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ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES

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Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. Let $A(n)$ be the number of elements of A not exceeding n, and denote by $R_2(n)$ the number of solutions of $a_i + a_j = n, i \leqslant j$. In 1986, Erdős, Sárközy and Sós proved that if $(n - A(n))/\log n \to \infty (n \to \infty)$, then

$$\limsup \sum_{k=1}^{N} \bigl(R_2(2k) - R_2(2k+1)\bigr) = +\infty.$$

In this paper, we generalise this theorem and give its quantitative form. For example, one of our conclusions implies that if $\limsup(n - A(n))/\log n = \infty$, then

$$\max_{n \leqslant N^2} \sum_{k=1}^{n} \bigl(R_2(2k) - R_2(2k+1)\bigr) \geqslant 0.004 \min\Bigl\{A(N), \bigl(N - A(N)\bigr)/\log N\Bigr\}$$

for infinitely many positive integers N.

1. INTRODUCTION

Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. Put $A(n) = \sum_{a \leqslant n, a \in A} 1$. For each positive integer n, let $R(n), R_1(n), R_2(n)$ denote the number of solutions of

$$\begin{aligned} x + y = n, &\quad x, y \in A, \\ x + y = n, &\quad x < y, x, y \in A, \\ x + y = n, &\quad x \leqslant y, x, y \in A, \end{aligned}$$

respectively. In [**3, 4**], Erdős, Sárközy examined the possible order of growth of the function $R(n)$ in comparison with that of functions such as $\log n$ or $\log n \log\log n$. In [**7**], Erdős, Sárközy and Sós showed that under certain assumptions on A, $|R(n+1) - R(n)|$ cannot be bounded. In [**5, 6**], Erdős et al studied the monotonicity properties of the functions $R(n), R_1(n), R_2(n)$. Continuing the work of Erdős, Sárközy and

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Sós; Balasubramanian [1] concluded: If $R_2(n+1) \geqslant R_2(n)$ for large n, then $A(N) = N + O(\log N)$, and if $R_1(n+1) \geqslant R_1(n)$ for large n, then $\sum_{a\in A} e^{-a/N} \gg N/\log N$. For the other related problems, see [2, 8, 9].

Erdős, Sárközy and Sós [6] proved that if $(n - A(n))/\log n \to \infty (n \to \infty)$, then $\limsup \sum_{k=1}^{N} (R_2(2k) - R_2(2k+1)) = +\infty$. Balasubramanian [1] remarked that his method can be employed to prove the same theorem. In this paper, we generalise this theorem and give its quantitative form.

THEOREM. *Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers, $N_0 > e$ be a positive integer such that*

$$\max_{n\leqslant m(N)} \sum_{k\leqslant n} (R_2(2k) - R_2(2k+1)) < \frac{1}{36} A(N) \tag{1}$$

for all $N \geqslant N_0$, where $m(N) = N(\log N + \log\log N)$. Then there exists an N_1 such that

$$\max_{n\leqslant m(N)} \sum_{k\leqslant n} (R_2(2k) - R_2(2k+1)) \geqslant \frac{1}{80e} \frac{N - A(N)}{\log N} - \frac{11}{4} - \frac{1}{8} N_1$$

for all $N \geqslant N_1$.

From the theorem, we may easily derive the following corollaries:

COROLLARY 1. *Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers such that*

$$\lim_{N\to+\infty} \frac{N - A(N)}{\log N} = +\infty.$$

Then at least one of the following statements is true:

(i) *for infinitely many positive integers N, we have*

$$\max_{n\leqslant m(N)} \sum_{k\leqslant n} (R_2(2k) - R_2(2k+1)) \geqslant \frac{1}{36} A(N);$$

(ii) *for all sufficiently large positive integers N, we have*

$$\max_{n\leqslant m(N)} \sum_{k\leqslant n} (R_2(2k) - R_2(2k+1)) \geqslant \frac{1}{240} \frac{N - A(N)}{\log N}.$$

COROLLARY 2. *Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers such that*

$$\limsup_{N\to+\infty} \frac{N - A(N)}{\log N} = +\infty.$$

Then

$$\limsup_{N\to+\infty} \sum_{k=1}^{N} (R_2(2k) - R_2(2k+1)) = +\infty.$$

Corollary 3. *Let $A = \{a_1, a_2, \ldots\}$ ($a_1 < a_2 < \cdots$) be an infinite sequence of positive integers such that*

$$\limsup_{N\to+\infty} \frac{N - A(N)}{\log N} = +\infty.$$

Then

$$\max_{n\leqslant m(N)} \sum_{k\leqslant n} \big(R_2(2k) - R_2(2k+1)\big) \geqslant \min\left\{\frac{1}{36}A(N), \frac{1}{240}\frac{N-A(N)}{\log N}\right\}$$

for infinitely many positive integers N.

2. Proofs

Lemma 1. ([**1**, Lemma 5.11].) *We have*

$$(1-d_1)^{d_2} \geqslant 1 - 2d_1d_2, \qquad \text{if } 0 < d_1 < 1/2, d_2 > 0.$$

Lemma 2. *Define $f(\alpha) = \sum_{a\in A} \alpha^a$, $0 < |\alpha| < 1$. Then*

$$f(\alpha^2) = \frac{1-\alpha}{2\alpha}\big(f(\alpha)\big)^2 - \frac{1+\alpha}{2\alpha}\big(f(-\alpha)\big)^2 + 2\sum_{k=1}^{\infty}\big(R_2(2k) - R_2(2k+1)\big)\alpha^{2k}.$$

Proof: Let $\delta(n)$ be an arithmetic function such that $\delta(n) = 1$ if $n = 2a$ for some $a \in A$, otherwise, $\delta(n) = 0$. Since $\big(f(\alpha)\big)^2 = \sum_{k=2}^{\infty} R(k)\alpha^k$, we have

$$\begin{aligned}
\big(f(-\alpha)\big)^2 &= \sum_{k=1}^{\infty} R(2k)\alpha^{2k} - \sum_{k=1}^{\infty} R(2k+1)\alpha^{2k+1} \\
&= 2\sum R_2(2k)\alpha^{2k} - 2\sum R_2(2k+1)\alpha^{2k+1} - f(\alpha^2) \\
&= U(\alpha) + 2\sum R_2(2k)(\alpha^{2k} - \alpha^{2k+1}) \\
&= U(\alpha) + (1-\alpha)\Big(\sum R_2(2k)\alpha^{2k} + \sum R_2(2k+1)\alpha^{2k+1}\Big) \\
&\qquad + (1-\alpha)\Big(\sum R_2(2k)\alpha^{2k} - \sum R_2(2k+1)\alpha^{2k+1}\Big) \\
&= U(\alpha) + (1-\alpha)\sum R_2(k)\alpha^k + (1-\alpha)\sum R_2(k)(-\alpha)^k \\
&= U(\alpha) + \frac{1-\alpha}{2}\sum\big(R(k)\alpha^k + \delta(k)\alpha^k\big) + \frac{1-\alpha}{2}\sum\big(R(k)(-\alpha)^k + \delta(k)(-\alpha)^k\big) \\
&= U(\alpha) + \frac{1-\alpha}{2}\big(f(\alpha)\big)^2 + \frac{1-\alpha}{2}\big(f(-\alpha)\big)^2 + 2\times\frac{1-\alpha}{2}f(\alpha^2),
\end{aligned}$$

where $U(\alpha) = 2\sum_{k=1}^{\infty}\big(R_2(2k) - R_2(2k+1)\big)\alpha^{2k+1} - f(\alpha^2)$. Hence

$$f(\alpha^2) = \frac{1-\alpha}{2\alpha}\big(f(\alpha)\big)^2 - \frac{1+\alpha}{2\alpha}\big(f(-\alpha)\big)^2 + 2\sum_{k=1}^{\infty}\big(R_2(2k) - R_2(2k+1)\big)\alpha^{2k}.$$

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This completes the proof of Lemma 2. ∎

Lemma 3. *Let $x \geqslant e$ and $m(x) = x(\log x + \log\log x)$. Then*

$$\sum_{k=1}^{\infty}\big(R_2(2k)-R_2(2k+1)\big)e^{-(2k/x)} < \frac{5}{2} + \max_{n\leqslant m(x)}\sum_{k\leqslant n}\big(R_2(2k)-R_2(2k+1)\big).$$

Proof: Let l be an integer with $l-1 \leqslant m(x) < l$, and

$$\beta = e^{-2/x}, \quad \sigma_n = \sum_{k\leqslant n}\big(R_2(2k)-R_2(2k+1)\big), \quad n = 1,2\cdots,l-1.$$

Then

$$\beta^l < \beta^{m(x)} = e^{-2(\log x+\log\log x)} \leqslant x^{-2}(\log x)^{-1}.$$

By Abel's Lemma, we have

$$\begin{aligned}\sum_{k=1}^{l-1}\big(R_2(2k)-R_2(2k+1)\big)e^{-(2k/x)} &= (\beta-\beta^2)\sigma_1+\cdots+(\beta^{l-2}-\beta^{l-1})\sigma_{l-2}+\beta^{l-1}\sigma_{l-1}\\ &\leqslant (\beta-\beta^2+\cdots+\beta^{l-2}-\beta^{l-1}+\beta^{l-1})\max_{n\leqslant l-1}\sigma_n\\ &= \beta\max_{n\leqslant l-1}\sigma_n\\ &< \max_{n\leqslant m(x)}\sigma_n.\end{aligned}$$

Since $R_2(2k) \leqslant k$ and $1/(1-\beta) \leqslant x$, we have

$$\begin{aligned}\sum_{k=l}^{\infty}\big(R_2(2k)-R_2(2k+1)\big)e^{-(2k/x)} &\leqslant \sum_{k=l}^{\infty}k\beta^k = \frac{l\beta^l}{1-\beta}+\frac{\beta^{l+1}}{(1-\beta)^2}\\ &< \frac{x(m(x)+1)+x^2}{x^2\log x}\\ &= \frac{x(\log x+\log\log x+1)+1}{x\log x}\\ &< \frac{5}{2}.\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty}\big(R_2(2k)-R_2(2k+1)\big)e^{-(2k/x)} < \frac{5}{2} + \max_{n\leqslant m(x)}\sum_{k\leqslant n}\big(R_2(2k)-R_2(2k+1)\big).$$

This completes the proof of Lemma 3. ∎

Proof of Theorem: Let $\psi(x) = f(e^{-(1/x)})$, $x > 0$, where

$$f(\alpha) = \sum_{a\in A}\alpha^a, \ 0 < |\alpha| < 1.$$

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Put $\alpha = e^{-(1/N)}$. Then $f(\alpha) = \psi(N), f(\alpha^2) = \psi(N/2)$. Note that

$$\frac{1-\alpha}{\alpha} = \frac{1}{N} + \frac{1}{2!}\frac{1}{N^2} + \cdots \leqslant \frac{1}{N} + \frac{1}{N^2}(e-2) < \frac{1}{N} + \frac{1}{N^2},$$

$$\psi(N) = \sum_{a\in A}\alpha^a \leqslant \sum_{n=1}^{\infty}\alpha^n = \frac{\alpha}{1-\alpha} = \frac{1}{e^{1/N}-1} < N,$$

by Lemma 2 we have

$$\psi(N/2) < \frac{1}{2N}\big(\psi(N)\big)^2 + \frac{1}{2} + 2\sum_{k=1}^{\infty}\big(R_2(2k) - R_2(2k+1)\big)e^{-(2k/N)}.$$

Thus

$$\big(\psi(N)\big)^2 > 2N\psi(N/2) - N - 4N\sum_{k=1}^{\infty}\big(R_2(2k) - R_2(2k+1)\big)e^{-(2k/N)}.$$

Let

$$g(x) = 11 + 4\max_{n\leqslant m(x)}\sum_{k\leqslant n}\big(R_2(2k) - R_2(2k+1)\big), x \geqslant e.$$

It is clear that $g(x)$ is a monotone increasing function and $g(x) \geqslant g(e) > 0$. By Lemma 3, we have

$$(2) \qquad \big(\psi(N)\big)^2 > 2N\psi(N/2) - Ng(N).$$

Note that

$$\psi(N/2) = \sum_{a\in A}e^{(-2a/N)} > \sum_{a\in A, a\leqslant N}e^{(-2a/N)} > e^{-2}\sum_{a\in A, a\leqslant N}1 = e^{-2}A(N),$$

by (1) there exists an $N_2 > N_0$ such that for $N \geqslant N_2$ we have $\psi(N) > 1$ and

$$(3) \qquad g(N) < 11 + \frac{4}{36}A(N) < \frac{1}{8.8}A(N) < \psi(N/2).$$

Then by (2) and (3) we have

$$\psi(N) > N^{1/2}\psi(N/2)^{1/2}, \qquad \text{if } N \geqslant N_2,$$
$$\psi(N/2) > (N/2)^{1/2}\psi(N/4)^{1/2}, \qquad \text{if } N/2 \geqslant N_2$$

and so on. Choosing λ such that

$$N_2 \leqslant \frac{N}{2^\lambda} \leqslant 2N_2,$$

we have

$$\begin{aligned}\psi(N) &> N^{1/2}(N/2)^{1/4}\cdots(N/2^{\lambda-1})^{1/2^\lambda}\big(\psi(N/2^\lambda)\big)^{1/2^\lambda}\\ &\geqslant \frac{1}{2}N^{1-1/2^\lambda}\big(\psi(N/2^\lambda)\big)^{1/2^\lambda}\\ &\geqslant \frac{1}{2}N^{1-1/2^\lambda}.\end{aligned}$$

Noting that $N^{1/2^\lambda} \leqslant N^{2N_2/N} \leqslant 2$ for $N \geqslant N_3$, where N_3 is a constant with $N_3 > 2N_2$, we have $\psi(N) > N/4$ for all $N \geqslant N_3$. By (2), for all $N \geqslant 2N_3$,

$$\big(\psi(N)\big)^2 > 2N\psi(N/2)\Big(1 - \frac{g(N)}{2\psi(N/2)}\Big) > 2N\psi(N/2)\Big(1 - \frac{4g(N)}{N}\Big),$$

that is,

$$\psi(N) > (2N)^{1/2}(\psi(N/2))^{1/2}\Big(1 - \frac{4g(N)}{N}\Big)^{1/2}.$$

By (3) we have $g(N) < N/8.8$ for all $N \geqslant N_2$. So there exists an $N_4(\geqslant 2N_3)$ such that $4g(N) + 2N_4 \leqslant N/2$ for all $N \geqslant N_4$. Let $g_1(N) = 4g(N) + 2N_4$. Then for $N \geqslant N_4$,

$$\psi(N) > (2N)^{1/2}(\psi(N/2))^{1/2}\Big(1 - \frac{g_1(N)}{N}\Big)^{1/2}.$$

For $N \geqslant N_4$, choose an integer μ such that

$$1/4 < \frac{2^{\mu-1}g_1(N)}{N} \leqslant 1/2.$$

Then

$$\frac{N}{2^\mu} \geqslant g_1(N) > 2N_4.$$

Since $g_1(N) \leqslant N/2$, we have $\mu \geqslant 1$. If $\mu \geqslant 2$, then

$$\psi(N/2) > N^{1/2}\Big(\psi(N/4)\Big)^{1/2}\Big(1 - \frac{2g_1(N/2)}{N}\Big)^{1/2} \geqslant N^{1/2}\Big(\psi(N/4)\Big)^{1/2}\Big(1 - \frac{2g_1(N)}{N}\Big)^{1/2}.$$

By Lemma 1, we have

$$\psi(N) > (2N)^{1/2}(N)^{1/4}\Big(\psi(N/4)\Big)^{1/4}\Big(1 - \frac{g_1(N)}{N}\Big)^2.$$

Proceeding similarly, by Lemma 1, we have

$$\begin{aligned}
\psi(N) &> (2N)^{1/2}N^{1/4}(N/2)^{1/8}\cdots\Big(N/2^{\mu-2}\Big)^{1/2^\mu}\Big(\psi(N/2^\mu)\Big)^{1/2^\mu}\Big(1 - \frac{g_1(N)}{N}\Big)^\mu \\
&\geqslant 2^{\mu/2^\mu}N^{1-(1/2^\mu)}\Big(1 - \frac{g_1(N)}{N}\Big)^\mu \\
&\geqslant 2^{\mu/2^\mu}N^{1-(1/2^\mu)}\Big(1 - \frac{2\mu g_1(N)}{N}\Big) \\
&> N\Big(1 - \frac{1}{2^\mu}\log N\Big)\Big(1 - \frac{2\mu g_1(N)}{N}\Big) \\
&> N - \frac{N}{2^\mu}\log N - 2\mu g_1(N) \\
&\geqslant N - 2g_1(N)\log N - \frac{2}{\log 2}g_1(N)\log N \\
&> N - 5g_1(N)\log N.
\end{aligned}$$

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Let $\chi(n)$ be the characteristic function of set A. Then

$$\sum_{n=1}^{\infty}\chi(n)e^{-(n/N)} > N - 5g_1(N)\log N > \sum_{n=1}^{\infty} e^{-(n/N)} - 5g_1(N)\log N.$$

Hence

$$e^{-1}\sum_{n\leqslant N}\big(1-\chi(n)\big) \leqslant \sum_{n\leqslant N}\big(1-\chi(n)\big)e^{-(n/N)} < 5g_1(N)\log N.$$

Thus

$$g_1(N) > \frac{1}{5e}\frac{N-A(N)}{\log N}.$$

That is,

$$\max_{n\leqslant m(N)}\sum_{k\leqslant n}\big(R_2(2k)-R_2(2k+1)\big) > \frac{1}{80e}\frac{N-A(N)}{\log N} - \frac{11}{4} - \frac{1}{8}N_4$$

for all $N \geqslant N_4$.

This completes the proof of the theorem. ∎

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