n) \in \mathrm{relint}(\Delta_l)$ converging to the vertex $\mathrm{Val}(\xi_Y)$ of Δ_l. By compactness of U_l, we may assume that x_n converges to some $x \in U_l$. By continuity, we have $\mathrm{Val}(x) = \mathrm{Val}(\xi_Y)$ and hence (c) again shows that $\pi(x)$ is not contained in any other irreducible component than Y. It is a basic fact for units in an affinoid

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algebra that this implies that $|f(x)| = |f(\xi_Y)|$ (see [Gub03, Proposition 7.6]). We conclude that

$$|f(\xi_Y)| = |f(x)| = \lim_{n\to\infty} |f(x_n)| = \lim_{n\to\infty} |t(x_n)| = |t(x)| = |t(\xi_Y)|.$$

If Y ranges over all irreducible components of $\tilde{\mathscr{U}}_l$, then the points ξ_Y form the Shilov boundary of U_l (see [Ber90, Proposition 2.4.4]). We conclude that $\mu := ft^{-1}$ is a unit in $\mathscr{O}(U_l)^\circ = \mathscr{O}(\mathscr{U}_l)$, proving (e). □

Remark A.3. By (46), the coordinate $u_{ij}^{(k)}$ is identically zero on Δ_l if and only if $a_i^{(k)} \in \mathscr{O}(\mathscr{U}_{k-1})^\times$. The corresponding $z_{ij}^{(k)}$ is a formal unit on $\mathscr{U}_l$. We deduce easily from Proposition A.2(e) that

$$\mathscr{O}(U_l)^\times = \mathscr{O}(\mathscr{U}_l)^\times \times \prod_{k,i,j} (z_{ij}^{(k)})^{\mathbb{Z}},$$

where the indices of the basis range over $1 \leqslant k \leqslant l$, $1 \leqslant i \leqslant p_k$ with $a_i^{(k)} \notin \mathscr{O}(\mathscr{U}_{k-1})^\times$ and $1 \leqslant j \leqslant n_i^{(k)}$.

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