On the primality of totally ordered q-factorization graphs

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Abstract. We introduce the combinatorial notion of a q-factorization graph intended as a tool to study and express results related to the classification of prime simple modules for quantum affine algebras. These are directed graphs equipped with three decorations: a coloring and a weight map on vertices, and an exponent map on arrows (the exponent map can be seen as a weight map on arrows). Such graphs do not contain oriented cycles and, hence, the set of arrows induces a partial order on the set of vertices. In this first paper on the topic, beside setting the theoretical base of the concept, we establish several criteria for deciding whether or not a tensor product of two simple modules is a highest-ℓ-weight module and use such criteria to prove, for type A, that a simple module whose q-factorization graph has a totally ordered vertex set is prime.

1 Introduction

The simple finite-dimensional modules for an affine Kac–Moody algebra $\tilde{\mathfrak{g}}$ were classified by Chari and Pressley [5, 10] in terms of tensor products of simple evaluation modules, which are built from simple finite-dimensional $\mathfrak{g}$-modules. Moreover, the factorization of such simple $\tilde{\mathfrak{g}}$-modules in terms of evaluation modules is unique, up to permutation of the factors. In fact, the finite-dimensional simple evaluation modules are exactly the finite-dimensional prime simple $\tilde{\mathfrak{g}}$-modules, that is, those that cannot be factored as a nontrivial tensor product.

As in the classical case, the simple finite-dimensional modules for the associated Drinfeld–Jimbo quantum group $U_q(\tilde{\mathfrak{g}})$ were also classified by Chari and Pressley [11, 12]. However, in this context, the classification was described in terms of their highest-ℓ-weights (or Drinfeld polynomials), with no mention to prime simple modules, except in the case, the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$ is of type A_1. In that case, the simple prime modules are again evaluation modules and every simple module can be uniquely expressed as a tensor product of prime ones (up to reordering). Thus, the question about finding a description of the simple modules in terms of tensor products of prime ones beyond rank one has intrigued the specialists since the early days of the study of the finite-dimensional representation theory

of quantum affine algebras. The situation is indeed much more complicated since evaluation modules exist only for type A but, even in that case, it is known [14] that there are prime simple modules which are not evaluation modules. The classification of prime simple modules remains open after more than three decades since the early works on the topic.

As further studies were made, several examples of families of prime simple modules started to appear in the literature such as the Kirillov–Reshetikhin (KR) modules or, more generally, minimal affinizations [6], and certain snake modules [28] (which contain the examples in the aforementioned [14]). However, the most important advent related to this topic was a theory introduced by Hernandez and Leclerc [20] connecting the finite-dimensional representations of quantum affine algebras to cluster algebras. In particular, as a consequence of their main conjecture (referred to as HL conjecture below), in principle, all real prime simple modules can be computed using the machinery of cluster mutations since they correspond to the cluster variables of certain explicitly prescribed cluster algebras. A real module is a simple module whose tensor square is also simple (only the trivial module is real in the classical setting, but they abound in the quantum setting). However, describing all cluster variables is not exactly a simple task in general. The combinatorics of cluster mutations for type A was rephrased in [4] in tableau-theoretic language and the resulting algorithm could be used to produce examples of prime, real, and nonreal modules. Most of the HL conjecture was proved for simply laced $\mathfrak{g}$ in [33], which built up on [30]. We refer to the survey [21] for an account on the status of the conjecture and the related literature. The papers [2, 3, 8, 15] have explicitly identified prime modules in certain HL subcategories and provided alternate proofs for parts of the HL conjecture. See also [1, 25, 31] for recent developments related to HL subcategories as well as [9] for a study of primality from a homological perspective.

The motivation for the present work is the problem of classifying the Drinfeld polynomials whose associated simple modules are prime and similarly for real modules. We will focus here on the former, leaving our first answers regarding the latter to appear in [27]. As it is clear from the above considerations, this is a difficult problem, so our goal is to gradually obtain general results toward such classification. In this sense, most of the original results of the present paper consist of criteria for deciding whether certain tensor products are highest-ℓ-weight modules or not. We use such criteria for proving the main result of the present paper, Theorem 3.5.5, as well as the main results of [27]. Such criteria allowed us to expand the number of examples of families of prime and real simple modules compared to the existing literature. In particular, they recover the primality and reality of minimal affinizations for all types and, for type A, the primality of snake modules arising from prime snakes, skew representations, and certain minimal affinizations by parts.

In order to describe Drinfeld polynomials which correspond to simple prime modules in an efficient manner, we propose a graph theoretical language based on the notion of q-factorization. The notion of q-factorization is already present in the literature and is based on the solution of this classification for $\mathfrak{g}$ of rank one. More precisely, for each simple root of $\mathfrak{g}$, one considers the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the corresponding loop-like generators and then the associated restriction of the Drinfeld polynomial. Since this subalgebra is of type $A_1^{(1)}$, this restricted polynomial

can then be factorized according to the decomposition of the associated simple module as a tensor product of prime modules. Each of these factors is said to be a q-factor of the original Drinfeld polynomial $\boldsymbol{\pi}$. Using the q-factorization, we define a decorated oriented graph $G(\boldsymbol{\pi})$ which we call the q-factorization graph of $\boldsymbol{\pi}$. The set of vertices of $G(\boldsymbol{\pi})$ is the multiset of q-factors (q-factors with multiplicities give rise to as many vertices). Each vertex is given two decorations: a "color" (the simple root which originated the vertex) and a "weight" (the degree of the polynomial). Given two vertices $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, $G(\boldsymbol{\pi})$ contains the arrow

$$\boldsymbol{\omega} \longrightarrow \boldsymbol{\omega}'$$

if and only if the tensor product $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')$ of the associated simple modules is reducible and highest-ℓ-weight. Since $L_q(\boldsymbol{\omega})$ is a KR module for every vertex $\boldsymbol{\omega}$ and KR modules are real, it follows that $G(\boldsymbol{\pi})$ has no loops. Moreover, if a tensor product of KR modules is not highest-ℓ-weight, the tensor product in the opposite order is. Hence, the determination of the arrows is equivalent to the solution of the problem of classifying the reducible tensor products of KR modules. Such classification gives rise to a decoration for the arrows: a positive integer which we call the exponent of the arrow. We recall that the roots of the polynomial $\boldsymbol{\omega}$ form a q-string. Let us say a is the center of such string and similarly let a' be the center of the string associated with $\boldsymbol{\omega}'$. Let us say that $\boldsymbol{\omega}$ is i-colored and has weight r while $\boldsymbol{\omega}'$ is j-colored and has weight s. Then, there exists a finite set of positive integers $\mathscr{R}_{i,j}^{r,s}$ such that $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')$ is reducible and highest-ℓ-weight if and only if $a = a'q^m$ for some $m \in \mathscr{R}_{i,j}^{r,s}$. This number m is then defined to be the exponent of the arrow. We visually express this set of data by the picture

$$\overset{r}{i} \overset{m}{\longrightarrow} \overset{s}{j}\,.$$

If $G(\boldsymbol{\pi})$ is connected, this data determines $\boldsymbol{\pi}$ uniquely up to uniform shift of all centers. Primeness and reality of the underlying simple modules are independent of such shift. Thus, the classification of prime simple modules can be rephrased as a classification of such decorated graphs. For instance, the result for $\mathfrak{g}$ of type A_1 can be phrased as: $L_q(\boldsymbol{\pi})$ is prime if and only if $G(\boldsymbol{\pi})$ has a single vertex. Also, for general $\mathfrak{g}$, if $G(\boldsymbol{\pi})$ has two vertices, then $L_q(\boldsymbol{\pi})$ is prime if and only if $G(\boldsymbol{\pi})$ is connected. This is not true in general: although $G(\boldsymbol{\pi})$ is connected if $L_q(\boldsymbol{\pi})$ is prime (Proposition 3.4.1), the converse is far from true. Henceforth, we say $G(\boldsymbol{\pi})$ is prime if $L_q(\boldsymbol{\pi})$ is prime. We remark that, by definition, $G(\boldsymbol{\pi})$ has no oriented cycles and, therefore, the structure of arrows induce a natural partial order on the set of vertices of $G(\boldsymbol{\pi})$. For instance, in the above picture, $\boldsymbol{\omega} > \boldsymbol{\omega}'$.

A precise description of the elements belonging to $\mathscr{R}_{i,j}^{r,s}$ can be read off the results of [32] for nonexceptional $\mathfrak{g}$ as well as for type G. Some of our results were proved without using such precise description and, hence, they are proved for all types. For instance, the main result of [14] describes a family of prime simple modules for type A_2. In the graph language that we are introducing here, this can be simply described by saying that $G(\boldsymbol{\pi})$ is prime if it is an oriented line (all arrows in the same direction):

$$\circ \longrightarrow \circ \longrightarrow \cdots.$$

In Theorem 3.5.4, we prove that this is true for all $\mathfrak{g}$. Even for type A_2, this does not cover all prime simple modules. For instance, one of the main results we present in [27] characterize all nonoriented lines with three vertices which are prime for type A. Another fact we prove here for all $\mathfrak{g}$ concerns the case that $G(\boldsymbol{\pi})$ is a tree, i.e., there are no (nonoriented) cycles. In that case, we prove that, if $G(\boldsymbol{\pi})$ is prime, then every connected subgraph of $G(\boldsymbol{\pi})$ is also prime. This not true if $G(\boldsymbol{\pi})$ is not a tree, and we give a counter example in [27], which is a paper dedicated to the study of several results concerning trees.

Beside the collection of criteria for deciding whether certain tensor products are highest-ℓ-weight modules or not, the main result of the present paper (Theorem 3.5.5) states that, if $\mathfrak{g}$ is of type A, $L_q(\boldsymbol{\pi})$ is prime if $G(\boldsymbol{\pi})$ is a totally ordered graph, i.e., the partial order on the set of vertices is a total order. In particular, this is the case if $G(\boldsymbol{\pi})$ is a tournament, i.e., if any pair of vertices is linked by an arrow. Thus, for type A, Theorem 3.5.5 is a strong generalization of the aforementioned Theorem 3.5.4. After solving the purely combinatorial problem of classifying all the totally ordered q-factorization graphs, Theorem 3.5.5 would then provide an explicit family of simple prime modules. We do not address this combinatorial problem here beyond type A_2, restricting ourselves to presenting a family of examples of q-factorization graphs with arbitrary number of vertices for type A which are afforded by tournaments in Example 3.6.1. For type A_2, Proposition 3.5.6 implies that a totally ordered q-factorization graph must be a tree and, hence, we are back to the context of [14] and Theorem 3.5.4.

The reason Theorem 3.5.5 is proved only for type A is that, differently from the proof of Theorem 3.5.4, the argument used here explicitly utilizes the description of the sets $\mathscr{R}_{i,j}^{r,s}$. Therefore, if the same approach is to be used for other types, a case-by-case analysis would have to be employed. Thus, we leave the analysis for other types to appear elsewhere.

The paper is organized as follows. In Section 2.1, we review the basic terminology and notation about directed graphs which we shall use, while in Section 2.2, we recall the concept of cuts of a graph as well as the definitions of special types of graphs such as trees and tournaments. The basic notation about classical and quantum affine algebras is fixed in Section 2.3, whereas the notions of Drinfeld polynomials, ℓ-weights, and q-factorization are reviewed in Section 2.4. This is sufficient to formalize the first part of the definition of q-factorization graphs. Thus, in Section 2.5, we define the concept of pre-factorization graph. Section 2.6 closes Section 2 by collecting some basic general facts about Hopf algebras and their representations theory.

The second part of the definition of q-factorization graphs, given in Section 3.4, concerns the sets $\mathscr{R}_{i,j}^{r,s}$, which are explained, alongside the definition of prime modules, in Section 3.3. The required representation theoretic background for these subsections is reviewed in Sections 3.1 and 3.2. The statements of our main results and conjectures are presented in Section 3.5, whereas Section 3.6 brings a few illustrative examples such as the aforementioned family of tournaments. The other two examples interpret the notions of snake and skew modules from the perspective of q-factorization graphs.

Section 4 brings the statements and proofs of the several criteria for deciding whether certain tensor products are highest-ℓ-weight modules or not. Its several subsections split them by the nature of the statements. Perhaps it is worth calling

attention to those criteria which are most used or play more crucial roles in the proof of Theorem 3.5.5 as well as in the proofs of the main results from [27]: Corollary 4.1.6, Proposition 4.3.1, and Proposition 4.5.1.

Section 5 is completely dedicated to the proof of Theorem 3.5.5. We begin by collecting a few technical lemmas concerned with arithmetic relations among the elements of $\mathscr{R}_{i,j}^{r,s}$ in Section 5.1. The key technical part of the proof of Theorem 3.5.5 is Lemma 5.2.1. All the criteria are then brought together to finalize the proof in Section 5.3.

2 Preliminaries

Throughout the paper, let $\mathbb{C}$ and $\mathbb{Z}$ denote the sets of complex numbers and integers, respectively. Let also $\mathbb{Z}_{\geq m}, \mathbb{Z}_{<m}$, etc. denote the obvious subsets of $\mathbb{Z}$. Given a ring $\mathbb{A}$, the underlying multiplicative group of units is denoted by $\mathbb{A}^{\times}$. The symbol $\cong$ means "isomorphic to." We shall use the symbol $\diamond$ to mark the end of remarks, examples, and statements of results whose proofs are postponed. The symbol $\blacksquare$ will mark the end of proofs as well as of statements whose proofs are omitted.

2.1 Directed graphs

In this section, we fix notation regarding the basic concepts of graph theory.

A directed graph is a pair $G = (\mathcal{V}_G, \mathcal{A}_G)$, where $\mathcal{V}_G$ is a set and $\mathcal{A}_G$ is a subset of $\mathcal{V}_G \times \mathcal{V}_G$ such that

$$(v, v') \in \mathcal{A}_G \;\Rightarrow\; (v', v) \notin \mathcal{A}_G.$$

We will typically simplify notation and write $\mathcal{V}$ and $\mathcal{A}$ instead of $\mathcal{V}_G$ and $\mathcal{A}_G$. An element of $\mathcal{V}$ is called a vertex and an element (v, v') of $\mathcal{A}$ is called an arrow from v to v'. We shall also say v' is the head of the arrow (v, v') while v is its tail. Given $a \in \mathcal{A}$, we write t_a for its tail end h_a for its head. As usual, the picture

$$v \longrightarrow v'$$

will mean that $(v, v') \in \mathcal{A}$. A loop in G is an element $a \in \mathcal{A}$ such that $t_a = h_a$. We will only consider graphs with no loops, so, henceforth, this is implicitly assumed. We also assume G is finite, i.e., $\mathcal{V}$ is a finite set.

Given a subset $\mathcal{V}'$ of $\mathcal{V}$, the subgraph $G' = G_{\mathcal{V}'}$ of G associated with $\mathcal{V}'$ is the pair $(\mathcal{V}', \mathcal{A}')$ with

$$\mathcal{A}' = \{a \in \mathcal{A} : t_a, h_a \in \mathcal{V}'\}.$$

In terms of pictures, $G_{\mathcal{V}'}$ is obtained from G by deleting the elements of $\mathcal{V}\backslash\mathcal{V}'$ as well as all the arrows starting at or heading to an element of $\mathcal{V}\backslash\mathcal{V}'$. It will often be convenient to write $G\backslash\mathcal{V}'$ instead of $G_{\mathcal{V}'}$.

Let $\mathscr{P}(\mathcal{V})$ be the power set of $\mathcal{V}$ and $\pi : \mathcal{A} \to \mathscr{P}(\mathcal{V})$ be given by $\pi(a) = \{t_a, h_a\}$. The (nondirected) graph associated with G is the pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{E} = \pi(\mathcal{A})$. The elements of $\mathcal{E}$ will be referred to as edges. By a (nondirected) path of length $m \in \mathbb{Z}_{\geq 0}$

in G, we mean a sequence $\rho = e_1, \ldots, e_m$ of edges in $\mathcal{E}$ such that

$$\#(e_j \cap e_{j+1}) = 1 \text{ for all } 1 \le j < m \text{ and } e_{j-1} \cap e_j \cap e_{j+1} = \varnothing \text{ for all } 1 < j < m.$$

This is equivalent to saying that there exists an underlying sequence of vertices $v_1, \ldots, v_{m+1}$ such that $e_j = \{v_j, v_{j+1}\}$ for all $1 \le j \le m$. This sequence is unique if $m > 1$. If $v_1 = v_{m+1}$, we say ρ is a cycle based on v_1. In that case, if $m = \min\{j > 1 : v_j = v_1\}$, we say ρ is an m-cycle. Note there does not exist m-cycles for $m \le 2$.

We shall often write $\rho = e_1 \ldots e_m$ instead of $\rho = e_1, \ldots, e_m$ and set $\ell(\rho) = m$. We also write $e \in \rho$ to mean that $e = e_j$ for some $1 \le j \le m$. Suppose $\rho' = e_1' \ldots e_{m'}'$ is another path such that $e_m \cap e_1' \ne \varnothing$ and either

$$e_m = e_1' \quad \text{or} \quad e_{m-1} \cap e_m \cap e_1' = \varnothing = e_m \cap e_1' \cap e_2'.$$

Then, the sequence obtained from $e_1 \ldots e_m e_1' \ldots e_{m'}'$ after successive deletion of any appearance of a substring of the form ee, $e \in \mathcal{E}$, is a path which we denote by $\rho * \rho'$. The path $\rho^- := e_m \ldots e_1$ will be referred to as the reverse path of ρ. In particular, $\rho * \rho^-$ is the empty sequence.

If $m = \ell(\rho) > 1$,

$$v \in e_1 \backslash e_2, \qquad \text{and} \qquad v' \in e_m \backslash e_{m-1},$$

we say ρ is a is a path from v to v'. If $\ell(\rho) = 1$, say, $\rho = e_1 = \pi(a)$ for some $a \in \mathcal{A}$, ρ can be regarded as a path from t_a to h_a and vice versa. We let $\mathscr{P}_{v,v'}$ be the set of all paths from v to v' and $\mathscr{P}_G$ be the set of all paths in G. If $\rho \in \mathscr{P}_{v,v'}$ and $\rho' \in \mathscr{P}_{v',v''}$, then $\rho * \rho' \in \mathscr{P}_{v,v''}$.

A subpath ρ' of ρ is subsequence such that

$$e_i, e_j \in \rho' \quad \text{with} \qquad i < j \qquad \Rightarrow \qquad e_k \in \rho' \quad \text{for all} \quad i \le k \le j.$$

We say ρ is a simple path if no subpath is a cycle. If $\rho = e_1 \ldots e_m$ is a path from v to v', $e_j = \pi(a_j)$, and $m > 1$, the signature of ρ is the element $\sigma_\rho = (s_1, \ldots, s_m) \in \mathbb{Z}^m$ given by

$$s_1 = \begin{cases} -1, & \text{if } v = t_{a_1}, \\ 1, & \text{if } v = h_{a_1}, \end{cases} \qquad \text{and} \qquad s_{j+1} = \begin{cases} s_j, & \text{if } t_{a_{j+1}} = h_{a_j} \text{ or } t_{a_j} = h_{a_{j+1}}, \\ -s_j, & \text{otherwise,} \end{cases}$$

for all $1 \le j < m$. If $m = 1$, the signature will be 1 or -1 depending on whether it is regarded as a path from h_{a_1} to t_{a_1} or the other way round, respectively. We shall say ρ is monotonic or directed if $s_i = s_j$ for all $1 \le i, j \le m$. In that case, if $s_j = 1$ for all $1 \le j \le m$, we say it is increasing. Otherwise, it is decreasing. If ρ is increasing, we set $h_\rho = h_{a_1}$ and $t_\rho = t_{a_m}$. If it is decreasing, then $t_\rho = t_{a_1}$ and $h_\rho = h_{a_m}$. If $s_{j+1} = -s_j$ for all $1 \le j < m$, we say ρ is alternating. Clearly, $\sigma_{\rho^-} = (-s_m, \ldots, -s_1)$. We shall refer to a monotonic cycle as an oriented cycle. We will denote by $\mathscr{P}_{v,v'}^+$ (resp. $\mathscr{P}_{v,v'}^-$) be the set of increasing (resp. decreasing) monotonic paths from v to v'. For instance,

$$v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} v_3 \in \mathscr{P}_{v_1,v_3}^- \qquad \text{while} \qquad v_1 \xleftarrow{a_1} v_2 \xleftarrow{a_2} v_3 \in \mathscr{P}_{v_1,v_3}^+.$$

On the other hand, $v_1 \xrightarrow{a_1} v_2 \xleftarrow{a_2} v_3 \in \mathscr{P}_{v_1,v_3}$, but is neither in $\mathscr{P}_{v_1,v_3}^+$ nor in $\mathscr{P}_{v_1,v_3}^-$.

A graph G is said to be connected if, for every pair of vertices $v \neq v'$, there exists a path from v to v'. If G is connected, we can consider the distance function $d : \mathcal{V} \to \mathbb{Z}$ defined by, $d(v, v) = 0$ for all $v \in \mathcal{V}$ and
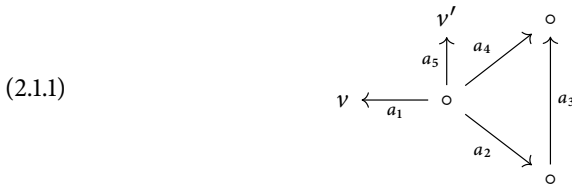
$$d(v, v') = \min\{\ell(\rho) : \rho \text{ is a path from } v \text{ to } v'\} \qquad \text{if} \qquad v \neq v'.$$

If $d(v, v') = 1$ we say v and v' are adjacent. Also, for two subsets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, define

$$d(\mathcal{V}_1, \mathcal{V}_2) = \min\{d(v_1, v_2) : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}.$$

Set $d(v, v') = \infty$ if v and v' belong to distinct connected components.

Example 2.1.1 The following path $\rho = e_1 \ldots e_5$ from v to v' has signature $(1, -1, -1, 1, -1)$ and contains the 3-cycle $e_2 e_3 e_4$. The circles denote other arbitrary elements in $\mathcal{V}$.

(2.1.1)

Note $d(v, v') \leq 2$ since $a_1 a_5$ is a path from v to v'. The subpath $e_2 e_3$ is decreasing, while $e_3 e_4 e_5$ is an alternating subpath. The path $e_1 e_5$ is alternating while $e_3 e_2$ is increasing, but they are not subpaths of ρ. $\diamond$

Every path gives rise to a subgraph associated with the set

$$\mathcal{V}^\rho = \{v \in \mathcal{V} : v \in e \text{ for some } e \in \rho\}.$$

Given $v \in \mathcal{V}$, set $\mathcal{A}_v = \{v' \in \mathcal{V} : d(v, v') = 1\}$,

$$(2.1.2) \qquad \mathcal{A}_v^1 = \{v' \in \mathcal{A}_v : (v', v) \in \mathcal{A}\}, \qquad \text{and} \qquad \mathcal{A}_v^{-1} = \{v' \in \mathcal{A}_v : (v, v') \in \mathcal{A}\}.$$

The valence of v is defined as $\#\mathcal{A}_v$. If this number is 0, we say v is an isolated vertex, if it is 1, we say v is monovalent, and if it is at least 3, we say v is multivalent. Set

$$\mathring{G} = \{v \in \mathcal{V} : \#\mathcal{A}_v > 1\} \qquad \text{and} \qquad \partial G = G \backslash \mathring{G}.$$

Elements of ∂G will be referred to as boundary vertices while those of $\mathring{G}$ will be referred to as inner vertices. A vertex v is said to be a source if there are no incoming arrows toward it or, equivalently,

$$\mathcal{A}_v \subseteq \mathcal{A}_v^{-1},$$

whereas it is a sink if

$$\mathcal{A}_v \subseteq \mathcal{A}_v^1.$$

In particular, isolated vertices are sinks and sources at the same time and a non-isolated vertex cannot be a sink and a source concomitantly. We will say a vertex is extremal if it is either a sink or a source. Note the middle circle in (2.1.1) is a source, the upper one is a sink, and the lower one is neither.

2.2 Cuts and special kinds of graphs

A cut of a directed graph G is a pair of subgraphs (G', G'') such that

$$\mathcal{V} = \mathcal{V}' \sqcup \mathcal{V}'', \quad \mathcal{A}' = \{a \in \mathcal{A} : h_a, t_a \in \mathcal{V}'\}, \text{ and } \mathcal{A}'' = \{a \in \mathcal{A} : h_a, t_a \in \mathcal{V}''\}.$$

The set

$$\mathcal{A} \backslash (\mathcal{A}' \cup \mathcal{A}'')$$

is called the associated cut-set. Note the cut can be recovered from its cut-set if G is connected. Elements of the cut-set are said to cross the cut. An element $a \in \mathcal{A}$ is said to be a bridge if the number of connected components of $(\mathcal{V}, \mathcal{A} \backslash \{a\})$ is larger than that of G. If G is connected, this is equivalent to saying that $\{a\}$ is the cut-set of a cut. We shall say a cut (G', G'') is connected if both G' and G'' are connected.

A connected graph with no cycles is said to be a tree. We shall refer to a tree with no multivalent vertex as a line. We will say G is a monotonic line if $\mathcal{V} = \mathcal{V}^\rho$ for some simple monotonic path ρ. Note every tree with more than one vertex has at least two monovalent vertices, a fact which is false in general, as seen in the following examples.

(2.2.1)
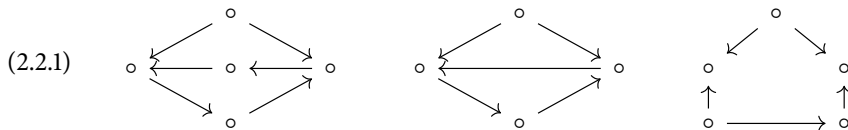
Note that, in the first two graphs, the subgraphs obtained by removing the upper vertex are formed by directed cycles, while the cycle corresponding to the subgraphs obtained by removing the lower vertex are not. Note also that an arrow a is a bridge if and only if it is not contained in a cycle. In particular, the above graphs are bridgeless. A forest is a graph whose connected components are trees or, equivalently, every arrow is a bridge. We also recall that a tournament is a graph whose underlying set of edges is complete, i.e., $\{v, v'\} \in \mathcal{E}$ for every $v, v' \in \mathcal{V}, v \neq v'$. In that case, the underlying nondirected graph is said to be complete. None of the above graphs is a tournament, but the middle one is missing only one arrow to become a tournament.

Let us record some elementary properties of trees.

Lemma 2.2.1 *The following are equivalent for a graph G.*

(i) *G is a tree.*
(ii) *G is connected and the graph obtained by removing any edge has two connected components.*
(iii) *$\#\mathscr{P}_{v,v'} = 1$ for all vertices $v, v' \in G$.*

In light of (iii) of the above lemma, given vertices v, v' in a tree, we denote by $[v, v']$ the set of vertices of the unique element of $\mathscr{P}_{v,v'}$. In particular, $[v', v] = [v, v']$. Evidently, if $v \neq v'$, $\#([v, v'] \cap \mathcal{A}_v) = 1$. Given $m \in \mathbb{Z}_{>0}$, set

(2.2.2) $$\mathcal{A}_v^{\pm m} = \{v' \in \mathcal{V} : d(v, v') = m \text{ and } [v, v'] \cap \mathcal{A}_{v'}^{\mp 1} \neq \varnothing\}.$$

This clearly coincides with the sets defined in (2.1.2) when $m = 1$. Set also $\mathcal{A}_v^0 = \{v\}$ and

$$(2.2.3) \qquad \mathcal{A}_v^\pm = \bigcup_{m \in \mathbb{Z}_{>0}} \mathcal{A}_v^{\pm m}.$$

Lemma 2.2.2 *Assume G is a tree.*

(a) *$\partial G \neq \varnothing$, $\#\partial G = 1$ iff G is a singleton, and $\#\partial G = 2$ iff G is a nontrivial path.*
(b) *If H is a subgraph, then H is a tree. Moreover, if H is connected and proper, $\partial G \backslash \mathcal{V}_H \neq \varnothing$.*
(c) *If H is a connected subgraph and $k = \#\mathcal{V}_G - \#\mathcal{V}_H$, there exist $v_1, \ldots, v_k \in \mathcal{V}_G$ such that $v_j \in \partial(G \backslash \{v_i : i < j\})$ and $H = G \backslash \{v_i : 1 \leq i \leq k\}$.*
(d) *If $\mathcal{V}_1 \cup \mathcal{V}_2$ is a nontrivial partition of $\mathcal{V}_G$ such that $G_{\mathcal{V}_i}$ is connected for $i = 1, 2$, there exists unique $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ such that $d(v_1, v_2) = 1$.*
(e) *For all $v \in \mathcal{V}$, the sets $\mathcal{A}_v^m, m \in \mathbb{Z}$ are disjoint and $\mathcal{V} = \mathcal{A}_v^+ \cup \mathcal{A}_v^0 \cup \mathcal{A}_v^-$.*

We will be interested in graphs with no oriented cycles. In that case, the set of arrows $\mathcal{A}$ induces a partial order on $\mathcal{V}$ by the transitive extension of the strict relation

$$h_a < t_a \quad \text{for} \quad a \in \mathcal{A}.$$

Note

$$(2.2.4) \qquad \mathcal{P}_{v,v'}^+ \neq \varnothing \iff v < v' \quad \text{and} \quad \mathcal{P}_{v,v'}^- \neq \varnothing \iff v' < v.$$

Set $D(v, v') = 0$ if $v = v'$,

$$(2.2.5) \qquad \begin{aligned} D(v, v') &= \min\{\ell(\rho) : \rho \in \mathcal{P}_{v,v'}^+\} \text{ if } v < v', \\ D(v, v') &= -\min\{\ell(\rho) : \rho \in \mathcal{P}_{v,v'}^-\} \text{ if } v' < v, \end{aligned}$$

and $D(v, v') = \infty$ if v and v' are not comparable by $\leq$. Given $m \in \mathbb{Z}$, set

$$(2.2.6) \qquad \mathcal{N}_G^m(v) = \{v \in \mathcal{V} : D(v, v') = m\} \quad \text{and} \quad \mathcal{N}_G^\pm(v) = \bigcup_{m \in \mathbb{Z}_{\geq 0}} \mathcal{N}_G(v)^{\pm m}.$$

If no confusion arises, we simplify notation and write $\mathcal{N}^m(v)$ and $\mathcal{N}^\pm(v)$.

We shall say G is a totally ordered graph if $\leq$ is a total order on $\mathcal{V}$. The following lemma is easily established.

Lemma 2.2.3 (a) *Every totally ordered graph is connected and has a unique sink and a unique source.*

(b) *If G is a totally ordered graph and $v \in \mathcal{V}$ is an extremal vertex, the subgraph associated with $\mathcal{V} \backslash \{v\}$ is also totally ordered.*
(c) *A totally ordered tree is an monotonic line.*
(d) *Every tournament with no oriented cycles is totally ordered.*

Only the last graph in (2.2.1) does not contain an directed cycle so $\leq$ is defined, but it is not totally ordered. The following are examples of totally ordered graphs:
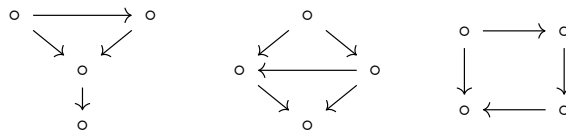
2.3 Classical and quantum algebras

Let I be the set of nodes of a finite-type connected Dynkin diagram. By regarding I as the set of vertices of the undirected graph whose edges are the sets of adjacent nodes of the diagram, we can use the notions of graph theory from the previous sections. By abuse of language, we refer to any subset J of I as a subdiagram (subgraph). In particular, we have defined $d(i,j)$ and $[i,j]$ for all $i,j \in I$ as well as ∂J and $\mathring{J}$ for any $J \subseteq I$. Let also $\bar{J}$ be the minimal connected subdiagram of I containing J. This is well defined since I is a tree.

Let $\mathfrak{g}$ be the simple Lie algebra over $\mathbb{C}$ corresponding to the given Dynkin diagram, fix a Cartan subalgebra $\mathfrak{h}$ and a set of positive roots R^+ and let $\mathfrak{g}_{\pm\alpha}, \alpha \in R^+$, and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the associated root spaces and triangular decomposition. The simple roots will be denoted by α_i, the fundamental weights by ω_i, $i \in I$, while Q, P, Q^+, P^+ will denote the root and weight lattices with corresponding positive cones, respectively. Let also $h_\alpha \in \mathfrak{h}$ be the co-root associated with $\alpha \in R^+$. If $\alpha = \alpha_i$ is simple, we often simplify notation and write h_i. Let $C = (c_{i,j})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$, i.e., $c_{i,j} = \alpha_j(h_i)$, and d_i, $i \in I$, be such that $d_i c_{i,j} = d_j c_{j,i}$, $i, j \in I$. The Weyl group is denoted by $\mathcal{W}$ and its longest element by w_0. We also denote by w_0 the involution on I induced by w_0 and set $i^* = w_0(i)$. The dual Coxeter number and the lacing number of $\mathfrak{g}$ will be denoted by $h^\vee$ and $r^\vee$, respectively. In particular, $r^\vee = \max\{d_i : i \in I\}$.

For a subdiagram $J \subseteq I$, let $\mathfrak{g}_J$ be the subalgebra of $\mathfrak{g}$ generated by the corresponding simple root vectors, $\mathfrak{h}_J = \mathfrak{h} \cap \mathfrak{g}_J$, and so on. Let also Q_J be the subgroup of Q generated by α_j, $j \in J$, $Q_J^+ = Q^+ \cap Q_J$, and $R_J^+ = R^+ \cap Q_J$. Given $\lambda \in P$, let λ_J denote the restriction of λ to $\mathfrak{h}_J^*$. For $\mu \in P$, define also

$$\operatorname{supp}(\mu) = \{i \in I : \mu(h_i) \neq 0\}.$$

For a Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, let $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ be its loop algebras and identify $\mathfrak{a}$ with the subalgebra $\mathfrak{a} \otimes 1$. Then, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra.

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, fix $q \in \mathbb{F}^\times$ which is not a root of 1, and set $q_i = q^{d_i}$, $i \in I$. Let also $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ be the associated Drinfeld–Jimbo quantum groups over $\mathbb{F}$. We use the notation as in [26, Section 1.2]. In particular, the Drinfeld loop-like generators of $U_q(\tilde{\mathfrak{g}})$ are denoted by $x_{i,r}^\pm, h_{i,s}, k_i^{\pm 1}, i \in I, r, s \in \mathbb{Z}, s \neq 0$. Also, $U_q(\mathfrak{g})$ is the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $x_i^\pm = x_{i,0}^\pm, k_i^{\pm 1}, i \in I$, and the subalgebras $U_q(\mathfrak{n}^\pm), U_q(\mathfrak{h}), U_q(\tilde{\mathfrak{n}}^\pm), U_q(\tilde{\mathfrak{h}})$ are defined in the expected way.

Given $J \subseteq I$, let $U_q(\mathfrak{a}_J)$, with $\mathfrak{a} = \mathfrak{g}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}$, etc. be the respective quantum groups associated with $\mathfrak{a}_J$. Let also $U_q(\mathfrak{a})_J$ be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the generators corresponding to J. It is well known that there is an algebra isomorphism

$$U_q(\mathfrak{a})_J \cong U_{q_J}(\mathfrak{a}_J), \qquad \text{where} \qquad q_J = q^{d_J} \qquad \text{with} \qquad d_J = \min\{d_j : j \in J\}.$$

This is a Hopf algebra isomorphism only if $\mathfrak{a} \subseteq \mathfrak{g}$. We shall always implicitly identify $U_q(\mathfrak{a})_J$ with $U_{q_J}(\mathfrak{a}_J)$ without further notice. When $J = \{j\}$ is a singleton, we simply write $U_q(\mathfrak{a})_j$ instead of $U_q(\mathfrak{a})_{\{j\}}$, and so on.

2.4 The ℓ-weight lattice

The ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}})$ is the multiplicative group $\mathcal{P}$ of n-tuples of rational functions $\boldsymbol{\varpi} = (\boldsymbol{\varpi}_i(u))_{i\in I}$ with values in $\mathbb{F}$ such that $\boldsymbol{\varpi}_i(0) = 1$ for all $i \in I$.

The elements of the submonoid $\mathcal{P}^+$ of $\mathcal{P}$ consisting of n-tuples of polynomials will be referred to as dominant ℓ-weights or Drinfeld polynomials. If $\boldsymbol{\pi}, \boldsymbol{\omega} \in \mathcal{P}^+$ satisfy $\boldsymbol{\pi}\boldsymbol{\omega}^{-1} \in \mathcal{P}^+$, we shall say $\boldsymbol{\omega}$ divides $\boldsymbol{\pi}$ and write $\boldsymbol{\omega}|\boldsymbol{\pi}$.

Given $a \in \mathbb{F}^\times$ and $\mu \in P$, let $\boldsymbol{\omega}_{\mu,a} \in \mathcal{P}$ be the element whose ith rational function is

$$(1 - au)^{\mu(h_i)}, \quad i \in I.$$

In the case that $\mu = \omega_i$ for some i, we simplify notation and write $\boldsymbol{\omega}_{i,a}$. Since $\mathcal{P}$ is a (multiplicative) free abelian group on the set $\{\boldsymbol{\omega}_{i,a} : i \in I, a \in \mathbb{F}^\times\}$, there exists a unique group homomorphism $\mathrm{wt} : \mathcal{P} \to P$ determined by setting $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$. Set

$$\mathrm{supp}(\boldsymbol{\omega}) = \mathrm{supp}(\mathrm{wt}(\boldsymbol{\omega})), \quad \boldsymbol{\omega} \in \mathcal{P}.$$

There exists an injective map $\mathcal{P} \to (U_q(\tilde{\mathfrak{h}}))^*$ (see [26]) and, hence, we identify $\mathcal{P}$ with its image in $(U_q(\tilde{\mathfrak{h}}))^*$.

Given $i \in I, a \in \mathbb{F}^\times, m \in \mathbb{Z}_{\geq 0}$, define $q_i = q^{d_i}$ and

$$\boldsymbol{\omega}_{i,a,r} = \prod_{p=0}^{r-1} \boldsymbol{\omega}_{i,aq_i^{r-1-2p}}.$$

Note that $\mathrm{wt}(\boldsymbol{\omega}_{i,a,r}) = r\omega_i$. We shall refer to Drinfeld polynomials of the form $\boldsymbol{\omega}_{i,a,r}$ as polynomials of KR type. Every Drinfeld polynomial can be written uniquely as a product of KR type polynomials such that, for every two factors supported at i, say $\boldsymbol{\omega}_{i,a,r}$ and $\boldsymbol{\omega}_{i,b,s}$, the following holds:

$$(2.4.1) \qquad\qquad \frac{a}{b} \neq q_i^{r+s-2p} \qquad \text{for all} \qquad 0 \leq p < \min\{r, s\}.$$

Such factorization is said to be the q-factorization of $\boldsymbol{\pi}$ and the corresponding factors are called the q-factors of $\boldsymbol{\pi}$. By abuse of language, whenever we mention the set of q-factors of $\boldsymbol{\pi}$ we actually mean the associated multiset of q-factors counted with multiplicities in the q-factorization. We shall say that $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$ have dissociate q-factorizations if the set of q-factors of $\boldsymbol{\pi}\boldsymbol{\pi}'$ is the union of the sets of q-factors of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$. It will also be convenient to work with factorizations in KR type polynomials which not necessarily satisfy (2.4.1). Such factorization will be referred to as pseudo q-factorizations and the associated factors as the corresponding pseudo q-factors.

For $\boldsymbol{\omega} \in \mathcal{P}$ and $J \subseteq I$, let $\boldsymbol{\omega}_J$ be the associated J-tuple of rational functions, and let $\mathcal{P}_J = \{\boldsymbol{\omega}_J : \boldsymbol{\omega} \in \mathcal{P}\}$. Similarly define $\mathcal{P}_J^+$. Notice that $\boldsymbol{\omega}_J$ can be regarded as an element of the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}})_J$. Let $\pi_J : \mathcal{P} \to \mathcal{P}_J$ denote the map $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}_J$. If $J = \{j\}$ is a singleton, we write π_j instead of π_J.

Given $i \in I, a \in \mathbb{F}^\times$, the following elements are known as simple ℓ-roots:

$$(2.4.2) \qquad\qquad \boldsymbol{\alpha}_{i,a} = (\boldsymbol{\omega}_{i,aq_i,2})^{-1} \prod_{j \neq i} \boldsymbol{\omega}_{j,aq_i,-c_{j,i}}.$$

The subgroup of $\mathcal{P}$ generated by them is called the ℓ-root lattice of $U_q(\tilde{\mathfrak{g}})$ and will be denoted by $\mathcal{Q}_q$. Let also $\mathcal{Q}_q^+$ be the submonoid generated by the simple ℓ-roots. Quite clearly, $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$. Define a partial order on $\mathcal{P}$ by

$$\boldsymbol{\varpi} \leq \boldsymbol{\omega} \quad if \quad \boldsymbol{\omega}\boldsymbol{\varpi}^{-1} \in \mathcal{Q}_q^+.$$

2.5 Pre-factorization graphs

Given a set I, an I-coloring of a graph $G = (\mathcal{V}, \mathcal{A})$ is a function $c : \mathcal{V} \to I$. Given an I-coloring c and $i \in I$, let $\mathcal{V}_i = \{x \in \mathcal{V} : c(x) = i\}$. By a colored graph, we will mean an oriented graph G with a choice of coloring $c : \mathcal{V} \to I$.

We shall also decorate the vertices and arrows of graphs by positive integers. We will refer to a function $\lambda : \mathcal{V} \to \mathbb{Z}_{>0}$ as a weight and to a function $\varepsilon : \mathcal{A} \to \mathbb{Z}_{>0}$ as an exponent on G. The number $\varepsilon(a)$ will be referred to as the exponent of a. We will always assume that ε satisfies the following compatibility condition:

$$(2.5.1) \qquad \varepsilon_\rho = \varepsilon_{\rho'} \qquad \text{for all} \qquad \rho, \rho' \in \mathscr{P}_{v,v'}, v, v' \in \mathcal{V},$$

where, if $\rho = e_1 \dots e_m$ is such that $\sigma_\rho = (s_1, \dots, s_m)$ and $e_j = \pi(a_j)$,

$$\varepsilon_\rho := \sum_{j=1}^{m} s_j \varepsilon(a_j).$$

Evidently, $\varepsilon_{\rho^-} = -\varepsilon_\rho$ and one easily checks $\varepsilon_{\rho * \rho'} = \varepsilon_\rho + \varepsilon_{\rho'}$. Set

$$(2.5.2) \qquad \mathscr{P}_G^+ = \{\rho \in \mathscr{P}_G : \varepsilon_\rho > 0\} \quad \text{and} \quad \mathscr{P}_G^- = \{\rho \in \mathscr{P}_G : \varepsilon_\rho < 0\}.$$

We shall refer to the data $(G, c, \lambda, \varepsilon)$ formed by a colored oriented graph, a weight λ, and an exponent ε on G as a pre-factorization graph. We shall abuse of language and simply say G is a pre-factorization graph. We locally illustrate the structures maps of a pre-factorization graph with the following picture:

$$\overset{r}{i} \xrightarrow{\ m\ } \overset{s}{j},$$

where i and j are the colors at the corresponding vertices, r and s are their associated weights, and m is the exponent associated with the given arrow. The following is an obvious consequence of (2.5.1).

Lemma 2.5.1 *If G is a pre-factorization graph, then G contains no oriented cycles.*

In particular, only the last graph in (2.2.1) can be equipped with a pre-factorization graph structure. If I is as in Section 2.3 and G is a connected pre-factorization graph, for each choice of $(v_0, a) \in \mathcal{V} \times \mathbb{F}^\times$, we can associate a Drinfeld polynomial as follows. Define

$$(2.5.3) \qquad a_{v_0} = a \quad \text{and} \quad a_v = a q^{\varepsilon_\rho} \quad \text{if} \quad \rho \in \mathscr{P}_{v_0, v}.$$

Condition (2.5.1) guarantees this is well defined. Then, define

$$(2.5.4) \qquad \pi_{G, v_0, a} = \prod_{v \in \mathcal{V}} \omega_{c(v), a_v, \lambda(v)}.$$

One can easily check that

$$(2.5.5) \qquad \pi_{G, v_0, a' q^{-\varepsilon_\rho}} = \pi_{G, v_0', a'} \qquad \text{for all} \qquad (v_0', a') \in \mathcal{V} \times \mathbb{F}^\times, \rho \in \mathscr{P}_{v_0, v_0'}.$$

Therefore, up to a uniform modification on the centers of the factors in the right-hand side of (2.5.4), the definition is independent of the choice of (v_0, a). We will often write π_G to shorten notation when the knowledge of precise centers is not relevant.

Example 2.5.2   Assume $\mathfrak{g}$ is of type A_2, so $I = \{1, 2\}$, and consider the following pre-factorization graph:

$$\overset{2}{1} \xrightarrow{\ 3\ } \overset{2}{2} \xleftarrow{\ 4\ } \overset{1}{1}.$$

If we select the middle vertex to define $\boldsymbol{\pi} = \boldsymbol{\pi}_G$, we get

$$\boldsymbol{\pi} = \boldsymbol{\omega}_{2,a,2}\ \boldsymbol{\omega}_{1,aq^3,2}\ \boldsymbol{\omega}_{1,aq^4}.$$

Note that, in this case, the factors in (2.5.4) are the q-factors of $\boldsymbol{\pi}$. However, this may not be the case as the following trivial example shows:

$$\overset{1}{1} \xrightarrow{\ 2\ } \overset{1}{1}.$$

In this case, if we choose the first vertex as the base for the definition, the factors in (2.5.4) are $\boldsymbol{\omega}_{1,a}$ and $\boldsymbol{\omega}_{1,aq^2}$, which combine to form a single q-factor.

2.6   Hopf algebra facts

We recall some general facts about Hopf algebras (see, for instance, [16] and the references therein).

Given a Hopf algebra $\mathcal{H}$ over $\mathbb{F}$, its category $\mathcal{C}$ of finite-dimensional representations is an abelian monoidal category and we denote the (right) dual of a module V by V^*. More precisely, the action of $\mathcal{H}$ of V^* is given by

$$(2.6.1) \qquad (hf)(v) = f(S(h)v) \quad \text{for} \quad h \in \mathcal{H}, f \in V^*, v \in V.$$

The evaluation map $V^* \otimes V \to \mathbb{F}$ is a module map, where $\mathbb{F}$ is regarded as the trivial module by using the counit map. Moreover if, $v_1, \dots, v_n$ is a basis of V and $f_1, \dots, f_n$ is the corresponding dual basis, there exists a unique homomorphism of modules

$$\mathbb{F} \to V \otimes V^*, \qquad 1 \mapsto \sum_{i=1}^{n} v_i \otimes f_i,$$

called the coevaluation map. We denote the evaluation and coevaluation maps associated with a module V by ev_V and coev_V, respectively, or simply by ev and coev if no confusion arises. In particular,

$$\mathrm{Hom}_{\mathcal{H}}(\mathbb{F}, V \otimes V^*) \neq 0 \qquad \text{and} \qquad \mathrm{Hom}_{\mathcal{H}}(V^* \otimes V, \mathbb{F}) \neq 0.$$

If the antipode is invertible, the notion of left dual module is obtained by replacing S by S^{-1} in (2.6.1). The left dual of V will be denoted by $^*V$ and we have

$$^*(V^*) \cong (^*V)^* \cong V.$$

Given $\mathcal{H}$-modules V_1, V_2, V_3, we have

$$(2.6.2) \qquad \begin{aligned} \mathrm{Hom}_{\mathcal{C}}(V_1 \otimes V_2, V_3) &\cong \mathrm{Hom}_{\mathcal{C}}(V_1, V_3 \otimes V_2^*), \\ \mathrm{Hom}_{\mathcal{C}}(V_1, V_2 \otimes V_3) &\cong \mathrm{Hom}_{\mathcal{C}}(V_2^* \otimes V_1, V_3), \end{aligned}$$

and

$$(2.6.3) \qquad (V_1 \otimes V_2)^* \cong V_2^* \otimes V_1^*.$$

For instance, an isomorphism for the first statement in (2.6.2) is given by

$$f \mapsto (f \otimes \mathrm{id}_{V_2^*}) \circ (\mathrm{id}_{V_1} \otimes \mathrm{coev}_{V_2}) \circ \gamma_{V_1}$$

and has the inverse

$$g \mapsto \gamma'_{V_3} \circ (\mathrm{id}_{V_3} \otimes \mathrm{ev}_{V_2}) \circ (g \otimes \mathrm{id}_{V_2}),$$

where $\gamma_V : V \to V \otimes \mathbb{F}$ and $\gamma'_V : V \otimes \mathbb{F} \to V$ are the canonical maps. Note also that every short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

gives rise to another short exact sequence of the form

(2.6.4) $$0 \to V_3^* \to V_2^* \to V_1^* \to 0.$$

We shall use the following lemma in the same spirit as in [24] (a proof can also be found in [34]).

Lemma 2.6.1 *Let $V_1, V_2, V_3 \in \mathcal{C}$ and suppose M is a submodule of $V_1 \otimes V_2$ and N is a submodule of $V_2 \otimes V_3$ such that*

$$M \otimes V_3 \subseteq V_1 \otimes N.$$

Then, there exists a submodule W of V_2 such that

$$M \subseteq V_1 \otimes W \qquad and \qquad W \otimes V_3 \subseteq N.$$

Similarly, if $V_1 \otimes N \subseteq M \otimes V_3$, there exists a submodule W of V_2 such that

$$N \subseteq W \otimes V_3 \qquad and \qquad V_1 \otimes W \subseteq M.$$

Lemma 2.6.2 *Let V_1, V_2, V_3, L_1, L_2 be $\mathcal{H}$-modules and assume V_2 is simple. If*

$$\varphi_1 : L_1 \to V_1 \otimes V_2 \quad and \quad \varphi_2 : V_2 \otimes V_3 \to L_2$$

are nonzero homomorphisms, the composition

$$L_1 \otimes V_3 \xrightarrow{\varphi_1 \otimes \mathrm{id}_{V_3}} V_1 \otimes V_2 \otimes V_3 \xrightarrow{\mathrm{id}_{V_1} \otimes \varphi_2} V_1 \otimes L_2$$

does not vanish. Similarly, if

$$\varphi_1 : V_1 \otimes V_2 \to L_1 \quad and \quad \varphi_2 : L_2 \to V_2 \otimes V_3$$

are nonzero homomorphisms, the composition

$$V_1 \otimes L_2 \xrightarrow{\mathrm{id}_{V_1} \otimes \varphi_2} V_1 \otimes V_2 \otimes V_3 \xrightarrow{\varphi_1 \otimes \mathrm{id}_{V_3}} L_1 \otimes V_3$$

does not vanish.

Proof We will write down the details for the first claim only, as the second can be proved similarly. Assume

$$(\mathrm{id}_{V_1} \otimes \varphi_2) \circ (\varphi_1 \otimes \mathrm{id}_{V_3}) = 0,$$

i.e.,

$$\mathrm{Im}(\varphi_1) \otimes V_3 = \mathrm{Im}(\varphi_1 \otimes \mathrm{id}_{V_3}) \subseteq \mathrm{Ker}(\mathrm{id}_{V_1} \otimes \varphi_2) = V_1 \otimes \mathrm{Ker}(\varphi_2).$$

Lemma 2.6.1 implies there exists a submodule $W \subseteq V_2$ such that

$$\mathrm{Im}(\varphi_1) \subseteq V_1 \otimes W \quad \text{and} \quad W \otimes V_3 \subseteq \mathrm{Ker}(\varphi_2).$$

Since V_2 is simple, either $W = 0$ or $W = V_2$. If $W = 0$, then $\mathrm{Im}(\varphi_1) \subseteq V_1 \otimes W = 0$, which is a contradiction, since φ_1 is nonzero. On the other hand, if $W = V_2$, it follows that $V_2 \otimes V_3 = \mathrm{Ker}(\varphi_2)$, yielding a contradiction, since φ_2 is nonzero. ∎

3 Representation theory and *q*-factorization graphs

We start this section reviewing the relevant representation theoretic background for our purposes. This will lead to the main definition of the paper: that of q-factorization graphs. We then state the main results of the paper and end the section with a few illustrative examples.

3.1 Finite-dimensional representations

Let $\mathcal{C}$ be the category of all finite-dimensional (type-1) weight modules of $U_q(\mathfrak{g})$. Thus, a finite-dimensional $U_q(\mathfrak{g})$-module V is in $\mathcal{C}$ if

$$V = \bigoplus_{\mu \in P} V_\mu, \qquad \text{where} \qquad V_\mu = \{v \in V : k_i v = q_i^{\mu(h_i)} v \text{ for all } i \in I\}.$$

The following theorem summarizes the basic facts about $\mathcal{C}$.

Theorem 3.1.1 *Let V be an object of $\mathcal{C}$. Then:*

(a) $\dim V_\mu = \dim V_{w\mu}$ *for all $w \in \mathcal{W}$.*
(b) *V is completely reducible.*
(c) *For each $\lambda \in P^+$, the $U_q(\mathfrak{g})$-module $L_q(\lambda)$ generated by a vector v satisfying*

$$x_i^+ v = 0, \qquad k_i v = q^{\lambda(h_i)} v, \qquad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \forall\, i \in I,$$

is irreducible and finite-dimensional. If $V \in \mathcal{C}$ is irreducible, then V is isomorphic to $L_q(\lambda)$ for some $\lambda \in P^+$.

If $J \subseteq I$ we shall denote by $L_q(\lambda_J)$ the simple $U_q(\mathfrak{g})_J$-module of highest weight λ_J. Since $\mathcal{C}$ is semisimple, it is easy to see that, if $\lambda \in P^+$ and $v \in L_q(\lambda)_\lambda$ is nonzero, then $U_q(\mathfrak{g})_J v \cong L_q(\lambda_J)$.

Let $\widetilde{\mathcal{C}}$ the category of all finite-dimensional ℓ-weight modules of $U_q(\tilde{\mathfrak{g}})$. Thus, a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V is in $\widetilde{\mathcal{C}}$ if

$$V = \bigoplus_{\boldsymbol{\varpi} \in \mathcal{P}} V_{\boldsymbol{\varpi}},$$

where

$$v \in V_{\boldsymbol{\varpi}} \quad \Leftrightarrow \quad \exists\, k \gg 0 \;\; \text{s.t.} \;\; (\eta - \boldsymbol{\varpi}(\eta))^k v = 0 \;\; \text{for all} \;\; \eta \in U_q(\tilde{\mathfrak{h}}).$$

$V_{\boldsymbol{\varpi}}$ is called the ℓ-weight space of V associated with $\boldsymbol{\varpi}$. Note that if $V \in \widetilde{\mathcal{C}}$, then $V \in \mathcal{C}$ and

$$V_\mu = \bigoplus_{\boldsymbol{\varpi}:\mathrm{wt}(\boldsymbol{\varpi})=\mu} V_{\boldsymbol{\varpi}}.$$

If $V \in \widetilde{\mathcal{C}}$, the q-character of V is the following element of the group ring $\mathbb{Z}[\mathcal{P}]$:

$$\mathrm{qch}(V) = \sum_{\boldsymbol{\varpi} \in \mathcal{P}} \dim(V_{\boldsymbol{\varpi}}) \boldsymbol{\varpi}.$$

A nonzero vector $v \in V_{\boldsymbol{\varpi}}$ is said to be a highest-ℓ-weight vector if

$$\eta v = \boldsymbol{\varpi}(\eta)v \quad \text{for every} \quad \eta \in U_q(\tilde{\mathfrak{h}}) \quad \text{and} \quad x_{i,r}^+ v = 0 \quad \text{for all} \quad i \in I, r \in \mathbb{Z}.$$

V is said to be a highest-ℓ-weight module if it is generated by a highest-ℓ-weight vector. Evidently, every highest-ℓ-weight module has a maximal proper submodule and, hence, a unique irreducible quotient. In particular, if two simple modules are highest-ℓ-weight, then they are isomorphic if and only if the highest ℓ-weights are the same. This is also equivalent to saying that they have the same q-character. The following was proved in [11].

Theorem 3.1.2 *Every simple object of $\widetilde{\mathcal{C}}$ is a highest-ℓ-weight module. There exists a simple object of $\widetilde{\mathcal{C}}$ of highest ℓ-weight $\boldsymbol{\pi}$ if and only if $\boldsymbol{\pi} \in \mathcal{P}^+$.*

It follows that $\mathrm{qch}(V)$ completely determines the irreducible factors of V. We shall denote by $L_q(\boldsymbol{\pi})$ any representative of the isomorphism class of simple modules with highest ℓ-weight $\boldsymbol{\pi}$. For $J \subseteq I$, we shall denote by $L_q(\boldsymbol{\pi}_J)$ the simple $U_q(\tilde{\mathfrak{g}})_J$-module of highest weight $\boldsymbol{\pi}_J$.

If V is a highest-ℓ-weight module with highest-ℓ-weight vector v and $J \subset I$, we let V_J denote the $U_q(\tilde{\mathfrak{g}})_J$-submodule of $L_q(\boldsymbol{\pi})$ generated by v. Evidently, if $\boldsymbol{\pi}$ is the highest-ℓ-weight of V, then V_J is highest-ℓ-weight with highest ℓ-weight $\boldsymbol{\pi}_J$. Moreover, we have the following well-known facts:

$$(3.1.1) \qquad V_J = \bigoplus_{\eta \in Q_J^+} V_{\mathrm{wt}(\boldsymbol{\pi})-\eta} = \bigoplus_{\eta \in Q_J} V_{\mathrm{wt}(\boldsymbol{\pi})+\eta}.$$

Lemma 3.1.3 *If V is simple, so is V_J.*

3.2 Tensor products and duality for $U_q(\tilde{\mathfrak{g}})$-modules

It is well known that $U_q(\tilde{\mathfrak{g}})$ is a Hopf algebra with invertible antipode. For the proof of following proposition, see [6, Propositions 1.5 and 1.6] (part (b) has not been proved there, but the proof is similar to that of part (c)).

Proposition 3.2.1 (a) *Given $a \in \mathbb{C}^\times$, there exists a unique Hopf algebra automorphism τ_a of $U_q(\tilde{\mathfrak{g}})$ such that*

$$\tau_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm, \quad \tau_a(h_{i,s}) = a^s h_{i,s}, \quad \tau_a(k_i^\pm) = k_i^\pm, \quad i \in I, \ r,s \in \mathbb{Z}, s \neq 0.$$

(b) *There exists a unique Hopf algebra automorphism σ of $U_q(\tilde{\mathfrak{g}})$ such that*

$$\sigma(x_{i,r}^\pm) = x_{i^*,r}^\pm, \quad \sigma(h_{i,s}) = h_{i^*,s}, \quad \sigma(k_i^\pm) = k_{i^*}^\pm, \quad i \in I, \ r,s \in \mathbb{Z}, s \neq 0.$$

(c) *There exists a unique algebra automorphism κ of $U_q(\tilde{\mathfrak{g}})$ such that*

$$\kappa(x_{i,r}^{\pm}) = -x_{i,-r}^{\mp}, \quad \kappa(h_{i,s}) = -h_{i,-s}, \quad \kappa(k_i^{\pm 1}) = k_i^{\mp 1}, \quad i \in I, \ r, s \in \mathbb{Z}, s \neq 0.$$

Moreover $(\kappa \otimes \kappa) \circ \Delta = \Delta^{op} \circ \kappa$, where Δ^{op} is the opposite comultiplication of $U_q(\tilde{\mathfrak{g}})$.[1]

Given $\pi \in \mathcal{P}^+$, define $\pi^{\tau_a} \in \mathcal{P}^+$ by $\pi_i^{\tau_a}(u) = \pi_i(au)$. One easily checks that the pullback $L_q(\omega)^{\tau_a}$ of $L_q(\pi)$ by τ_a satisfies

$$(3.2.1) \qquad\qquad\qquad L_q(\pi)^{\tau_a} \cong L_q(\pi^{\tau_a}).$$

Define also $\pi^\sigma, \pi^* \in \mathcal{P}^+$ by

$$(3.2.2) \qquad \pi_i^\sigma(u) = \pi_{i^*}(u) \quad \text{for} \quad i \in I, \quad \text{and} \quad \pi^* = (\pi^\sigma)^{\tau_{q^{-r^\vee h^\vee}}} = (\pi^{\tau_{q^{-r^\vee h^\vee}}})^\sigma.$$

It is well known that

$$(3.2.3) \qquad\qquad\qquad L_q(\pi)^* \cong L_q(\pi^*).$$

We denote by V^σ and V^κ the pull-back of V by σ and κ, respectively. In particular,

$$(3.2.4)$$
$$(V_1 \otimes V_2)^{\tau_a} \cong V_1^{\tau_a} \otimes V_2^{\tau_a}, \quad (V_1 \otimes V_2)^\sigma \cong V_1^\sigma \otimes V_2^\sigma, \quad \text{and} \quad (V_1 \otimes V_2)^\kappa \cong V_2^\kappa \otimes V_1^\kappa.$$

Also, for any short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0,$$

we have short exact sequences

$$(3.2.5) \qquad\qquad 0 \to V_1^f \to V_2^f \to V_3^f \to 0 \qquad \text{with} \qquad f = \tau_a, \sigma, \kappa.$$

Moreover, if $\pi \in \mathcal{P}^+$ with $\pi_i(u) = \prod_j(1 - a_{i,j}u)$, where $a_{i,j} \in \mathbb{F}$, and $\pi^- \in \mathcal{P}^+$ is defined by $\pi_i^-(u) = \prod_j(1 - a_{i,j}^{-1}u)$, we have

$$(3.2.6) \qquad L_q(\pi)^\sigma \cong L_q(\pi^\sigma) \quad \text{and} \quad L_q(\pi)^\kappa \cong L_q(\pi^\kappa), \quad \text{where} \quad \pi^\kappa = (\pi^-)^*.$$

It was proved in [17] that

$$(3.2.7) \qquad\qquad\qquad \text{qch}(V \otimes W) = \text{qch}(V)\text{qch}(W).$$

In particular, we have the following proposition.

Proposition 3.2.2 *Let $\pi, \omega \in \mathcal{P}^+$. Then, $L_q(\pi) \otimes L_q(\omega)$ is simple if and only if $L_q(\omega) \otimes L_q(\pi)$ is simple and, in that case, $L_q(\pi) \otimes L_q(\omega) \cong L_q(\pi\omega) \cong L_q(\omega) \otimes L_q(\pi)$.*

Given a connected subdiagram J, since $U_q(\tilde{\mathfrak{g}})_J$ is not a sub-coalgebra of $U_q(\tilde{\mathfrak{g}})$, if M and N are $U_q(\tilde{\mathfrak{g}})_J$-submodules of $U_q(\tilde{\mathfrak{g}})$-modules V and W, respectively, it is in general not true that $M \otimes N$ is a $U_q(\tilde{\mathfrak{g}})_J$-submodule of $V \otimes W$. Recalling that

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[1]The automorphism κ is most often denoted by $\hat{\omega}$ in the literature and its restriction to $U_q(\mathfrak{g})$, typically denoted by ω, is referred to as the Cartan automorphism of $U_q(\mathfrak{g})$. We chose to modify the notation to avoid visual confusion with our most often used symbol for a Drinfeld polynomial: ω.

we have an algebra isomorphism $U_q(\tilde{\mathfrak{g}})_J \cong U_{q_J}(\tilde{\mathfrak{g}}_J)$, we shall denote by $M \otimes_J N$ the $U_q(\tilde{\mathfrak{g}})_J$-module obtained by using the coalgebra structure from $U_{q_J}(\tilde{\mathfrak{g}}_J)$. The next result describes a special situation on which $M \otimes N$ is a submodule isomorphic to $M \otimes_J N$. Recall the notation defined in the paragraph preceding Lemma 3.1.3.

Proposition 3.2.3 ([13, Proposition 2.2]) *Let V and W be finite-dimensional highest-ℓ-weight modules with highest ℓ-weights $\boldsymbol{\pi}, \boldsymbol{\omega} \in \mathcal{P}^+$, respectively, and let $J \subseteq I$ be a connected subdiagram. Then, $V_J \otimes W_J$ is a $U_q(\tilde{\mathfrak{g}})_J$-submodule of $V \otimes W$ isomorphic to $V_J \otimes_J W_J$ via the identity map.*

Corollary 3.2.4 *In the notation of Proposition 3.2.3, if $V \otimes W$ is highest-ℓ-weight, so is $V_J \otimes W_J$. Moreover, if $V \otimes W$ is simple, so is $V_J \otimes W_J$.*

Proof As shown in the proof of Proposition 3.2.3, we have

$$(3.2.8) \qquad V_J \otimes W_J = \bigoplus_{\eta \in Q_J^+} (V \otimes W)_{\mathrm{wt}(\boldsymbol{\pi}) + \mathrm{wt}(\boldsymbol{\omega}) - \eta}.$$

Thus, if $V \otimes W$ is highest-ℓ-weight, any nonzero vector in $V_J \otimes W_J$ is a linear combination of vectors of the form $x_{i_1,r_1}^- \dots x_{i_l,r_l}^- (v \otimes w)$ for some $l \geq 0$, $i_k \in I$, $r_k \in \mathbb{Z}$, $1 \leq k \leq l$. But the weight of such vector is

$$\mathrm{wt}(\boldsymbol{\pi}) + \mathrm{wt}(\boldsymbol{\omega}) - \sum_{k=1}^{l} \alpha_{i_k}$$

and, hence, we must have $i_k \in J$ for all $1 \leq k \leq l$, which implies the first claim. The second claim follows from the first together with Lemma 3.1.3. ∎

3.3 Simple prime modules and *q*-factors

A finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V is said to be prime if it is not isomorphic to a tensor product of two nontrivial modules. Evidently, any finite-dimensional simple module can be written as a tensor product of (simple) prime modules. If a prime module P appears in some factorization of a simple module S, we shall say that P is a prime factor of S.

In particular, in light of (3.2.7), in order to understand the q-characters of the simple modules, it suffices to understand those of the simple prime modules. However, the only case, the classification of simple prime modules is completely understood is for $\mathfrak{g} = \mathfrak{sl}_2$. In that case, the classification is given by the following theorem, proved in [11].

Theorem 3.3.1 *If $\mathfrak{g} = \mathfrak{sl}_2$, $\boldsymbol{\pi} \in \mathcal{P}^+$, and the q-factors of $\boldsymbol{\pi}$ are $\boldsymbol{\pi}^{(j)}$, $1 \leq j \leq m$, then*

$$L_q(\boldsymbol{\pi}) \cong L_q(\boldsymbol{\pi}^{(1)}) \otimes \cdots \otimes L_q(\boldsymbol{\pi}^{(m)}).$$

Moreover, up to re-ordering, $L_q(\boldsymbol{\pi})$ has a unique factorization as tensor product of prime modules. In particular, $L_q(\boldsymbol{\pi})$ is prime if and only if it has a unique q-factor.

If $\boldsymbol{\pi} \in \mathcal{P}^+$ has a unique q-factor, the module $L_q(\boldsymbol{\pi})$ is called a KR module. It is well known (see [7, 32] and the references therein) that, given $(i, r), (j, s) \in I \times \mathbb{Z}_{>0}$, there exists a finite set $\mathscr{R}_{i,j}^{r,s} \subseteq \mathbb{Z}_{>0}$ such that

$$(3.3.1) \quad L_q(\boldsymbol{\omega}_{i,a,r}) \otimes L_q(\boldsymbol{\omega}_{j,b,s}) \text{ is reducible} \qquad \Leftrightarrow \qquad \frac{a}{b} = q^m \text{ with } |m| \in \mathscr{R}_{i,j}^{r,s}.$$

Moreover, in that case,

(3.3.2) $L_q(\boldsymbol{\omega}_{i,a,r}) \otimes L_q(\boldsymbol{\omega}_{j,b,s})$ is highest-ℓ-weight $\Leftrightarrow$ $m > 0$.

It follows from Proposition 3.2.2 and (2.6.3) that

(3.3.3) $\mathscr{R}^{s,r}_{j,i} = \mathscr{R}^{r,s}_{i,j} = \mathscr{R}^{r,s}_{i^*,j^*}.$

Theorem 3.3.2   *If $\mathfrak{g}$ is of type A and $i, j \in I, r, s \in \mathbb{Z}_{>0}$, we have*

$$\mathscr{R}^{r,s}_{i,j} = \{r + s + d(i, j) - 2p : -d([i, j], \partial I) \leq p < \min\{r, s\}\}.$$

The above was essentially proved in [7] and can be read off the results of [32], from where the description for other types can also be extracted (see also [23]).

Given a connected subdiagram J such that $[i, j] \subseteq J$, let $\mathscr{R}^{r,s}_{i,j,J}$ be determined by

$$L_q((\boldsymbol{\omega}_{i,a,r})_J) \otimes L_q((\boldsymbol{\omega}_{j,b,s})_J) \text{ is reducible} \qquad \Leftrightarrow \qquad \frac{a}{b} = q^m \text{ with } |m| \in \mathscr{R}^{r,s}_{i,j,J}.$$

Note this is not the same set obtained by considering the corresponding module for the algebra $U_{q_J}(\tilde{\mathfrak{g}}_J) \cong U_q(\tilde{\mathfrak{g}})_J$. Indeed, if we denote the latter by $\mathscr{R}^{r,s}_{i,j}[J]$, we have

$$m \in \mathscr{R}^{r,s}_{i,j}[J] \qquad \Leftrightarrow \qquad d_J m \in \mathscr{R}^{r,s}_{i,j,J}.$$

Note also that Corollary 3.2.4 implies

(3.3.4) $\mathscr{R}^{r,s}_{i,j,J} \subseteq \mathscr{R}^{r,s}_{i,j,K} \quad \text{if} \quad J \subseteq K.$

Finally, set

(3.3.5) $\mathscr{R}^{r,s}_{i} = \mathscr{R}^{r,s}_{i,i,\{i\}}.$

Corollary   3.3.3   *For   every $i \in I, r, s \in \mathbb{Z}_{>0}, \quad \mathscr{R}^{r,s}_{i} = \{d_i(r + s - 2p) : 0 \leq p < \min\{r, s\}\}.$*

Proposition 3.3.4   *If $\boldsymbol{\pi}, \boldsymbol{\varpi} \in \mathcal{P}^+$ are such that $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\varpi})$ is simple, then they have dissociate q-factorizations.*

Proof   If the q-factorizations are not dissociate, it follows from Theorem 3.3.1 that there exists $i \in I$ and q-factors $\boldsymbol{\omega}$ of $\boldsymbol{\pi}$ and $\boldsymbol{\omega}'$ of $\boldsymbol{\pi}'$, both supported at i, such that $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')$ is reducible. Moreover, writing $\boldsymbol{\pi} = \tilde{\boldsymbol{\pi}}\boldsymbol{\omega}$ and $\boldsymbol{\pi}' = \tilde{\boldsymbol{\pi}}'\boldsymbol{\omega}'$, it follows that

$$L_q(\boldsymbol{\pi}_i) \otimes L_q(\boldsymbol{\pi}'_i) \cong L_q(\tilde{\boldsymbol{\pi}}_i) \otimes L_q(\boldsymbol{\omega}_i) \otimes L_q(\boldsymbol{\omega}'_i) \otimes L_q(\tilde{\boldsymbol{\pi}}'_i)$$

which is reducible, yielding a contradiction with Corollary 3.2.4.   ∎

Corollary 3.3.5   *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. $L_q(\boldsymbol{\pi})$ is prime if and only if for every decomposition $\boldsymbol{\pi} = \boldsymbol{\omega}\boldsymbol{\varpi}, \boldsymbol{\omega}, \boldsymbol{\varpi} \in \mathcal{P}^+$, such that $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}$ have dissociate q-factorizations, $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi})$ is reducible.*

Proof   If $L_q(\boldsymbol{\pi})$ is not prime, by definition, there exists a nontrivial decomposition $\boldsymbol{\pi} = \boldsymbol{\omega}\boldsymbol{\varpi}$ such that $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi})$ is simple and Proposition 3.3.4 implies $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}$ have dissociate q-factorizations. If $L_q(\boldsymbol{\pi})$ is prime, by definition, $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi})$ is reducible for any nontrivial decomposition $\boldsymbol{\pi} = \boldsymbol{\omega}\boldsymbol{\varpi}$.   ∎

Given $\boldsymbol{\pi} \in \mathcal{P}^+$, consider a nontrivial 2-set partition of its set of q-factors and let $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}$ be the products of the q-factors in each of the parts. The above corollary tells us that the task of deciding the primality of $L_q(\boldsymbol{\pi})$ can be phrased as a task of testing the reducibility of $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi})$ for every such partition. Thus, one can think of organizing the level of complexity of the task by the number of q-factors of $\boldsymbol{\pi}$. The answer for the two first levels is given by:

Corollary 3.3.6 *Every KR module is prime. Moreover, if* $\boldsymbol{\pi} \in \mathcal{P}^+$ *has exactly two q-factors, say* $\boldsymbol{\omega}_{i,a,r}$ *and* $\boldsymbol{\omega}_{j,b,s}$*, then* $L_q(\boldsymbol{\pi})$ *is prime if and only if* $\frac{a}{b} = q^m$ *with* $|m| \in \mathcal{R}_{i,j}^{r,s}$.

3.4 Factorization graphs

Recall the definition of the sets $\mathcal{R}_{i,j}^{r,s}$ in (3.3.1), as well as (3.3.5), and (2.5.2). We shall say that a pre-factorization graph G is a q-factorization graph if, for every $i \in I$,

$$(3.4.1) \qquad v, v' \in \mathcal{V}_i, \ \rho \in \mathscr{P}_{v,v'} \qquad \Rightarrow \qquad |\varepsilon_\rho| \notin \mathcal{R}_i^{\lambda(v),\lambda(v')}$$

and

$$(3.4.2) \qquad \rho \in \mathscr{P}_{v,v'} \cap \mathscr{P}_G^+ \quad \text{with} \quad \varepsilon_\rho \in \mathcal{R}_{c(v),c(v')}^{\lambda(v),\lambda(v')} \qquad \Rightarrow \qquad (v',v) \in \mathcal{A}.$$

Condition (3.4.1) ensures that the factors in the right-hand side of (2.5.4) are the q-factors of $\boldsymbol{\pi}$. On the other hand, (3.4.2) guarantees that no pre-factorization graph can be obtained by adding an arrow to G. We will refer to a pre-factorization graph satisfying (3.4.2) as a pseudo q-factorization graph.

We shall now see that any pseudo q-factorization of a Drinfeld polynomial gives rise to a pseudo q-factorization graph which is a q-factorization graph if and only if it is the q-factorization. Thus, fix a Drinfeld polynomial $\boldsymbol{\pi}$, and let $\mathcal{V}$ be the corresponding multiset of pseudo q-factors. For $i \in I$, let

$$\mathcal{V}_i = \{\boldsymbol{\omega} \in \mathcal{V} : \text{supp}(\boldsymbol{\omega}) = \{i\}\}.$$

This gives rise to a coloring $c : \mathcal{V} \to I$ defined by declaring $\mathcal{V}_i = c^{-1}(\{i\})$. The weight map $\lambda : \mathcal{V} \to \mathbb{Z}_{>0}$ is defined by

$$(3.4.3) \qquad \lambda(\boldsymbol{\omega}) = \text{wt}(\boldsymbol{\omega})(h_i) \qquad \text{for all} \qquad \boldsymbol{\omega} \in \mathcal{V}_i.$$

In particular,

$$(3.4.4) \qquad \sum_{i \in I} \sum_{\boldsymbol{\omega} \in \mathcal{V}_i} \lambda(\boldsymbol{\omega})\omega_i = \text{wt}(\boldsymbol{\pi}).$$

The set of arrows $\mathcal{A} = \mathcal{A}(\boldsymbol{\pi})$ is defined as the set of ordered pairs of q-factors, say $(\boldsymbol{\omega}_{i,a,r}, \boldsymbol{\omega}_{j,b,s})$, such that

$$(3.4.5) \qquad a = bq^m \quad \text{for some} \quad m \in \mathcal{R}_{i,j}^{r,s}.$$

In representation theoretic terms, this is equivalent to saying:

$$L_q(\boldsymbol{\omega}_{i,a,r}) \otimes L_q(\boldsymbol{\omega}_{j,b,s}) \quad \text{is reducible and highest-}\ell\text{-weight.}$$

Note that, in the case of the actual q-factorization, we necessarily have $m \notin \mathcal{R}_i^{r,s}$ when $i = j$. The value of the exponent $\varepsilon : \mathcal{A} \to \mathbb{Z}_{>0}$ at an arrow satisfying (3.4.5) is set to be m.

Quite clearly, $G = (\mathcal{V}, \mathcal{A})$ with the above choice of coloring, weight, and exponent is a pseudo q-factorization graph and $\pi_G = \pi$. We refer to G as a pseudo q-factorization graph over π. In the case this construction was performed using the q-factorization of π, then G will be called the q-factorization graph of π and we denote it by $G(\pi)$.

It is now natural to seek for the classification of the prime q-factorization graphs, i.e., those for which $L_q(\pi_G)$ is prime. In type A_1, this is the case if and only if the q-factorization graph of π has a single vertex, which is also equivalent to saying that the graph is connected. For higher rank, the story is much more complicated. We still have the following proposition which will be proved in Section 4.1.

Proposition 3.4.1 *Let $\pi \in \mathcal{P}^+$. If $G_1, \ldots, G_k$ are the connected components of $G(\pi)$ and $\pi^{(j)} \in \mathcal{P}^+, 1 \le j \le k$, are such that $\pi = \prod_{j=1}^{k} \pi^{(j)}$ and $G_j = G(\pi^{(j)})$, then*

$$L_q(\pi) \cong L_q(\pi^{(1)}) \otimes \cdots \otimes L_q(\pi^{(k)}).$$

In particular, $G(\pi)$ is connected if $L_q(\pi)$ is prime.

However, even for type A_2, the converse is not true and counter examples can be found in [27], for instance.

We also introduce duality notions for pre-factorization graphs. Given a graph G, we denote by G^- the graph obtained from G by reversing all the arrows[2] and keeping the rest of structure of (pre)-factorization graph. In light of (3.3.3), G^- is a factorization graph as well, which we refer to as the arrow-dual of G. Similarly, the graph G^*, called the color-dual of G, obtained by changing the coloring according to the rule $i \mapsto i^*$ for all $i \in I$, is a factorization graph. Moreover,

$$(3.4.6) \qquad \pi_{G^-, v, a^{-1}} = \pi_{G, v, a}^- \qquad \text{and} \qquad \pi_{G^*, v, aq^{-r^\vee h^\vee}} = \pi_{G, v, a}^*.$$

Given $\pi, \pi' \in \mathcal{P}^+$, the graph $G(\pi\pi')$ may have no relation to $G(\pi)$ and $G(\pi')$. However, if π and π' have dissociate q-factorizations, then $G(\pi)$ and $G(\pi')$ determine a cut of $G(\pi\pi')$. We will consider several times the situation that the corresponding cut-set is a singleton. More generally, given pseudo q-factorization graphs G and G', let $G \otimes G'$ be the unique pseudo q-factorization graph whose set of vertices is $\mathcal{V} \sqcup \mathcal{V}'$ preserving the original coloring and weight map. The following is trivially established.

Lemma 3.4.2 *If G and G' are pseudo q-factorization graphs over π and π', respectively, then $G \otimes G'$ is a pseudo q-factorization graph over $\pi\pi'$. Moreover, if $G = G(\pi)$ and $G' = G(\pi')$, then $G \otimes G' = G(\pi\pi')$ if and only if π and π' have dissociate q-factorizations.*

If G and G' are pseudo q-factorization graphs over π and π', respectively, we shall say that the pair (G, G') or, equivalently, that $G \otimes G'$ is simple if so is $L_q(\pi) \otimes L_q(\pi')$. Otherwise we say it is reducible. Evidently, $G \otimes G' = G' \otimes G$, regardless if $L_q(\pi) \otimes L_q(\pi')$ is simple or not.

3.5 Main conjectures and results

Throughout this section, we let $G = G(\pi) = (\mathcal{V}, \mathcal{A})$ be a q-factorization graph. We say G is prime if $L_q(\pi)$ is prime. One could hope that the complexity of prime graphs

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[2]The underlying directed graph is usually called the transpose of G.

is incremental in the sense that all prime graphs with $N + 1$ vertices are obtained by adding a vertex in some particular ways to some prime graph with N vertices. In other words:

Conjecture 3.5.1 *If G is prime and $\#\mathcal{V} > 1$, there exists $v \in \mathcal{V}$ such that $G_{\mathcal{V}\backslash\{v\}}$ is also prime.*

Remark 3.5.2 In Section 4.2, we give a proof that the conclusion of Conjecture 3.5.1 holds for every $v \in \partial G$ using the main result of [19] as stated there (see Remark 4.1.7 for more precise comments) by exploring the role of sinks and sources. In particular, this would prove the conjecture holds if G is a tree since, in that case, $\partial G \neq \varnothing$. Together with general combinatorial properties of trees (Lemma 2.2.2(c)), we would then have the following corollary.

Corollary 3.5.3 *If G is a prime tree, every of its proper connected subgraphs are prime.*

Trees are the simplest kinds of directed graphs and, among them, totally ordered lines are the simplest. The following is the first of our main results.

Theorem 3.5.4 *If G is a totally ordered line, then G is prime.*

For $\mathfrak{g}$ of type A_2, this was the main result of [14]. It will be proved here for general $\mathfrak{g}$, by a different argument, as a corollary of Proposition 4.3.1. In particular, differently than the proof in [14], our proof does not use any information about the elements belonging to the sets $\mathscr{R}_{i,j}^{r,s}$. For $\mathfrak{g}$ of type A, we also prove the following generalization, which is the main result of the present paper.

Theorem 3.5.5 *If $\mathfrak{g}$ is of type A, every totally ordered q-factorization graph is prime.*

In particular, a q-factorization graph afforded by a tournament is prime. After Theorem 3.5.5, it becomes natural the purely combinatorial problem of classifying all q-factorization graphs which are totally ordered since this leads to the explicit construction of a family of Drinfeld polynomials whose corresponding simple modules are prime. We shall not pursue a general answer for this combinatorial problem here. However, for illustrative purposes, we do present two results in this direction. One is Example 3.6.1, where we describe an infinite family of examples of q-factorization graphs afforded by tournaments. The other is the following proposition whose proof, given in Section 5.2, is essentially a byproduct of some technical lemmas extracted from the proof of Theorem 3.5.5 in Section 5.1. In particular, it solves this combinatorial problem for type A_2 since, in that case, $I = \partial I$.

Proposition 3.5.6 *Suppose $\mathfrak{g}$ is of type A and $\pi \in \mathcal{P}^+$ is such that $G = G(\pi)$ is totally ordered. If $c(\mathcal{V}_G) \subseteq \partial I$, then G is a line whose vertices are alternately colored.*

We have the following rephrasing of Corollary 3.3.5 in the language of cuts: G is prime if and only if every nontrivial cut of G is reducible. Several partial results in the direction of proving our main results provide criteria for checking the reducibility of certain special cuts, which we deem to be interesting results in their own right. One of these, which is an immediate consequence of Corollary 4.3.3, is stated here in the language of cuts.

Theorem 3.5.7 *Let (G', G'') be a cut of G, and suppose there exist vertices v' of G' and v'' of G'' satisfying the following conditions:*

(i) *v' and v'' are adjacent in G.*
(ii) *v' and v'' are extremal in G' and G'', respectively.*
(iii) *v' is extremal in G only if v' is an isolated vertex of G' and similarly for v''.*

Then, (G', G'') is reducible.

The following corollary about triangles is immediate.

Corollary 3.5.8 *Suppose G is a triangle, and let (G', G'') be a cut such that G' is a singleton containing an extremal vertex of G. Then, (G', G'') is reducible.*

This corollary implies there is only one cut for a triangle which may be simple: the one whose singleton contains the vertex which is not extremal. Theorem 3.5.5 implies this is not so for type A. However, the present proof utilizes the precise description of the sets $\mathscr{R}_{i,j}^{r,s}$ and, hence, in order to extend it to other types, it requires a case by case analysis, which will appear elsewhere.

We also have the following general criterion for primality.

Proposition 3.5.9 *G is prime if, for any cut (G', G'') of G, there exist $\boldsymbol{\varpi}' \in \mathcal{V}_{G'}$, $\boldsymbol{\varpi}'' \in \mathcal{V}_{G''}$ such that one of the following two conditions holds:*

(i) *$(\boldsymbol{\varpi}'', \boldsymbol{\varpi}') \in \mathcal{A}_G$ and $L_q(\boldsymbol{\omega}') \otimes L_q(\boldsymbol{\omega}'')^*$ is simple for all $(\boldsymbol{\omega}', \boldsymbol{\omega}'') \in \mathcal{N}_{G'}^+(\boldsymbol{\varpi}') \times \mathcal{N}_{G''}^-(\boldsymbol{\varpi}'')\setminus\{(\boldsymbol{\varpi}', \boldsymbol{\varpi}'')\}$.*
(ii) *$(\boldsymbol{\varpi}', \boldsymbol{\varpi}'') \in \mathcal{A}_G$ and $L_q(\boldsymbol{\omega}')^* \otimes L_q(\boldsymbol{\omega}'')$ is simple for all $(\boldsymbol{\omega}', \boldsymbol{\omega}'') \in \mathcal{N}_{G'}^-(\boldsymbol{\varpi}') \times \mathcal{N}_{G''}^+(\boldsymbol{\varpi}'')\setminus\{(\boldsymbol{\varpi}', \boldsymbol{\varpi}'')\}$.*

Proposition 3.5.9 will be proved as a corollary to Proposition 4.5.3.

3.6 Examples

The first example provides a family of tournaments for type A and, hence, a family of simple prime modules.

Example 3.6.1 Given $N > 1$, let $\mathfrak{g}$ be of type A_n, $n \geq 3N - 4$, identify I with the integer interval $[1, n]$ as usual, and consider

$$\boldsymbol{\pi} = \prod_{i=1}^{N} \boldsymbol{\omega}_{i+N-2,q^{3(i-1)}}.$$

Checking that $G(\boldsymbol{\pi})$ is a tournament with N vertices amounts to showing that

$$3(j - i) \in \mathscr{R}_{i+N-2,j+N-2}^{1,1} \quad \text{for all} \quad 1 \leq i < j \leq N,$$

which, by Theorem 3.3.2, is equivalent to

$$3(j - i) = 2 + d(i + N - 2, j + N - 2) - 2p \quad \text{with} \quad -d([i + N - 2, j + N - 2], \partial I) \leq p \leq 0.$$

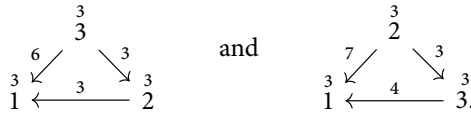Since $d(i + N - 2, j + N - 2) = j - i$ and $3(j - i) = 2 + (j - i) - 2(1 - (j - i))$, we need to check

$$-d([i + N - 2, j + N - 2], \partial I) \leq 1 - (j - i) \leq 0.$$

The second inequality is immediate from $1 \le i < j \le N$. On the other hand,

$$d\big([i+N-2, j+N-2], \partial I\big) = \min\{(i+N-2) - 1, n - (j+N-2)\} \ge N-2,$$

from where the first inequality easily follows.

Although the above family affords triangles only for rank at least 5, it is easy to build q-factorization graphs which are triangles for $n \ge 3$. For instance,

Example 3.6.2 We now examine prime snake modules for type A from the perspective of q-factorization graphs. The notion of snake and snake modules was introduced in [28] while that of prime snakes was introduced in [29] and revised in [15]. We now rephrase these definitions in terms of the sets $\mathscr{R}^{r,s}_{i,j}$. A (type A) snake of length k is a sequence $(i_j, m_j) \in \mathcal{I} \times \mathbb{Z}, 1 \le j \le k$, such that

$$m_{j+1} - m_j = 2 + d(i_j, i_{j+1}) - 2p_j \quad \text{for some} \quad p_j \in \mathbb{Z}_{\le 0} \quad \text{and all} \quad 1 \le j < k.$$

The snake is said to be prime if $-d\big([i_j, i_{j+1}], \partial I\big) \le p_j$ for all $1 \le j < k$. In other words, the snake is prime if and only if

$$m_{j+1} - m_j \in \mathscr{R}^{1,1}_{i_j, i_{j+1}} \quad \text{for all} \quad 1 \le j < k.$$

Given a snake and $a \in \mathbb{F}^\times$, the associated snake module is $L_q(\boldsymbol{\pi})$ with

$$(3.6.1) \qquad \boldsymbol{\pi} = \prod_{j=1}^{k} \boldsymbol{\omega}_{i_j, aq^{m_j}}.$$

For a general snake, $G(\boldsymbol{\pi})$ may be disconnected and, hence, not prime. However, if the snake is prime and we regard the definition of $\boldsymbol{\pi}$ as a pseudo q-factorization, the associated pseudo q-factorization graph is totally ordered. It is then easy to see that $G(\boldsymbol{\pi})$, the actual q-factorization graph, is also totally ordered and, hence, prime by Theorem 3.5.5. Thus, Theorem 3.5.5, together with Proposition 3.4.1, recovers [29, Proposition 3.1].

The following is the q-factorization graph arising from the following prime snake for type A_5: $(4, -2), (3, 1), (2, 4), (3, 7)$.
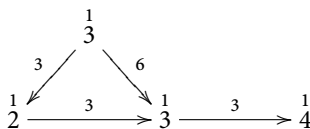
Example 3.6.3 Snake modules also arise in the study of the so-called skew representations associated with skew tableaux $\lambda \backslash \mu$ [22, Section 4], which we now review. Fix $m \in \mathbb{Z}_{\geq 0}$, as well as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+n+1}) \in \mathbb{Z}^{m+n+1}$ and $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m$ such that

$$\lambda_i \geq \lambda_{i+1}, \quad \mu_l \geq \mu_{l+1}, \quad \text{and} \quad \lambda_k \geq \mu_k \geq \lambda_{k+n+1}$$

for all $1 \leq i \leq m + n, 1 \leq l < m, 1 \leq k \leq m$. Set $\mu_0 = +\infty, \mu_{m+1} = -\infty$, and, for $1 \leq i \leq n + 1$ and $1 \leq l \leq m + 1$, let $v_{i,l}$ be the middle value among μ_{l-1}, μ_l, and λ_{i+l-1}. The skew module of $U_q(\hat{\mathfrak{g}})$ associated with the skew tableaux $\lambda \backslash \mu$ is the simple module $L_q(\pi^{\lambda,\mu})$, where

$$(3.6.2) \qquad \pi_i^{\lambda,\mu}(u) = \prod_{l=1}^{m+1} \omega_{i, q^{v_{i,l} + v_{i+1,l} - 2l+1-i}, v_{i,l} - v_{i+1,l}}.$$

For $\mu = \varnothing$, this is the Drinfeld polynomial of an evaluation module. With a little patience, one can check each factor of the above definition is a q-factor of $\pi^{\lambda,\mu}$. Moreover, each connected component of $G(\pi^{\lambda,\mu})$ is totally ordered and, hence, corresponds to a prime simple module by Theorem 3.5.5.

In order to explore specific examples, let us organize the table:

Values of $v_{i,l}$				
$l\backslash i$	1	2	$\cdots$	$n+1$
1	$v_{1,1}$	$v_{2,1}$	$\cdots$	$v_{n+1,1}$
$\vdots$				$\vdots$
$m+1$	$v_{1,m+1}$	$\cdots$	$\cdots$	$v_{n+1,m+1}$

Plugging this information in (3.6.2), each row will produce at most one q-factor for each $i \in I$. Moreover, the centers of the associated q-strings are of the form q^k for some exponent $k \in \mathbb{Z}$. We can then form a table with the corresponding exponents and lengths.

Exponents and lengths				
$l\backslash i$	1	2	$\cdots$	n
1	$k_{1,1} \mid r_{1,1}$	$k_{2,1} \mid r_{2,1}$	$\cdots$	$k_{n,1} \mid r_{n,1}$
$\vdots$				$\vdots$
$m+1$	$k_{1,m+1} \mid r_{1,m+11}$	$\cdots$	$\cdots$	$k_{n,m+1} \mid r_{n,m+11}$

For instance, if $\lambda = (20, 16, 10, 7, 2, 0)$ and $\mu = (17, 5)$, so $m = 2$ and $n = 3$, we have

	Values of $v_{i,l}$					Exponents and Lengths		
$l\backslash i$	1	2	3	4	$l\backslash i$	1	2	3
1	20	17	17	17	1	35 $\mid$ 3	31 $\mid$ 0	30 $\mid$ 0
2	16	10	7	5	2	22 $\mid$ 6	12 $\mid$ 3	6 $\mid$ 2
3	5	5	2	0	3	4 $\mid$ 0	0 $\mid$ 3	-6 $\mid$ 2

Organizing the vertices of $G(\pi^{\lambda,\mu})$ following the rows of the last table, we get
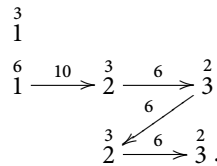
Thus, this example leads to a graph with two connected components: a singleton and an oriented line. If $\lambda = (6, 6, 6, 4, 2, 1, 1)$ and $\mu = (5)$, so $m = 1$ and $n = 5$, then

	Values of $v_{i,l}$							Exponents and Lengths				
$l\backslash i$	1	2	3	4	5	6	$l\backslash i$	1	2	3	4	5
1	6	6	6	5	5	5	1	10 $\mid$ 0	9 $\mid$ 0	7 $\mid$ 1	5 $\mid$ 0	4 $\mid$ 0
2	5	5	4	3	1	1	2	6 $\mid$ 0	4 $\mid$ 1	0 $\mid$ 2	-3 $\mid$ 2	-7 $\mid$ 0

and $G(\pi^{\lambda,\mu})$ is connected:
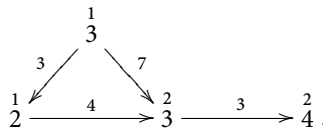
Although the underlying directed graph is the same as the one in Example 3.6.2, $L_q(\pi^{\lambda,\mu})$ is not a snake module. Indeed, $\pi^{\lambda,\mu}$ can be constructed as in (3.6.1) by using the sequence:

$$(4, -4), (4, -2), (3, -1), (3, 1), (2, 4), (3, 7).$$

However, this is not a snake because $(i_2, m_2) = (4, -2)$, $(i_3, m_3) = (3, -1)$, and $m_3 - m_2 = 1 \notin \mathscr{R}_{4,3}^{1,1}$. One can easily check no reordering of this sequence is a snake.

4 Highest-ℓ-weight criteria

In this section, we prove several criteria for deciding whether a tensor product of simple module is highest-ℓ-weight or not. In particular, the results proved here can be regarded as the backbone of the arguments in the proof of Theorem 3.5.5. Moreover, they will also be prominently used to prove the main results of [27].

4.1 Background on highest-ℓ-weight tensor products

The following is easily established.

Lemma 4.1.1 *Let $m \in \mathbb{Z}_{>0}$ and $V_k \in \widetilde{\mathbb{C}}, 1 \le k \le m$. Then, $V_1 \otimes \cdots \otimes V_m$ is highest-ℓ-weight (resp. simple) only if V_k is highest-ℓ-weight (resp. simple) for all $1 \le k \le m$.*

Lemma 4.1.2 *Let $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$, $V = L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$, and $W = L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\pi})$.*

(a) *V contains a submodule isomorphic to $L_q(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathcal{P}^+$, if and only there exists an epimorphism $W \to L_q(\boldsymbol{\omega})$.*
(b) *If W is highest-ℓ-weight, the submodule of V generated by its top weight space is simple.*
(c) *If V is not highest-ℓ-weight, there exists an epimorphism $V \to L_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}^+$ such that $\boldsymbol{\omega} < \boldsymbol{\pi}\boldsymbol{\pi}'$.*

Proof Assume we have a monomorphism $L_q(\boldsymbol{\omega}) \to V$. It follows from (2.6.4) and (2.6.3) that we have an epimorphism

$$L_q(\boldsymbol{\pi}')^* \otimes L_q(\boldsymbol{\pi})^* \to L_q(\boldsymbol{\omega})^*.$$

In particular, letting $\psi = \sigma \circ \tau_{q^{r^\vee h^\vee}}$ and using (3.2.4) and (3.2.5), we get an epimorphism

$$(L_q(\boldsymbol{\pi}')^*)^\psi \otimes (L_q(\boldsymbol{\pi})^*)^\psi \to (L_q(\boldsymbol{\omega})^*)^\psi.$$

One easily checks using (3.2.1), (3.2.3), and (3.2.6) that the domain of the latter epimorphism is isomorphic to W and $(L_q(\boldsymbol{\omega})^*)^\psi \cong L_q(\boldsymbol{\omega})$. The converse in part (a) is proved by reversing this argument. Part (b) is immediate from (a) since the assumption on W implies we have an epimorphism $W \to L_q(\boldsymbol{\pi}\boldsymbol{\pi}')$ and the top weight space of V is one-dimensional and equal to $V_{\boldsymbol{\pi}\boldsymbol{\pi}'}$.

For proving (c), let V' be the submodule of V generated by its top weight space. The assumption is equivalent to saying that V' is a proper submodule. Since V is finite-dimensional, the set of proper submodules of V containing V' is nonempty and contains a maximal element, say U, which is necessarily also maximal in the set of all proper submodules of V. Hence, $V/U \cong L_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}^+$ and, since $V_{\boldsymbol{\pi}\boldsymbol{\pi}'} \subseteq U$, we have $\boldsymbol{\omega} \ne \boldsymbol{\pi}\boldsymbol{\pi}'$. Since $\boldsymbol{\omega}$ is an ℓ-weight of V and every ℓ-weight of V is smaller than $\boldsymbol{\pi}\boldsymbol{\pi}'$, we have $\boldsymbol{\omega} < \boldsymbol{\pi}\boldsymbol{\pi}'$.   ∎

The following fact is well known (a proof can be found in [26]).

Proposition 4.1.3 *Let V be finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module. Then, V is simple if and only if V and V^* are highest-ℓ-weight.*

Corollary 4.1.4 *Let $\boldsymbol{\pi}, \boldsymbol{\omega} \in \mathcal{P}^+$. Then, $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\omega})$ is simple if and only if both $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\omega})$ and $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\pi})$ are highest-ℓ-weight.*

Proof If $U := L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\omega})$ is simple, Proposition 3.2.2 implies $U \cong W := L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\pi})$. In particular, U and W are both highest-ℓ-weight. Conversely, assume U and W are both highest-ℓ-weight. Since U is highest-ℓ-weight, Proposition 4.1.3, (2.6.3), and (3.2.3), imply that it suffices to show

(4.1.1) $L_q(\boldsymbol{\omega}^*) \otimes L_q(\boldsymbol{\pi}^*)$ is highest-ℓ-weight.

Using (3.2.5) with $V_2 = W$, it follows from (3.2.4) and (3.2.6) that $W^\sigma \cong L_q(\boldsymbol{\varpi}^\sigma) \otimes L_q(\boldsymbol{\pi}^\sigma)$ is highest-ℓ-weight. Setting $a = q^{-r^\vee h^\vee}$, (3.2.6) implies $\boldsymbol{\pi}^* = (\boldsymbol{\pi}^\sigma)^{\tau_a}$ and similarly for $\boldsymbol{\varpi}$. The proof of (4.1.1) is then completed by using (3.2.4), (3.2.1) and (3.2.5) with $V_2 = W^\sigma$ and $f = \tau_a$. ∎

We now state one of the main tools we shall use in the proofs of our main results.

Theorem 4.1.5 *Let $S_1, \ldots, S_m$ be simple $U_q(\tilde{\mathfrak{g}})$-modules. If $S_i \otimes S_j$ is highest-ℓ-weight, for all $1 \le i \le j \le m$, then $S_1 \otimes \cdots \otimes S_m$ is highest-ℓ-weight. Conversely, if $S_1 \otimes \cdots \otimes S_m$ is highest-ℓ-weight, then $S_i \otimes S_j$ is highest-ℓ-weight for all $1 \le i < j \le m$.*

Proof The first claim, which is the part we need below, is the main result of [19] (see also Remark 4.1.7). We now prove the second by induction on m. Thus, suppose $S_1 \otimes \cdots \otimes S_m$ is highest-ℓ-weight and note there is nothing to prove if $m \le 2$. Assume $m > 2$ and note Lemma 4.1.1 implies $S_i \otimes \cdots \otimes S_j$ is highest-ℓ-weight for all $1 \le i \le j \le m$. Together with the induction hypothesis, this implies $S_i \otimes S_j$ is highest-ℓ-weight for all $1 \le i < j \le m$ except if $(i, j) = (1, m)$.

To prove that $S_1 \otimes S_m$ is also highest-ℓ-weight, thus completing the proof, let S be a simple quotient of $T := S_2 \otimes \cdots \otimes S_{m-1}$ and consider the associated epimorphism $\pi : T \to S$. This implies we have an epimorphism

$$ S_1 \otimes T \otimes S_m \xrightarrow{\mathrm{id}_{S_1} \otimes \pi \otimes \mathrm{id}_{S_m}} S_1 \otimes S \otimes S_m, $$

which, together with the assumption that $S_1 \otimes \cdots \otimes S_m$ is highest-ℓ-weight, implies $S_1 \otimes S \otimes S_m$ is highest-ℓ-weight as well. The inductive argument is completed if $m > 3$.

If $m = 3$, let $\lambda_i \in P^+$ be the highest weight of $S_i, 1 \le i \le 3$. Since $S_1 \otimes S_2$ is highest-ℓ-weight, Lemma 4.1.2(b) implies the submodule M generated by the top weight space of $S_2 \otimes S_1$ is simple and we have a monomorphism

$$ M \otimes S_3 \to S_2 \otimes S_1 \otimes S_3. $$

If $S_1 \otimes S_3$ were not highest-ℓ-weight, an application of Lemma 4.1.2(c) would gives us an epimorphism

$$ S_1 \otimes S_3 \to N, $$

where N is a simple module whose highest weight λ satisfies $\lambda < \lambda_1 + \lambda_3$. Lemma 2.6.2 says these maps can be used to obtain a nonzero map

$$ M \otimes S_3 \to S_2 \otimes N. $$

The highest weight of $M \otimes S_3$ is $\lambda_1 + \lambda_2 + \lambda_3$ while that of $S_2 \otimes N$ is $\lambda_2 + \lambda$. To reach a contradiction, it then suffices to show $M \otimes S_3$ is highest-ℓ-weight.

Indeed, we have an epimorphism $S_1 \otimes S_2 \to M$ and, hence, an epimorphism

$$ S_1 \otimes S_2 \otimes S_3 \to M \otimes S_3. $$

The assumption that $S_1 \otimes S_2 \otimes S_3$ is highest-ℓ-weight then implies that so is $M \otimes S_3$, as desired. ∎

Corollary 4.1.6 *Given $\pi, \widetilde{\pi} \in \mathcal{P}^+$, $L_q(\pi) \otimes L_q(\widetilde{\pi})$ is highest-ℓ-weight if there exist $\pi^{(k)} \in \mathcal{P}^+, 1 \le k \le m$, $\widetilde{\pi}^{(k)} \in \mathcal{P}^+, 1 \le k \le \tilde{m}$, such that*

$$\pi = \prod_{k=1}^{m} \pi^{(k)}, \quad \widetilde{\pi} = \prod_{k=1}^{\tilde{m}} \widetilde{\pi}^{(k)},$$

and the following tensor products are highest-ℓ-weight:

$$L_q(\pi^{(k)}) \otimes L_q(\pi^{(l)}), \quad L_q(\widetilde{\pi}^{(k)}) \otimes L_q(\widetilde{\pi}^{(l)}), \quad for\ k \le l,$$
$$and \quad L_q(\pi^{(k)}) \otimes L_q(\widetilde{\pi}^{(l)}) \quad for\ all\ k, l.$$

Moreover, if all these tensor products are irreducible, then so is $L_q(\pi) \otimes L_q(\widetilde{\pi})$.

Proof It follows from Theorem 4.1.5 that

$$W := L_q(\pi^{(1)}) \otimes \cdots \otimes L_q(\pi^{(m)}), \quad \tilde{W} := L_q(\widetilde{\pi}^{(1)}) \otimes \cdots \otimes L_q(\widetilde{\pi}^{(\tilde{m})}), \quad and \quad W \otimes \tilde{W}$$

are highest-ℓ-weight. Therefore, we have epimorphisms $p : W \to L_q(\pi)$, $\tilde{p} : \tilde{W} \to L_q(\widetilde{\pi})$, and, hence, $p \otimes \tilde{p} : W \otimes \tilde{W} \to L_q(\pi) \otimes L_q(\widetilde{\pi})$. For the last claim, the extra assumption implies we reach the same conclusion with the tensor products above in reversed order. Hence, we are done by Corollary 4.1.4. ∎

Remark 4.1.7 The first claim in Theorem 4.1.5 was stated with $i < j$ in [19] instead of $i \le j$, as it is stated above. In other words, the version stated above includes the assumption that all tensor factors are real modules. This version for real factors also follows from the results of [24] (see also [18]). In every instance, we use Corollary 4.1.6, we use it with the factors being of KR type which are well known to be real modules. Hence, in such situations, the $k \le l$ in the statement can be replaced by $k < l$. The strong version of Theorem 4.1.5 without the assumption the factors are real is used only in Section 4.2, as explained in Remark 3.5.2. The results of Section 4.2 are not used anywhere else in the paper and are presented here since we believe they are interesting in their on right.

We are now able to prove Proposition 3.4.1.

Proof of Proposition 3.4.1 Let G' and G'' be nonempty unions of distinct connected components of $G(\pi)$. Let also $\pi', \pi'' \in \mathcal{P}^+$ be such that $G' = G(\pi')$ and $G'' = G(\pi'')$. If ω' is a vertex of G' and ω'' is a vertex of G'', they belong to different connected components of G and, hence, $L_q(\omega') \otimes L_q(\omega'')$ is simple. It then follows from Corollary 4.1.6 that $L_q(\pi') \otimes L_q(\pi'')$ is simple. An obvious inductive argument proves Proposition 3.4.1. ∎

4.2 On the removal of boundary vertices

Lemma 4.2.1 *Let $\pi \in \mathcal{P}^+$ and suppose ω is an extremal vertex in $G(\pi)$. Let $\bar{\omega} = \pi \omega^{-1}$ and assume there exists a nontrivial factorization $\bar{\omega} = \bar{\omega}^{(1)} \bar{\omega}^{(2)}$ such that*

$$L_q(\bar{\omega}) \cong L_q(\bar{\omega}^{(1)}) \otimes L_q(\bar{\omega}^{(2)})$$

and every q-factor of π adjacent to ω in $G(\pi)$ lies in $G(\bar{\omega}^{(1)})$. Then,

$$L_q(\pi) \cong L_q(\omega \bar{\omega}^{(1)}) \otimes L_q(\bar{\omega}^{(2)}).$$

Proof Up to arrow dualization, we can assume $\boldsymbol{\omega}$ is a source and, hence,

$$L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}') \quad \text{is highest-}\ell\text{-weight for every other } q\text{-factor } \boldsymbol{\omega}' \text{ of } \boldsymbol{\pi}.$$

By Corollary 4.1.4, it suffices to prove that both

$$L_q(\boldsymbol{\omega}\boldsymbol{\varpi}^{(1)}) \otimes L_q(\boldsymbol{\varpi}^{(2)}) \qquad \text{and} \qquad L_q(\boldsymbol{\varpi}^{(2)}) \otimes L_q(\boldsymbol{\omega}\boldsymbol{\varpi}^{(1)})$$

are highest-ℓ-weight. Consider the tensor product

$$W_1 = L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi}^{(1)}) \otimes L_q(\boldsymbol{\varpi}^{(2)}).$$

We claim that W_1 is highest-ℓ-weight. Indeed, we are assuming that $L_q(\boldsymbol{\varpi}^{(1)}) \otimes L_q(\boldsymbol{\varpi}^{(2)})$ is simple and the hypothesis on $G(\boldsymbol{\varpi}^{(1)})$ together with Corollary 4.1.6 implies $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi}^{(2)})$ is also simple. Also, since $\boldsymbol{\omega}$ is a source, Corollary 4.1.6 implies $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi}^{(1)})$ is highest-ℓ-weight. Hence, the strong version of Theorem 4.1.5 implies W_1 is highest-ℓ-weight, as well as its quotient $L_q(\boldsymbol{\omega}\boldsymbol{\varpi}^{(1)}) \otimes L_q(\boldsymbol{\varpi}^{(2)})$. These facts also imply $L_q(\boldsymbol{\varpi}^{(2)}) \otimes L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\varpi}^{(1)})$ is highest-ℓ-weight, showing that $L_q(\boldsymbol{\varpi}^{(2)}) \otimes L_q(\boldsymbol{\omega}\boldsymbol{\varpi}^{(1)})$ is highest-ℓ-weight. ∎

Lemma 4.2.2 *Suppose $L_q(\boldsymbol{\pi})$ is prime and that $\boldsymbol{\omega}$ is an extremal vertex in $G(\boldsymbol{\pi})$. Let $\boldsymbol{\varpi} = \boldsymbol{\pi}\boldsymbol{\omega}^{-1}$. Then, either $L_q(\boldsymbol{\varpi})$ is prime or there exists a nontrivial factorization $\boldsymbol{\varpi} = \boldsymbol{\varpi}^{(1)}\boldsymbol{\varpi}^{(2)}$ such that*

$$L_q(\boldsymbol{\varpi}) \cong L_q(\boldsymbol{\varpi}^{(1)}) \otimes L_q(\boldsymbol{\varpi}^{(2)})$$

and both $\boldsymbol{\varpi}^{(1)}$ and $\boldsymbol{\varpi}^{(2)}$ contain q-factors of $\boldsymbol{\pi}$ adjacent to $\boldsymbol{\omega}$ in $G(\boldsymbol{\pi})$.

Proof Immediate from Lemma 4.2.1. ∎

We can now give the proof mentioned in Remark 3.5.2. Write $G = G(\boldsymbol{\pi})$, let $\boldsymbol{\omega} = v$, and $\boldsymbol{\varpi} = \boldsymbol{\pi}\boldsymbol{\omega}^{-1}$. In particular, $G_{\mathcal{V}\setminus\{v\}} = G(\boldsymbol{\varpi})$. Since G is connected by Proposition 3.4.1, v must be monovalent and, hence, there exists a unique q-factor $\boldsymbol{\omega}'$ of $\boldsymbol{\varpi}$ such that $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')$ is reducible. In particular, $\boldsymbol{\omega}$ is extremal in G. The claim then follows immediately from Lemma 4.2.2.

4.3 A key highest-ℓ-weight criterion and Theorem 3.5.4

We now establish a criterion for a tensor product to be highest-ℓ-weight which is the heart of the proof of Theorem 3.5.4 and will also be used to deduce further criteria which will be used in the proof of Theorem 3.5.5.

Proposition 4.3.1 *Let $\boldsymbol{\lambda}, \boldsymbol{v} \in \mathcal{P}^+$ and $V = L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{v})$. Then, V is highest-ℓ-weight provided there exists $\boldsymbol{\mu} \in \mathcal{P}^+$ such that one of the following conditions holds:*

(i) *$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{v})$ and $L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu})$ are both highest-ℓ-weight.*
(ii) *$L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu}\boldsymbol{v})$ and $L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{v})$ are both highest-ℓ-weight.*

Proof We write the details only in case (i) holds since the other case is similar. So, assume that V is not highest-ℓ-weight. In particular, there exists $\boldsymbol{\xi} \in \mathcal{P}^+$ such that $\boldsymbol{\xi} < \boldsymbol{\lambda}\boldsymbol{v}$ together with an epimorphism $V \xrightarrow{f} L_q(\boldsymbol{\xi})$. Therefore, there also exists an

epimorphism

$$L_q(\boldsymbol{\mu}) \otimes V = L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\nu}) \xrightarrow{\mathrm{id}_{L_q(\boldsymbol{\mu})} \otimes f} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\xi}).$$

On the other hand, since $L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu})$ is highest-ℓ-weight, there exist monomorphisms

$$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \xrightarrow{g} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \quad \text{and} \quad L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}) \xrightarrow{g \otimes \mathrm{id}_{L_q(\boldsymbol{\nu})}} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\nu}).$$

Lemma 2.6.2 implies the composition

$$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}) \xrightarrow{g \otimes \mathrm{id}_{L_q(\boldsymbol{\nu})}} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\nu}) \xrightarrow{\mathrm{id}_{L_q(\boldsymbol{\mu})} \otimes f} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\xi})$$

is nonzero. Then, since $L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu})$ is highest-ℓ-weight, the image of its highest-ℓ-weight vector under this composition must be a nonzero vector with ℓ-weight $\boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\nu}$. However, since $\boldsymbol{\xi} < \boldsymbol{\lambda}\boldsymbol{\nu}$, we also have $\boldsymbol{\xi}\boldsymbol{\mu} < \boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\nu}$ and, hence, there is no vector in $L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\xi})$ with ℓ-weight $\boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\nu}$, yielding a contradiction. ∎

One of the applications of the above result gives a partial answer to the following question. Suppose $\boldsymbol{\pi}, \boldsymbol{\pi}', \boldsymbol{\omega} \in \mathcal{P}^+$ are such that $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight and $\boldsymbol{\omega}$ divides $\boldsymbol{\pi}$. Under which further assumptions $L_q(\boldsymbol{\pi}\boldsymbol{\omega}^{-1}) \otimes L_q(\boldsymbol{\pi}')$ is also highest-ℓ-weight? A similar question can be made in the case $\boldsymbol{\omega}$ divides $\boldsymbol{\pi}'$.

Corollary 4.3.2 *Suppose G and G' are pseudo-q-factorization graphs over $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$, respectively. Assume $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight, and let $\boldsymbol{\omega} \in \mathcal{V}_G$ and $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$. Then:*

(a) *If $\boldsymbol{\omega}$ is a sink in $\mathcal{V}_G$, $L_q(\boldsymbol{\pi}\boldsymbol{\omega}^{-1}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight.*
(b) *If $\boldsymbol{\omega}'$ is a source in $\mathcal{V}_{G'}$, $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}'(\boldsymbol{\omega}')^{-1})$ is highest-ℓ-weight.*

Proof If $\boldsymbol{\omega}$ is a sink in $\mathcal{V}_G$, Corollary 4.1.6 implies that $L_q(\boldsymbol{\pi}\boldsymbol{\omega}^{-1}) \otimes L_q(\boldsymbol{\omega})$ is highest-ℓ-weight, showing that (i) of Proposition 4.3.1 holds with $\boldsymbol{\lambda} = \boldsymbol{\pi}\boldsymbol{\omega}^{-1}$, $\boldsymbol{\mu} = \boldsymbol{\omega}$, and $\boldsymbol{\nu} = \boldsymbol{\pi}'$. Part (b) follows similarly. ∎

Corollary 4.3.3 *Let $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \mathcal{P}^+$ with dissociate factorizations and $\boldsymbol{\pi} = \boldsymbol{\pi}'\boldsymbol{\pi}''$. Let also $G = G(\boldsymbol{\pi}), G' = G(\boldsymbol{\pi}'), G'' = G(\boldsymbol{\pi}'')$, and suppose $\boldsymbol{\omega}', \boldsymbol{\omega}'' \in \mathcal{P}^+$ satisfy*

$$\boldsymbol{\omega}' \text{ is a source in } G', \quad \boldsymbol{\omega}'' \text{ is a sink in } G'', \quad \text{and} \quad (\boldsymbol{\omega}'', \boldsymbol{\omega}') \in \mathcal{A}_G.$$

Then, $L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\pi}'')$ is not highest-ℓ-weight.

Proof The first two assumptions, together with Corollary 4.1.6, imply that

$$L_q(\boldsymbol{\omega}') \otimes L_q(\boldsymbol{\pi}'(\boldsymbol{\omega}')^{-1}) \quad \text{and} \quad L_q(\boldsymbol{\pi}''(\boldsymbol{\omega}'')^{-1}) \otimes L_q(\boldsymbol{\omega}'') \quad \text{are highest-ℓ-weight.}$$

On the other hand, the last assumption implies that $L_q(\boldsymbol{\omega}') \otimes L_q(\boldsymbol{\omega}'')$ is not highest-ℓ-weight. Letting $\boldsymbol{\lambda} = \boldsymbol{\omega}'$, $\boldsymbol{\mu} = \boldsymbol{\pi}'(\boldsymbol{\omega}')^{-1}$ and $\boldsymbol{\nu} = \boldsymbol{\omega}''$, Proposition 4.3.1(i) implies $L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\omega}'')$ is not highest-ℓ-weight. Then, an application of Proposition 4.3.1(ii) with $\boldsymbol{\mu} = \boldsymbol{\pi}''(\boldsymbol{\omega}'')^{-1}$, $\boldsymbol{\nu} = \boldsymbol{\omega}''$, and $\boldsymbol{\lambda} = \boldsymbol{\pi}'$ completes the proof. ∎

Theorem 3.5.7 is easily deduced from Corollary 4.3.3. Moreover, we can also give the following proof.

Proof of Theorem 3.5.4   Let $\pi \in \mathcal{P}^+$ and assume $G = G(\pi)$ is a totally ordered line. By Proposition 3.3.4, we need to show that $L_q(\pi') \otimes L_q(\pi'')$ is reducible for any nontrivial decomposition $\pi = \pi'\pi''$ such that $G = G' \otimes G''$ with $G' = G(\pi')$ and $G'' = G(\pi'')$ connected. In particular, G' and G'' are also totally ordered lines. Without loss of generality, assume G' contains the sink of G. Then, if ω' is the source of G' and ω'' is the sink of G'', the fact that G, G', and G'' are totally ordered lines implies that $(\omega'', \omega') \in \mathcal{A}_G$ and, hence, $L_q(\pi') \otimes L_q(\pi'')$ is not simple by Corollary 4.3.3. ∎

4.4 Highest-ℓ-weight criteria via monotonic paths

Recall (2.2.6).

Lemma 4.4.1   *Let $\pi \in \mathcal{P}^+$ and suppose G is a pseudo q-factorization graph over π. Given $\bar{\omega} \in \mathcal{V}_G$, consider*

$$\pi_\pm = \prod_{\omega \in \mathcal{N}_G^\pm(\bar{\omega})} \omega'.$$

Then, the following tensor products are highest-ℓ-weight:

$$L_q(\pi_+) \otimes L_q(\pi\pi_+^{-1}) \quad and \quad L_q(\pi\pi_-^{-1}) \otimes L_q(\pi_-).$$

Proof   For the first tensor product, Corollary 4.1.6. implies it suffices to show $L_q(\omega) \otimes L_q(\omega')$ is highest-ℓ-weight for any $\omega \in \mathcal{N}_G^+(\bar{\omega})$ and any $\omega' \in \mathcal{V}_G \setminus \mathcal{N}_G^+(\bar{\omega})$. Indeed, if this failed for some choice of such ω and ω', it would follow that $a := (\omega', \omega) \in \mathcal{A}_G$. Since $\omega \in \mathcal{N}_G^+(\bar{\omega})$, we can chose $\rho \in \mathscr{P}_{\bar{\omega},\omega}^+$ and it would follow that $\rho * a \in \mathscr{P}_{\bar{\omega},\omega'}^+$, contradicting the assumption $\omega' \notin \mathcal{N}_G^+(\bar{\omega})$. The second tensor product is treated similarly. ∎

The next lemma will play a role in the proofs of Proposition 4.5.3 and Theorem 3.5.5.

Lemma 4.4.2   *Let $\pi, \pi' \in \mathcal{P}^+$ and suppose G and G' are pseudo q-factorization graphs over π and π', respectively. Let $\bar{\omega} \in \mathcal{V}_G$ and $\bar{\omega}' \in \mathcal{V}_{G'}$ and consider*

$$\pi_+ = \prod_{\omega \in \mathcal{N}_G^+(\bar{\omega})} \omega \quad and \quad \pi_-' = \prod_{\omega \in \mathcal{N}_{G'}^-(\bar{\omega}')} \omega.$$

If $V = L_q(\pi) \otimes L_q(\pi')$ is highest-ℓ-weight, so are the following tensor products:

$$L_q(\pi_+) \otimes L_q(\pi'), \quad L_q(\pi) \otimes L_q(\pi_-'), \quad and \quad L_q(\pi_+) \otimes L_q(\pi_-').$$

Proof   Let $\lambda = \pi_+, \mu = \pi\pi_+^{-1}$, and $\nu = \pi'$. By assumption, $L_q(\lambda\mu) \otimes L_q(\nu)$ is highest-ℓ-weight, while Lemma 4.4.1 implies that so is $L_q(\lambda) \otimes L_q(\mu)$. The claim about the first tensor product then follows from Proposition 4.3.1. The other two cases are treated similarly. ∎

4.5 A highest-ℓ-weight criterion from duality

We now deduce the main technical part behind the proof of Proposition 3.5.9 which will also be used for proving Theorem 3.5.5.

Proposition 4.5.1 *Let $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^+$. Let also $V = L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\nu})^*$,*

$$T_1 = L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}), \quad U_1 = L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu}), \quad W_1 = L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}),$$

$$T_2 = L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu}\boldsymbol{\nu}), \quad U_2 = L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}), \quad W_2 = L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\mu}).$$

Then, W_i is highest-ℓ-weight provided T_i and U_i are highest-ℓ-weight, $i \in \{1, 2\}$, and V is simple.

Proof We write the details for $i = 1$ only. Since V is simple, we have $V \cong L_q(\boldsymbol{\nu})^* \otimes L_q(\boldsymbol{\lambda})$. If W_1 were not highest-ℓ-weight, there would exist $\boldsymbol{\xi} \in \mathcal{P}^+$ such that $\boldsymbol{\xi} < \boldsymbol{\mu}\boldsymbol{\nu}$, together with an epimorphism $W_1 \xrightarrow{f} L_q(\boldsymbol{\xi})$. Then, (2.6.2) implies there would also exist monomorphisms

$$L_q(\boldsymbol{\mu}) \xrightarrow{g} L_q(\boldsymbol{\xi}) \otimes L_q(\boldsymbol{\nu})^* \qquad \text{and}$$

$$L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \xrightarrow{g \otimes \mathrm{id}_{L_q(\boldsymbol{\lambda})}} L_q(\boldsymbol{\xi}) \otimes L_q(\boldsymbol{\nu})^* \otimes L_q(\boldsymbol{\lambda}) \cong L_q(\boldsymbol{\xi}) \otimes V.$$

On the other hand, since U_1 is highest-ℓ-weight, there exists a monomorphism

$$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \xrightarrow{h} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}).$$

Therefore, the following composition would also be injective:

$$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \xrightarrow{h} L_q(\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\lambda}) \xrightarrow{g \otimes \mathrm{id}_{L_q(\boldsymbol{\lambda})}} L_q(\boldsymbol{\xi}) \otimes V,$$

yielding a monomorphism

$$L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \hookrightarrow L_q(\boldsymbol{\xi}) \otimes L_q(\boldsymbol{\lambda}) \otimes L_q(\boldsymbol{\nu})^*.$$

Finally, by (2.6.2), this implies there would exist a nonzero homomorphism

$$T_1 = L_q(\boldsymbol{\lambda}\boldsymbol{\mu}) \otimes L_q(\boldsymbol{\nu}) \to L_q(\boldsymbol{\xi}) \otimes L_q(\boldsymbol{\lambda}).$$

Since $\boldsymbol{\xi} < \boldsymbol{\mu}\boldsymbol{\nu}$ and, therefore, $\boldsymbol{\lambda}\boldsymbol{\xi} < \boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\nu}$, this yields a contradiction with the assumption that T_1 is highest-ℓ-weight. ∎

Corollary 4.5.2 *Suppose G and G' are pseudo-q-factorization graphs over $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$, respectively, and assume $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight.*

(a) *If $\boldsymbol{\omega} \in \mathcal{V}_G$ is a source in G such that $L_q(\boldsymbol{\omega}')^* \otimes L_q(\boldsymbol{\omega})$ is simple for any $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$, then $L_q(\boldsymbol{\pi}\boldsymbol{\omega}^{-1}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight.*
(b) *If $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$ is a sink in G' such that $L_q(\boldsymbol{\omega}') \otimes {}^*L_q(\boldsymbol{\omega})$ is simple for any $\boldsymbol{\omega} \in \mathcal{V}_G$, then $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}'(\boldsymbol{\omega}')^{-1})$ is highest-ℓ-weight.*

Proof If $\boldsymbol{\omega} \in \mathcal{V}_G$ is a source in G, Corollary 4.1.6 implies $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\pi}\boldsymbol{\omega}^{-1})$ is highest-ℓ-weight. In its turn, since $L_q(\boldsymbol{\omega}')^* \otimes L_q(\boldsymbol{\omega})$ is simple for any $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$, Corollary 4.1.6 implies $L_q(\boldsymbol{\pi}')^* \otimes L_q(\boldsymbol{\omega})$ is simple. Part (a) then follows from Proposition 4.5.1 with $\boldsymbol{\lambda} = \boldsymbol{\omega}$, $\boldsymbol{\mu} = \boldsymbol{\pi}\boldsymbol{\omega}^{-1}$, $\boldsymbol{\nu} = \boldsymbol{\pi}'$, and $i = 1$. Part (b) is proved similarly. ∎

The latter criteria leads to the following criterion for proving that a tensor product is not highest-ℓ-weight.

Proposition 4.5.3 *Assume* $\pi, \pi' \in \mathcal{P}^+$ *have dissociate q-factorizations, and let* $G = G(\pi), G' = G(\pi'), G'' = G(\pi\pi')$. *Suppose that there exist* $\boldsymbol{\omega} \in \mathcal{V}_G$, $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$ *such that* $(\boldsymbol{\omega}', \boldsymbol{\omega}) \in \mathcal{A}_{G''}$ *and*

(4.5.1) $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')^*$ *is simple* $\forall \, (\boldsymbol{\omega}, \boldsymbol{\omega}') \in \mathcal{N}_G^+(\boldsymbol{\omega}) \times \mathcal{N}_{G'}^-(\boldsymbol{\omega}') \setminus \{(\boldsymbol{\omega}, \boldsymbol{\omega}')\}$.

Then, $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$ *is not highest-ℓ-weight.*

Proof   Suppose $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$ is highest-ℓ-weight, and let $\boldsymbol{\pi}_+, \boldsymbol{\pi}'_-$ be defined as in Lemma 4.4.2. In particular, $L_q(\boldsymbol{\pi}_+) \otimes L_q(\boldsymbol{\pi}'_-)$ is also highest-ℓ-weight. On the other hand, it follows from (4.5.1) and Corollary 4.1.6 that

$$L_q(\boldsymbol{\pi}_+) \otimes L_q(\boldsymbol{\pi}'_-(\boldsymbol{\omega}')^{-1})^* \quad \text{is simple.}$$

Proposition 4.5.1 with $\boldsymbol{\lambda} = \boldsymbol{\pi}_+, \boldsymbol{\mu} = \boldsymbol{\omega}', \boldsymbol{\nu} = \boldsymbol{\pi}'_-(\boldsymbol{\omega}')^{-1}$ and $i = 2$ then implies that

$$L_q(\boldsymbol{\pi}_+) \otimes L_q(\boldsymbol{\omega}') \quad \text{is highest-ℓ-weight.}$$

Assumption (4.5.1) also implies $L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}')^*$ is simple for all $\boldsymbol{\omega} \in \mathcal{N}_G^+(\boldsymbol{\omega}) \setminus \{\boldsymbol{\omega}\}$ and, hence, it follows from Corollary 4.1.6 that

$$L_q(\boldsymbol{\pi}_+ \boldsymbol{\omega}^{-1}) \otimes L_q(\boldsymbol{\omega}')^* \quad \text{is simple.}$$

Therefore, Proposition 4.5.1 with $\boldsymbol{\lambda} = \boldsymbol{\pi}_+ \boldsymbol{\omega}^{-1}, \boldsymbol{\mu} = \boldsymbol{\omega}, \boldsymbol{\nu} = \boldsymbol{\omega}'$, and $i = 1$ implies that

$$L_q(\boldsymbol{\omega}) \otimes L_q(\boldsymbol{\omega}') \quad \text{is highest-ℓ-weight,}$$

which contradicts the assumption $(\boldsymbol{\omega}', \boldsymbol{\omega}) \in \mathcal{A}_{G''}$.   ∎

We are ready for:

Proof of Proposition 3.5.9   Assume that G is not prime, so we have a nontrivial factorization

(4.5.2) $$L_q(\boldsymbol{\pi}) \cong L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\pi}'').$$

By Proposition 3.3.4, $\boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$ have dissociate q-factorizations and, hence, if $G' = G(\boldsymbol{\pi}')$ and $G'' = G(\boldsymbol{\pi}'')$, (G', G'') is a cut of G. Therefore, by assumption there exist $\boldsymbol{\omega}' \in \mathcal{V}_{G'}$ and $\boldsymbol{\omega}'' \in \mathcal{V}_{G''}$ such that either (i) or (ii) holds. If it is (i), Proposition 4.5.3 implies that $L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\pi}'')$ is not highest-ℓ-weight, yielding a contradiction. If it is (ii), the same conclusion is reached by interchanging the roles of $\boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$.   ∎

5   Totally ordered graphs

In this section, we prove Theorem 3.5.5 and, hence, assume $\mathfrak{g}$ is of type A.

5.1   Some combinatorics

In this section, we deduce a few technical lemmas concerned with arithmetic relations among the elements of $\mathscr{R}_{i,j}^{r,s}$. In particular, they are useful for detecting whether a pseudo q-factorization graph is a tournament.

Lemma 5.1.1   *If* $i, j \in I, r, s \in \mathbb{Z}_{>0}, m \in \mathscr{R}_{i,j}^{r,s} \setminus \mathscr{R}_{i,j,[i,j]}^{r,s}$, *and* $a \in \mathbb{F}^\times$, *then*

$$L_q(\boldsymbol{\omega}_{i, aq^m, r}) \otimes L_q(\boldsymbol{\omega}_{j, a, s})^* \quad \text{is simple.}$$

Proof The assumptions imply $m = r + s + d(i, j) - 2p$ for some $-d([i, j], \partial I) \le p < 0$, while the claim follows if we show that $m + h^\vee \notin \mathscr{R}_{i, j^*}^{r, s}$. Since

$$0 < m + h^\vee = r + s + d(i, j^*) - 2p' \quad \text{with} \quad p' = p + \frac{d(i, j^*) - d(i, j) - h^\vee}{2},$$

it suffices to show $p' < -d([i, j^*], \partial I)$.

Without loss of generality, assume I has been identified with $\{1, \dots, n\}$ so that $i \le j$ and recall that $j^* = n + 1 - j$. Suppose first that $d([i, j], \partial I) = d(i, \partial I)$. It follows that $d([i, j^*], \partial I) = d(i, \partial I)$ and either $d(i, \partial I) = i - 1$ or $i = j$ and $d(i, \partial I) = n - i$. In the former case, we have $j^* \ge i$ and

$$d(i, j^*) - d(i, j) - h^\vee = (n + 1 - j - i) - (j - i) - (n + 1) = -2j.$$

Therefore, since $p \le -1$, we see that

$$p' \le -1 - j \le -1 - i = -1 - (d([i, j^*], \partial I) + 1) = -d([i, j^*], \partial I) - 2,$$

thus completing the proof in this case. In the latter case, $j^* \le i = j$ and we have

$$d(i, j^*) - d(i, j) - h^\vee = (i - (n + 1 - i)) - (n + 1) = -2(n + 1 - i) = -2d([i, j^*], \partial I) - 2,$$

which also completes the proof.

It remains to consider the case $d([i, j], \partial I) = d(j, \partial I) = n - j$, which implies $j^* \le i$ and $d([i, j^*], \partial I) = d(j^*, \partial I) = n - j$. Hence,

$$d(i, j^*) - d(i, j) - h^\vee = (i - (n + 1 - j)) - (j - i) - (n + 1) = -2(n + 1 - i),$$

and we get,

$$p' \le -1 - (d([i, j^*], \partial I) + 1 + (j - i)) < -d([i, j^*], \partial I),$$

as desired. ∎

Lemma 5.1.2 *Let $N \in \mathbb{Z}_{>0}$ and $(m_k, r_k, i_k) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{>0} \times I, 1 \le k \le N$. Suppose*

(5.1.1)
$$|m_k - m_{k-1}| \in \mathscr{R}_{i_{k-1}, i_k}^{r_{k-1}, r_k} \qquad \text{for all} \qquad 1 < k \le N.$$

(a) *For all $1 \le k, l \le N$, there exists $p_{l,k} \in \mathbb{Z}$ such that $m_l - m_k = r_l + r_k + d(i_l, i_k) - 2p_{l,k}$.*

(b) *If $m_k > m_{k-1}$ for all $1 < k \le N$, then $p_{N,1} < \min\{r_1, r_N\}$, and*

$$p_{N,1} < p_{l,k} < \min\{r_k, r_l\}, \quad \text{for all} \quad 1 \le k < l \le N, \quad \text{with} \quad (k, l) \ne (1, N).$$

Similarly, if $m_k < m_{k-1}$ for all $1 < k \le N$, then $p_{1,N} < \min\{r_1, r_N\}$, and

$$p_{1,N} < p_{k,l} < \min\{r_k, r_l\}, \quad \text{for all} \quad 1 \le k < l \le N, \quad \text{with} \quad (k, l) \ne (1, N).$$

Proof The equality $m_k - m_l = r_k + r_l + d(i_k, i_l) - 2p_{k,l}$ clearly defines $p_{k,l} \in \mathbb{Q}$ and, moreover, one can easily check that

(5.1.2)
$$p_{k,l} + p_{j,k} = p_{j,l} + r_k + d_{i_j, i_l}^{i_k} \quad \text{for all} \quad 1 \le j, k, l \le N,$$

and

(5.1.3)
$$p_{k,l} + p_{l,k} = r_k + r_l + d(i_k, i_l) \quad \text{for all} \quad 1 \le k, l \le N.$$

We will use these to show $p_{k,l} \in \mathbb{Z}$ by induction on $|k - l| \geq 1$ (note we also have $p_{k,k} = r_k$). If $|k - l| = 1$, (5.1.1) implies either $p_{k,l}$ or $p_{l,k}$ is an integer. Then, (5.1.3) implies the same is true for the other one. If $|k - l| > 1$, the inductive step easily follows from (5.1.2) by choosing j in between k and l.

We prove (b) in the case $m_k > m_{k-1}$ by induction on $N > 1$ (the other case is similar). For $N = 2$, we have $p_{N,1} = p_{2,1}$ and, hence, the first claim follows from (5.1.1), while the second claim is vacuous. Note the first claim is a consequence of the second for $N > 2$, in which case, the inductive hypothesis implies

$$p_{N-1,1} < p_{l,k} < \min\{r_k, r_l\}, \quad \text{for all } 1 \leq k < l \leq N - 1, \text{ with } (k, l) \neq (1, N - 1),$$

as well as $p_{N-1,1} < \min\{r_1, r_{N-1}\}$. By (5.1.2), we have

(5.1.4)
$$\begin{aligned} p_{N,1} &= p_{N,N-1} + p_{N-1,1} - r_{N-1} - d_{i_1,i_N}^{i_{N-1}} \quad \text{and} \\ p_{N,1} &= p_{N,2} + p_{2,1} - r_2 - d_{i_1,i_N}^{i_1}. \end{aligned}$$

Moreover,

(5.1.5) $\quad p_{N,N-1} = p_N < \min\{r_N, r_{N-1}\} \leq r_{N-1} \quad \text{and} \quad p_{2,1} < \min\{r_2, r_1\} \leq r_2.$

Therefore,

$$p_{N,1} < p_{N-1,1} - d_{i_1,i_N}^{i_{N-1}} \leq p_{N-1,1} < p_{l,k} < \min\{r_k, r_l\},$$

for all $1 \leq k < l \leq N - 1$ with $(k, l) \neq (1, N - 1)$.

Thus, it remains to show that

$$p_{N,1} < p_{N,k} < \min\{r_k, r_N\} \quad \text{for all} \quad 1 < k < N.$$

Applying the inductive hypothesis to the sequence $(m_k, r_k, i_k), 1 < k \leq N$, we have

$$p_{N,1} < p_{l,k} < \min\{r_k, r_l\} \quad \text{for all} \quad 1 < k < l < N, \ (k, l) \neq (2, N).$$

The second claims in (5.1.4) and (5.1.5) imply $p_{N,1} < p_{N,2}$, thus completing the proof. ∎

Lemma 5.1.3 *Assume $m_k > m_{k-1}$ for all $1 < k \leq N$ in Lemma 5.1.2.*

(a) *If $p_{N,1} \geq -d([i_k, i_l], \partial I) - 1$ for some $1 \leq k < l \leq N, (l, k) \neq (1, N)$, then $m_l - m_k \in \mathscr{R}_{i_k,i_l}^{r_k,r_l}$. In particular, this is the case if $m_N - m_1 \in \mathscr{R}_{i_1,i_N}^{r_1,r_N}$ and $d([i_k, i_l], \partial I) \geq d([i_1, i_N], \partial I)$.*
(b) *If $m_N - m_1 \in \mathscr{R}_{i_1,i_N,[i_1,i_N]}^{r_1,r_N}$, then $m_l - m_k \in \mathscr{R}_{i_k,i_l,[i_k,i_l]}^{r_k,r_l}$ for all $1 \leq k < l \leq N$.*

Proof The initial assumption in (a), together with Lemma 5.1.2(b), implies $-d([i_k, i_l], \partial I) - 1 \leq p_{N,1} < p_{l,k}$. A second application of Lemma 5.1.2(b) then implies $-d([i_k, i_l], \partial I) \leq p_{l,k} < \min\{r_k, r_l\}$, thus proving the first claim in (a). The assumptions in the second part of (a) imply $p_{N,1} \geq -d([i_1, i_N], \partial I) \geq -d([i_k, i_l], \partial I)$, showing the second part follows from the first.

The assumption in (b) implies $0 \leq p_{N,1} < \min\{r_1, r_N\}$ and we want to show $0 \leq p_{l,k} < \min\{r_k, r_l\}$, which follows from Lemma 5.1.2(b). ∎

Assume, for instance, that the assumption in (b) of the last lemma holds. Then, if $J \subseteq I$ is connected and

$$\boldsymbol{\pi} = \prod_{k:i_k \in J} \boldsymbol{\omega}_{i_k, aq^{m_k}, r_k},$$

the pseudo q-factorization graph G for $U_q(\tilde{\mathfrak{g}})_J$ associated with this pseudo q-factorization of $\boldsymbol{\pi}$ is a tournament.

5.2   The main lemma

Fix $\boldsymbol{\pi} \in \mathcal{P}^+$ such that its q-factorization graph $G = G(\boldsymbol{\pi}) = (\mathcal{V}, \mathcal{A})$ is totally ordered, and let $N = \#\mathcal{V}$. Let $\boldsymbol{\omega}_{i,a,r}$ be the sink, and let $m_l, r_l \in \mathbb{Z}_{\geq 0}, 1 \leq l \leq N$, be such that $0 = m_1 < m_2 < \cdots < m_N$ and

$$\mathcal{V} = \{\boldsymbol{\omega}_{i_l, aq^{m_l}, r_l} : 1 \leq l \leq N\}.$$

To shorten notation, set $\boldsymbol{\omega}^{(l)} = \boldsymbol{\omega}_{i_l, aq^{m_l}, r_l}, 1 \leq l \leq N$. Note

$$\mathcal{A} \subseteq \{(\boldsymbol{\omega}^{(l)}, \boldsymbol{\omega}^{(k)}) : 1 \leq k < l \leq N\}$$

and

$$(\boldsymbol{\omega}^{(l)}, \boldsymbol{\omega}^{(k)}) \in \mathcal{A} \quad \Rightarrow \quad m_l - m_k \in \mathscr{R}_{i_k, i_l}^{r_k, r_l}.$$

Since $\mathfrak{g}$ is of type A, the latter is equivalent to

$$m_l - m_k = r_k + r_l + d(i_k, i_l) - 2p_{l,k}, \quad \text{for some} \quad -d([i_k, i_l], \partial I) \leq p_{l,k} < \min\{r_k, r_l\}.$$

Lemma 5.1.2 implies such an expression exists for $m_l - m_k$ for all $1 \leq k < l \leq N$ for some $p_{l,k} \in \mathbb{Z}$ and, moreover,

(5.2.1) $p_{l,k} < \min\{r_k, r_l\}$ for all $1 \leq k < l \leq N$.

Furthermore, Theorem 3.3.2 and (3.4.1) imply

(5.2.2) $p_{l,k} \geq 0 \quad \Leftrightarrow \quad m_l - m_k \in \mathscr{R}_{i_k, i_l, [i_k, i_l]}^{r_k, r_l} \quad \Rightarrow \quad i_l \neq i_k$.

Let us make a brief interlude and use the setup we have just fixed to give the:

Proof of Proposition 3.5.6   In light of (5.2.2), the assumption $c(\mathcal{V}) \subseteq \partial I$ implies

(5.2.3) $(\boldsymbol{\omega}^{(l)}, \boldsymbol{\omega}^{(k)}) \in \mathcal{A}$ only if $i_k \neq i_l$.

Note the claim about the vertices being alternately colored is immediate from this. Since a totally ordered tree is a line, if G were not a line, it would contain a cycle. In that case, let v be the maximal element of $\mathcal{V}$ which is part of a cycle. Suppose $a = (v, w)$ is the first arrow of this cycle and $a' = (v, w')$ is the last:

$$\cdots w \xleftarrow{\ a\ } v \xrightarrow{\ a'\ } w' \cdots.$$

Set $e = \pi(a)$ and $e' = \pi(a')$. Since G is totally ordered, we must have either $w < w'$ or $w > w'$. Without loss of generality, we assume it is the latter. This means there exists a (simple) monotonic path $\rho \in \mathscr{P}_{w,w'}$ and, moreover, $\rho * e \in \mathscr{P}_{v,w'}$ is a monotonic path. Furthermore, $e' * \rho * e$ is a cycle based on v and, by construction, the vertices in this

cycle satisfy all the assumptions in Lemma 5.1.3(b). As commented after that lemma, this implies the subgraph determined by this subset of vertices is a tournament which, by (5.2.3), have all of its vertices differently colored. This yields a contradiction since $\#\partial I = 2$ and there are no cycles with less than three vertices. ∎

The next lemma is the heart of the proof of Theorem 3.5.5.

Lemma 5.2.1 *Let $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \mathcal{P}^+$ have dissociate q-factorizations and assume $\boldsymbol{\pi} = \boldsymbol{\pi}'\boldsymbol{\pi}''$. Let also $G' = G(\boldsymbol{\pi}')$ and $G'' = G(\boldsymbol{\pi}'')$. Assume*

$$(5.2.4) \qquad L_q(\boldsymbol{\pi}') \otimes L_q(\boldsymbol{\pi}'') \quad \text{is highest-}\ell\text{-weight}$$

and that $1 \le j' < j'' \le N$ are such that $\boldsymbol{\omega}^{(j')} \in \mathcal{V}_{G'}$, $\boldsymbol{\omega}^{(j'')} \in \mathcal{V}_{G''}$, and

$$(5.2.5) \qquad m_{j''} - m_{j'} \in \mathscr{R}^{r_{j'},r_{j''}}_{i_{j'},i_{j''},J}, \quad \text{where} \quad J = [i_{j'}, i_{j''}].$$

Then, there exist $1 \le k'' < k' \le N$ such that $K := [i_{k'}, i_{k''}] \subsetneqq J$, $\boldsymbol{\omega}^{(k')} \in \mathcal{V}_{G'}$, $\boldsymbol{\omega}^{(k'')} \in \mathcal{V}_{G''}$, and $m_{k'} - m_{k''} \in \mathscr{R}^{r_{k'},r_{k''}}_{i_{k'},i_{k''},K}$.

Proof Assume, by contradiction, that there does not exist such pair (k', k'') and consider:

$$\mathcal{I}^+_{G'} = \{1 \le l \le N : \boldsymbol{\omega}^{(l)} \in \mathcal{N}^+_{G'}(\boldsymbol{\omega}^{(j')})\} = \{1 \le l \le N : \boldsymbol{\omega}^{(l)} \in \mathcal{V}_{G'}, \; l \ge j'\},$$

$$\mathcal{I}^-_{G''} = \{1 \le l \le N : \boldsymbol{\omega}^{(l)} \in \mathcal{N}^-_{G''}(\boldsymbol{\omega}^{(j'')})\} = \{1 \le l \le N : \boldsymbol{\omega}^{(l)} \in \mathcal{V}_{G''}, \; l \le j''\}.$$

Lemma 4.4.2, together with (5.2.4), implies

$$L_q(\boldsymbol{\pi}_+) \otimes L_q(\boldsymbol{\pi}_-) \quad \text{is highest-}\ell\text{-weight, where}$$
$$(5.2.6) \qquad \boldsymbol{\pi}_+ := \prod_{l \in \mathcal{I}^+_{G'}} \boldsymbol{\omega}^{(l)} \quad \text{and} \quad \boldsymbol{\pi}_- := \prod_{l \in \mathcal{I}^-_{G''}} \boldsymbol{\omega}^{(l)}.$$

If $\mathcal{I}^+_{G'} = \{j'\}$ and $\mathcal{I}^-_{G''} = \{j''\}$, then $\boldsymbol{\pi}_+ = \boldsymbol{\omega}^{(j')}, \boldsymbol{\pi}_- = \boldsymbol{\omega}^{(j'')}$, and (5.2.6) contradicts (5.2.5). Thus, henceforth assume that

$$\text{either} \quad \#\mathcal{I}^+_{G'} > 1 \quad \text{or} \quad \#\mathcal{I}^-_{G''} > 1.$$

Set

$$\mathcal{I}^{++}_{G'} = \{l \in \mathcal{I}^+_{G'} : l > j'', \, p_{l,j''} < 0\}, \qquad \mathcal{I}^{--}_{G''} = \{l \in \mathcal{I}^-_{G''} : l < j', \, p_{j',l} < 0\},$$

$$\boldsymbol{\pi}_{++} = \prod_{l \in \mathcal{I}^{++}_{G''}} \boldsymbol{\omega}^{(l)} \quad \text{and} \quad \boldsymbol{\pi}_{--} = \prod_{l \in \mathcal{I}^{--}_{G''}} \boldsymbol{\omega}^{(l)}.$$

Lemma 5.1.2 implies

$$(5.2.7) \qquad p_{l',l''} < p_{l',j''} < 0 \quad \text{for all} \quad l' \in \mathcal{I}^{++}_{G'}, \; l'' \in \mathcal{I}^-_{G''}\backslash\{j''\}$$

and

$$(5.2.8) \qquad p_{l',l''} < p_{j',l''} < 0 \quad \text{for all} \quad l'' \in \mathcal{I}^{--}_{G''}, \; l' \in \mathcal{I}^+_{G'}\backslash\{j'\}.$$

Moreover, Lemma 5.1.2 also implies

$$(5.2.9) \qquad l' \in \mathcal{I}_{G'}^+ \setminus \mathcal{I}_{G'}^{++} \quad \Rightarrow \quad l' < l \quad \text{for all} \quad l \in \mathcal{I}_{G'}^{++}.$$

Indeed, if it could be $l < l'$ for some $l \in \mathcal{I}_{G'}^{++}$, it would follow from Lemma 5.1.2 that

$$p_{l',j''} < p_{l,j''} < 0$$

which contradicts the assumption $l' \notin \mathcal{I}_{G'}^{++}$. Similarly,

$$(5.2.10) \qquad l'' \in \mathcal{I}_{G''}^- \setminus \mathcal{I}_{G''}^{--} \quad \Rightarrow \quad l'' > l \quad \text{for all} \quad l \in \mathcal{I}_{G''}^{--}.$$

Note that (5.2.7), together with Lemma 5.1.1, implies

$$(5.2.11) \qquad L_q(\boldsymbol{\omega}^{(l')}) \otimes L_q(\boldsymbol{\omega}^{(l'')})^* \quad \text{is simple for all} \quad l' \in \mathcal{I}_{G'}^{++}, \ l'' \in \mathcal{I}_{G''}^-$$

and, similarly, (5.2.8) implies

$$(5.2.12) \qquad L_q(\boldsymbol{\omega}^{(l')}) \otimes L_q(\boldsymbol{\omega}^{(l'')})^* \quad \text{is simple for all} \quad l'' \in \mathcal{I}_{G''}^{--}, \ l' \in \mathcal{I}_{G'}^+.$$

In their turn, (5.2.9) and (5.2.10) imply

$$(5.2.13) \quad L_q(\boldsymbol{\omega}^{(l)}) \otimes L_q(\boldsymbol{\omega}^{(l')}) \quad \text{is highest-}\ell\text{-weight for all} \quad l' \in \mathcal{I}_{G'}^+ \setminus \mathcal{I}_{G'}^{++}, \ l \in \mathcal{I}_{G'}^{++}$$

and

$$(5.2.14) \quad L_q(\boldsymbol{\omega}^{(l')}) \otimes L_q(\boldsymbol{\omega}^{(l)}) \quad \text{is highest-}\ell\text{-weight for all} \quad l' \in \mathcal{I}_{G''}^- \setminus \mathcal{I}_{G''}^{--}, \ l \in \mathcal{I}_{G''}^{--}.$$

We will check that these facts, together with (5.2.6), Corollary 4.1.6, and Proposition 4.5.1, imply

$$(5.2.15) \qquad \begin{aligned} M = L_q(\boldsymbol{\varpi}') \otimes L_q(\boldsymbol{\varpi}'') \quad &\text{is highest-}\ell\text{-weight, where} \\ \boldsymbol{\varpi}' = \boldsymbol{\pi}_+(\boldsymbol{\pi}_{++})^{-1} \quad &\text{and} \quad \boldsymbol{\varpi}'' = \boldsymbol{\pi}_-(\boldsymbol{\pi}_{--})^{-1}. \end{aligned}$$

Moreover, Corollary 3.2.4 implies that $M_J = L_q(\boldsymbol{\varpi}'_J) \otimes L_q(\boldsymbol{\varpi}''_J)$ is also highest-ℓ-weight. Using the initial assumption of the proof, we will see that this contradicts (5.2.5), thus completing the proof.

To check (5.2.15), we first use Proposition 4.5.1 with $i = 1$, $\boldsymbol{\lambda} = \boldsymbol{\pi}_{++}$, $\boldsymbol{\mu} = \boldsymbol{\varpi}'$, and $\boldsymbol{\nu} = \boldsymbol{\pi}_-$. In the terminology of Proposition 4.5.1, (5.2.6) means T_1 is highest-ℓ-weight, (5.2.11) and Corollary 4.1.6 imply V is simple, while (5.2.13) and Corollary 4.1.6 imply U_1 is highest-ℓ-weight. Hence, $W_1 = L_q(\boldsymbol{\varpi}') \otimes L_q(\boldsymbol{\pi}_-)$ is highest-ℓ-weight. A second application of Proposition 4.5.1 with $i = 2$, $\boldsymbol{\lambda} = \boldsymbol{\varpi}'$, $\boldsymbol{\mu} = \boldsymbol{\varpi}''$, and $\boldsymbol{\nu} = \boldsymbol{\pi}_{--}$, together with (5.2.12), (5.2.14), Corollary 4.1.6, and Corollary 4.1.6 gives (5.2.15).

Consider the following sets:

$$\begin{aligned} \mathcal{J}_{G'} &= (\mathcal{I}_{G'}^+ \setminus \mathcal{I}_{G'}^{++}) \cap \{1 \le l \le N : i_l \in J\} \quad \text{and} \\ \mathcal{J}_{G''} &= (\mathcal{I}_{G''}^- \setminus \mathcal{I}_{G''}^{--}) \cap \{1 \le l \le N : i_l \in J\}. \end{aligned}$$

Note

$$(5.2.16) \qquad \boldsymbol{\varpi}'_J = \prod_{l \in \mathcal{J}_{G'}} \boldsymbol{\omega}_J^{(l)} \quad \text{and} \quad \boldsymbol{\varpi}''_J = \prod_{l \in \mathcal{J}_{G''}} \boldsymbol{\omega}_J^{(l)}.$$

If $\mathcal{J}_{G'} = \{j'\}$ and $\mathcal{J}_{G''} = \{j''\}$, then $\boldsymbol{\varpi}'_J = \boldsymbol{\omega}_J^{(j')}$, $\boldsymbol{\varpi}''_J = \boldsymbol{\omega}_J^{(j'')}$ and $M_J = L_q(\boldsymbol{\omega}_J^{(j')}) \otimes L_q(\boldsymbol{\omega}_J^{(j'')})$, yielding a contradiction between (5.2.15) and (5.2.5). Thus, we must have

$$\text{either} \quad \#\mathcal{J}_{G'} > 1 \quad \text{or} \quad \#\mathcal{J}_{G''} > 1.$$

Consider also

$$\mathcal{J}_{G'}^+ := \{l \in \mathcal{J}_{G'} : l > j''\}, \qquad \mathcal{J}_{G'}^- := \{l \in \mathcal{J}_{G'} : l < j''\},$$

$$\mathcal{J}_{G''}^+ := \{l \in \mathcal{J}_{G''} : l > j'\}, \qquad \mathcal{J}_{G''}^- := \{l \in \mathcal{J}_{G''} : l < j'\}.$$

Obviously, $j' \in \mathcal{J}_{G'}^-$, $j'' \in \mathcal{J}_{G''}^+$, $\mathcal{J}_{G'}$ is the disjoint union of $\mathcal{J}_{G'}^\pm$, and similarly for $\mathcal{J}_{G''}$. We claim

$$\#\mathcal{J}_{G'}^+ \le 1 \quad \text{and} \quad \#\mathcal{J}_{G''}^- \le 1.$$

Indeed, by definition of let $\mathcal{J}_{G'}^+$, we have

(5.2.17) $$l \in \mathcal{J}_{G'}^+ \quad \Rightarrow \quad i_l \in J, \quad l > j'', \quad \text{and} \quad p_{l,j''} \ge 0.$$

In particular, together with (5.2.1) and (5.2.2), this implies

$$m_l - m_{j''} \in \mathscr{R}_{i_{j''}, i_l}^{r_{j''}, r_l} {}_{[i_{j''}, i_l]}, \quad i_{j''} \ne i_l, \quad \text{and} \quad [i_{j''}, i_l] \subseteq J.$$

If it were $[i_{j''}, i_l] \subsetneq J$, then $k' = l$ and $k'' = j''$ would be a pair of indices satisfying the conclusion of the lemma, contradicting the initial assumption in the proof. Hence, we must have

$$i_l = i_j \quad \text{for all} \quad l \in \mathcal{J}_{G'}^+.$$

If it were $\#\mathcal{J}_{G'}^+ > 1$, let $l, l' \in \mathcal{J}_{G'}^+$ with $l > l'$. Then, since $i_{l'} = i_l = i_{j'}$ and G is a q-factorization graph, we must have $p_{l,l'} < 0$. However, Lemma 5.1.2 implies that

$$p_{l,j''} < p_{l,l'} < 0,$$

contradicting (5.2.17). Similar arguments can be used to show that $\#\mathcal{J}_{G''}^- \le 1$ and that $i_l = i_{j''}$ if $l \in \mathcal{J}_{G''}^-$. Henceforth, let j^+ denote the unique element of $\mathcal{J}_{G'}^+$, if it exists, and let j^- be the unique element of $\mathcal{J}_{G''}^-$, if it exists. In particular,

(5.2.18) $$\mathcal{J}_{G'}^- = \mathcal{J}_{G'} \backslash \{j^+\}, \quad \mathcal{J}_{G''}^+ = \mathcal{J}_{G''} \backslash \{j^-\}, \quad i_{j^+} = i_{j'}, \quad i_{j^-} = i_{j''},$$
$$\text{and} \quad j^- < j' \le l \le j'' < j^+ \quad \text{for all} \quad l \in \mathcal{J}_{G'}^- \cup \mathcal{J}_{G''}^+.$$

Moreover, since $p_{j'',j'} \ge 0$ by (5.2.5), Lemma 5.1.2, and (5.2.2) imply that

(5.2.19) $\quad 0 \le p_{l,l'} < \min\{r_l, r_{l'}\} \quad \text{and} \quad i_l \ne i_{l'} \quad \text{for all} \quad l, l' \in \mathcal{J}_{G'}^- \cup \mathcal{J}_{G''}^+, \quad l > l'.$

It follows that a pair (k', k'') such that $k' \in \mathcal{J}_{G'}^-$, $k'' \in \mathcal{J}_{G''}^+$, and $k' > k''$ satisfies the conclusion of the lemma and, hence, does not exist by the initial assumption of the proof. Thus, we must have

$$l < l' \quad \text{for all} \quad l \in \mathcal{J}_{G'}^-, \ l' \in \mathcal{J}_{G''}^+.$$

Note also that

(5.2.20) $$\boldsymbol{\varpi}_J' = \boldsymbol{\omega}^{(j^+)} \prod_{l \in \mathcal{J}_{G'}^-} \boldsymbol{\omega}_J^{(l)} \quad \text{and} \quad \boldsymbol{\varpi}_J'' = \boldsymbol{\omega}^{(j^-)} \prod_{l \in \mathcal{J}_{G''}^+} \boldsymbol{\omega}_J^{(l)},$$

where we set $\boldsymbol{\omega}^{(j^\pm)} = 1$ if $j^\pm$ does not exist. Let us check that

(5.2.21) $$\mathcal{J}_{G'}^- = \{j'\} \quad \text{and} \quad \mathcal{J}_{G''}^+ = \{j''\}.$$

Indeed, assume $\mathcal{J}_{G'}^- \setminus \{j'\} \neq \varnothing$, choose $l' \in \mathcal{J}_{G'}^- \setminus \{j'\}$ and $l'' \in \mathcal{J}_{G''}^+$ such that $d(i_{l'}, i_{l''})$ is minimal, and let

$$\overline{J} = [i_{l'}, i_{l''}] \subsetneq J.$$

The choice of (l', l'') implies $\boldsymbol{\varpi}_{\overline{J}}' = \boldsymbol{\omega}_{\overline{J}}^{(l')}$, while

$$\boldsymbol{\varpi}_{\overline{J}}'' = \begin{cases} \boldsymbol{\omega}_{\overline{J}}^{(l'')}, & \text{if } l'' \neq j'', \\ \boldsymbol{\omega}_{\overline{J}}^{(j'')} \boldsymbol{\omega}_{\overline{J}}^{(j^-)}, & \text{if } l'' = j''. \end{cases}$$

As commented after (5.2.15), M_J is highest-ℓ-weight and, hence, so is

$$M_{\overline{J}} := L_q(\boldsymbol{\varpi}_{\overline{J}}') \otimes L_q(\boldsymbol{\varpi}_{\overline{J}}'') = L_q(\boldsymbol{\omega}_{\overline{J}}^{(l')}) \otimes L_q(\boldsymbol{\varpi}_{\overline{J}}'').$$

If $l'' \neq j''$, we have

$$M_{\overline{J}} = L_q(\boldsymbol{\omega}_{\overline{J}}^{(l')}) \otimes L_q(\boldsymbol{\omega}^{(l'')\overline{J}}),$$

which is not highest-ℓ-weight by (5.2.19), yielding a contradiction. If $l'' = j''$ (so $i_{l''} = i_{j''}$), (5.2.1) and (5.2.2) imply $p_{j'',j^-} < 0$ and, hence,

$$L_q(\boldsymbol{\varpi}_{\overline{J}}'') \cong L_q(\boldsymbol{\omega}_{\overline{J}}^{(j'')}) \otimes L_q(\boldsymbol{\omega}_{\overline{J}}^{(j^-)}).$$

Therefore,

$$M_{\overline{J}} \cong L_q(\boldsymbol{\omega}_{\overline{J}}^{(l')}) \otimes L_q(\boldsymbol{\omega}_{\overline{J}}^{(j'')}) \otimes L_q(\boldsymbol{\omega}_{\overline{J}}^{(j^-)}),$$

yielding a contradiction with (5.2.19) again. This proves the first claim in (5.2.21) and the second is proved similarly.

We have shown $\mathcal{J}_{G'} = \{j', j^+\}$ and $\mathcal{J}_{G''} = \{j'', j^-\}$, where we understand $j^\pm$ has not being listed if it does not exist. In particular,

$$J \cap \operatorname{supp}(\boldsymbol{\varpi}') = \{i_{j'}\} \quad \text{and} \quad J \cap \operatorname{supp}(\boldsymbol{\varpi}') = \{i_{j''}\},$$

which implies

$$M_J = L_q((\boldsymbol{\omega}^{(j')} \boldsymbol{\omega}^{(j^+)})_J) \otimes L_q((\boldsymbol{\omega}^{(j'')} \boldsymbol{\omega}^{(j^-)})_J).$$

Since $p_{j^+,j'} < 0$ and $p_{j'',j^-} < 0$, it follows that

$$M_J \cong L_q(\boldsymbol{\omega}_J^{(j^+)}) \otimes L_q(\boldsymbol{\omega}_J^{(j')}) \otimes L_q(\boldsymbol{\omega}_J^{(j'')}) \otimes L_q(\boldsymbol{\omega}_J^{(j^-)}).$$

However, $L_q(\boldsymbol{\omega}_J^{(j')}) \otimes L_q(\boldsymbol{\omega}_J^{(j'')})$ is not highest-ℓ-weight by (5.2.5), yielding the promised contradiction. ∎

5.3 Proof of Theorem 3.5.5

Let $\pi', \pi'' \in \mathcal{P}^+\backslash\{1\}$ be such that $\pi = \pi'\pi''$ and set

$$U = L_q(\pi') \otimes L_q(\pi'') \quad \text{and} \quad V = L_q(\pi'') \otimes L_q(\pi')$$

In light of Corollary 4.1.4, Theorem 3.5.5 follows if we show that either U or V is not highest-ℓ-weight. Moreover, by Corollary 3.2.4, we can assume π' and π'' have dissociate q-factorizations. The case $N = 1$ is obvious, while the case $N = 2$ follows from the definition of q-factorization graph and (3.3.2), since G is connected. Thus, henceforth, $N \geq 3$. We shall assume U and V are highest-ℓ-weight and reach a contradiction.

We will use the notation fixed before Lemma 5.2.1. Let also $G' = G(\pi') = (\mathcal{V}', \mathcal{A}')$ and $G'' = G(\pi'') = (\mathcal{V}'', \mathcal{A}'')$. Without loss of generality, assume $\omega := \omega^{(N)} \in \mathcal{V}''$ (ω is the source of G). We claim

$$(5.3.1) \qquad \#\mathcal{V}'' > 1 \quad \text{and, hence,} \quad \pi''\omega^{-1} \neq 1.$$

Indeed, if this were not the case, it would follow that $v := \pi'' \in \mathcal{V}$ and $\pi' = \pi v^{-1}$. Lemma 2.2.3 then implies G' is also totally ordered and, letting $\lambda = \omega^{(N-1)}$ be the source of G', it would follow that

$$(5.3.2) \qquad (v, \lambda) \in \mathcal{A}.$$

Set also $\mu = \pi'\lambda^{-1}$ and note $\mu \in \mathcal{P}^+\backslash\{1\}$ since $N \geq 3$ and we are assuming $\#\mathcal{V}'' = 1$. By assumption, $L_q(\lambda\mu) \otimes L_q(v) = U$ is highest-ℓ-weight. On the other hand, Corollary 4.1.6 implies $L_q(\lambda) \otimes L_q(\mu)$ is also highest-ℓ-weight. Together with Proposition 4.3.1, this implies $L_q(\lambda) \otimes L_q(v)$ is highest-ℓ-weight as well, yielding a contradiction with (5.3.2) and (3.3.2).

Note also that Corollary 4.3.2 implies that

$$\tilde{U} := L_q(\pi') \otimes L_q(\pi''\omega^{-1}) \quad \text{is highest-}\ell\text{-weight.}$$

Since $G(\pi\omega^{-1})$ is totally ordered by Lemma 2.2.3, an inductive argument on N then implies

$$(5.3.3) \qquad \tilde{V} := L_q(\pi''\omega^{-1}) \otimes L_q(\pi') \quad \text{is not highest-}\ell\text{-weight.}$$

Let

$$\mathcal{J}' = \{j : \omega^{(j)} \in \mathcal{V}'\}, \qquad \mathcal{J}'' = \{j : \omega^{(j)} \in \mathcal{V}''\},$$

and

$$\mathcal{J}'_> = \{j \in \mathcal{J}' : p_{N,j} \geq 0\}.$$

Let us show $\mathcal{J}'_> \neq \varnothing$. If it were $\mathcal{J}'_> = \varnothing$, i.e., $p_{N,j} < 0$ for all $j \in \mathcal{J}'$, it would follow from Lemma 5.1.1 that $L_q(\omega) \otimes L_q(\omega^{(j)})^*$ is simple for all $j \in \mathcal{J}'$ and, hence, $L_q(\omega) \otimes L_q(\pi')^*$ would be simple by Corollary 4.1.6. Let us show this contradicts Proposition 4.5.1. Indeed, let $\lambda = \omega$, $\mu = \pi''\omega^{-1}$, $v = \pi'$. We have just argued that $L_q(\lambda) \otimes L_q(v)^*$ would be simple if $\mathcal{J}'_> = \varnothing$. In the notation of Proposition 4.5.1, notice U_1 is highest-ℓ-weight by Corollary 4.1.6 since ω is the source and $T_1 = L_q(\pi'') \otimes L_q(\pi')$ is highest-

ℓ-weight by assumption. Hence, Proposition 4.5.1 would imply W_1 is also highest-ℓ-weight, contradicting (5.3.3).

If $j \in \mathcal{I}'_>$ and $k > j$, it follows from Lemma 5.1.2 that $p_{k,j} > p_{N,j} \geq 0$ and, hence, $i_j \neq i_k$ by (5.2.2). This shows

$$i_j \neq i_k \quad \text{for all} \quad j, k \in \mathcal{I}'_>, \ j \neq k,$$

and, therefore, there exists unique $j' \in \mathcal{I}'_>$ such that

$$0 < d(i_{j'}, i_N) = \min\{d(i_j, i_N) : j \in \mathcal{I}'_>\}.$$

Set

$$\mathcal{I}''_> = \{j \in \mathcal{I}'' : j' < j \text{ and } p_{j,j'} \geq 0\}$$

and note $N \in \mathcal{I}''_>$. Proceeding as above, one easily checks that

$$l \in \mathcal{I}''_> \text{ and } j' < k < l \quad \Rightarrow \quad i_k \neq i_l,$$

which implies $i_k \neq i_l$ for all $k, l \in \mathcal{I}''_>, k \neq l$. Let then $j'' \in \mathcal{I}''_>$ be the unique element such that

$$0 < d(i_{j'}, i_{j''}) = \min\{d(i_{j'}, i_j) : j \in \mathcal{I}''_>\}$$

and set $J = [i_{j'}, i_{j''}]$. By construction (5.2.5) holds. Since U is highest-ℓ-weight, Lemma 5.2.1 then implies there exist $j''_1 < j'_1$ such that $J_1 := [i_{j'_1}, i_{j''_1}] \subsetneqq J$, $\omega^{(j'_1)} \in \mathcal{V}_{G'}$, $\omega^{(j''_1)} \in \mathcal{V}_{G''}$, and $m_{j'_1} - m_{j''_1} \in \mathscr{R}^{r_{j'_1}, r_{j''_1}}_{i_{j'_1}, i_{j''_1}, J_1}$.

Since V is also highest-ℓ-weight, Lemma 5.2.1 with V in place of U, j''_1 in place of j' and j'_1 in place of j'', would imply there exist $j'_2 < j''_2$ such that $J_2 := [i_{j'_2}, i_{j''_2}] \subsetneqq J_1$, $\omega^{(j'_2)} \in \mathcal{V}_{G'}, \omega^{(j''_2)} \in \mathcal{V}_{G''}$, and $m_{j''_2} - m_{j'_2} \in \mathscr{R}^{r_{j'_2}, r_{j''_2}}_{i_{j'_2}, i_{j''_2}, J_2}$. The same lemma with j'_2 in place of j' and j''_2 in place of j'' and so on would give rise to an infinite sequence $J \supsetneqq J_1 \supsetneqq J_2 \supsetneqq \cdots$ and, hence, the desired contradiction.

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