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# Approximate Amenability of Segal Algebras II

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*Abstract.* We prove that no proper Segal algebra of a SIN group is approximately amenable.

A Banach algebra $A$ is called *amenable* if every bounded derivation $D\colon A \to X^*$ for any dual Banach $A$-bimodule $X^*$ is inner. A weaker version of amenability is approximate amenability. Precisely, a Banach algebra $A$ is *approximately amenable* if and only if every bounded derivation $D\colon A \to X^*$ for every Banach dual $A$-bimodule $X^*$ can be approximated by a net of inner derivations. The concept of approximate amenability was first introduced and studied in [6]. It has been shown in [7] that if $A$ is approximately amenable, then every bounded derivation from $A$ into any Banach $A$-bimodule $X$ can be approximated by a net of inner derivations.

Different algebras have been studied for their approximate amenability. These include *Segal algebras* (for the definition of Segal algebras and their basic properties, see [11]). In [4], it was shown that the *Feichtinger Segal algebra* on an infinite compact abelian group is not approximately amenable. In [5], Dales and Loy studied some specific Segal algebras on the commutative groups $\mathbb{T}$ and $\mathbb{R}$. They proved that these Segal algebras are not approximately amenable; they also conjectured that the same should be true for every *proper* Segal algebra on these groups. Here we call a Segal algebra proper if it is not equal to the group algebra.

Later, Choi and Ghahramani [3] proved this conjecture by showing that proper Segal algebras on $\mathbb{T}^d$ and $\mathbb{R}^d$ are not approximately amenable (for any dimension $d$). To do so, they developed a criterion for "ruling out approximate amenability" of Banach algebras. At the Banach algebra 2011 conference, Ghahramani conjectured that no proper Segal algebra of a locally compact group can be approximately amenable.

In [1], the author applied the criterion developed in [3] to show that, in fact, no proper Segal algebra of a locally compact abelian group is approximately amenable. Also, applying the hypergroup structure on the dual of compact groups, it was proved that for some classes of compact groups, including SU(2), no proper Segal algebra is approximately amenable.

In this short paper we prove that this conjecture is actually true for every SIN group. Recall that a locally compact group $G$ is called a *SIN group* if there exists a topological basis of conjugate invariant neighbourhoods of the identity element of the group $G$. This class of locally compact groups includes abelian, compact, and discrete groups.

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***Theorem 1*** *No proper Segal algebra of a SIN group is approximately amenable.*

We prove this theorem for a generalized version of Segal algebras called *abstract Segal algebras*. A Banach algebra $(B, \|\cdot\|_B)$ is an abstract Segal algebra of a Banach algebra $(A, \|\cdot\|_A)$ if $B$ is a dense left ideal in $\mathcal{A}$, there exists $C > 0$ such that $\|b\|_{\mathcal{A}} \le C\|b\|_B$ (for each $b \in B$), and there exists $M > 0$ such that $\|ab\|_B \le M\|a\|_{\mathcal{A}}\|b\|_B$ for all $a, b \in B$. We call $B$ a *proper* abstract Segal algebra of $A$ if in addition $B \ne A$. It is clear that every (proper) Segal algebra of a locally compact group $G$ is a (proper) abstract Segal algebra of $L^1(G)$.

Let $A$ be a commutative Banach algebra, and let $\Delta(A)$ denote the *Gelfand spectrum* of $A$. For each $a \in A$, let $\widehat{a}$ be the Gelfand transform of $a$. For definitions related to the Gelfand spectrum of commutative Banach algebras, we refer the reader to [8]. We denote by $A_c$ the set of all elements $a \in A$ such that $\operatorname{supp}(\widehat{a})$ is compact. A semisimple commutative Banach algebra $A$ is called a *Tauberian algebra* when $A_c$ is dense in $A$.

For a semisimple regular commutative Banach algebra $A$ with an abstract Segal algebra $B$, it is known that $B$ is also semisimple and regular and $\Delta(B) = \Delta(A)$; see [2]. Moreover, $B$ contains $A_c$. For a proof of the latter fact, see [1, Proposition 2.2].

For a Banach algebra $A$ and a constant $D > 0$, $A$ has a *D-bounded approximate identity* if there is a net $(e_\alpha)_\alpha \subseteq A$ such that for every $a \in A$, $\|ae_\alpha - a\|_A \to 0$, $\|e_\alpha a - a\|_A \to 0$, and $\sup_\alpha \|e_\alpha\|_A \le D$. Note that if $A$ is a unital commutative Banach algebra with the unit $e \in A$, then $e$ is constantly one on $\Delta(A)$. The following lemma shows that a bounded approximate identity approximately plays a similar role for a regular Tauberian algebra.

***Lemma 2*** *Let $A$ be a regular commutative Tauberian Banach algebra. Then $A$ has a $D$-bounded approximate identity if and only if for each compact set $K \subseteq \Delta(A)$ and $\epsilon > 0$, there is some $a_{K,\epsilon} \in A$ with a compact support such that $\|a_{K,\epsilon}\|_A \le D + \epsilon$ and $\widehat{a}_{K,\epsilon}|_K \equiv 1$.*

**Proof** Suppose that $(e_\alpha)_\alpha$ is a bounded approximate identity of $\mathcal{A}$ such that $\|e_\alpha\|_A \le D$ for some $D > 0$. For each compact subset $K \subseteq \Delta(A)$, let $b_K \in A_c$ such that $b_K|_K \equiv 1$ and let $I_K$ be the ideal $\{b \in A : \widehat{b}(K) = \{0\}\}$. Therefore, for each $b \in A$, $bb_K - b \in I_K$. Considering the quotient norm of $A/I_K$, one gets $\|b_K + I_K\|_{A/I_K} = \lim_\alpha \|b_K e_\alpha + I_K\|_{A/I_K} = \lim_\alpha \|e_\alpha + I_K\|_{A/I_K} \le D$. Recall that $I_K \cap A_c$ is dense in $I_K$. So, there is some $b \in A_c \cap I_K$ such that $\|b_K + b\|_A < D + \epsilon$. Note that for $a_K := (b_K + b)$, $a_K|_K \equiv 1$ and $a_K \in A_c$.

Conversely, for each $\epsilon > 0$ and $K \subseteq \Delta(A)$ compact, let $a_K \in A$ such that $\widehat{a}_K|_K \equiv 1$ and $\|a_K\|_A \le D(1+\epsilon)$. Define $e_{K,\epsilon} := (1+\epsilon)^{-1} a_K$. It is not hard to show that $(e_{K,\epsilon})_{K,\epsilon}$ forms an approximate identity of the Tauberian algebra $A$ that is $\|\cdot\|_A$-bounded by $D$, where $\epsilon \to 0$ and $K \to \Delta(A)$. ∎

Let $B$ be an abstract Segal algebra with respect to a Banach algebra $A$ and let $A$ have a $\|\cdot\|_A$-bounded approximate identity. Then, by an approximation argument, one can show that $A$ has a $\|\cdot\|_A$-bounded approximate identity that lies in $B$. If $A$ is a Tauberian algebra, the existence of a $\|\cdot\|_A$-bounded approximate identity of $A$ leads to a $\|\cdot\|_A$-bounded approximate identity that belongs to $A_c$, while we know

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that $A_c \subseteq B$. The density condition and relation of the norms imply that a proper abstract Segal algebra never has a bounded approximate identity.

For a Banach algebra $A$, let $ZA$ denote the *center* of $A$, which is the commutative subalgebra of $A$ consisting of all elements $a \in A$ such that $ab = ba$ for every $b \in A$. The following proposition proves the non-approximate amenability of Segal algebras with notable centres.

***Proposition 3*** *Let $B$ be a proper abstract Segal algebra of a Banach algebra $A$ that has a central bounded approximate identity and $ZB$ is dense in $ZA$. If $ZA$ is a semisimple regular Tauberian algebra, then $B$ is not approximately amenable.*

**Proof** To prove that such an abstract Segal algebra is not approximately amenable, we apply the criterion developed in [3]. To do so, we construct a sequence $(a_n)_{n\in\mathbb{N}}$ in $B$ that is $\|\cdot\|_A$-bounded, $\|\cdot\|_B$-unbounded, and satisfies $a_n a_{n+1} = a_{n+1}a_n = a_n$ for every $n \in \mathbb{N}$. Given a fixed $\epsilon > 0$ and $K_0 \subseteq \Delta(ZA)$, for each compact set $K$ such that $K_0 \subseteq K \subseteq \Delta(ZA)$, there is some $a_K \in ZA_c$ such that $\widehat{a}_K|_K \equiv 1$ and $\|a_K\|_A \leq D + \epsilon$, by Lemma 2. Consider the $\|\cdot\|_A$-bounded net $(a_K)_{K_0\subseteq K\subseteq\Delta(ZA)}$ directed by inclusion over compact sets $K$. Then $a_{K_1}a_{K_2} = a_{K_1}$ if $\operatorname{supp}(\widehat{a}_{K_1}) \subseteq K_2$.

Note that $ZB$ is an abstract Segal algebra of $ZA$, and hence $ZA_c \subseteq ZB$. Recall that for each $K$, $a_K$ is compactly supported; consequently, $a_K$ belongs to $ZB$. We claim that $(a_K)_{K_0\subseteq K\subseteq\Delta(ZA)}$ is $\|\cdot\|_B$-unbounded. Note that $ZA$ is a Tauberian algebra. Therefore, $A$ has a $\|\cdot\|_A$-bounded approximate identity $(e_\alpha) \subseteq ZA_c \subseteq ZB$. So, for each $\alpha$, for $K = \operatorname{supp}(e_\alpha)$, $e_\alpha a_K = e_\alpha$. Hence,

$$\|e_\alpha\|_B = \lim_{K_0\subseteq K\to\Delta(ZA)} \|e_\alpha a_K\|_B \leq \limsup_{K_0\subseteq K} \|e_\alpha\|_A \|a_K\|_B.$$

Therefore, if $(a_K)_{K_0\subseteq K}$ is $\|\cdot\|_B$-bounded, then $(e_\alpha)_\alpha$ is a $\|\cdot\|_B$-bounded approximate identity of $A$ that violates the properness of $B$.

To generate a sequence that satisfies the desired conditions mentioned before, fix a non-empty compact set $K_0 \subseteq \Delta(ZA)$. By our claim, we inductively construct a sequence of compact sets $K_0 \subset K_1 \subset \cdots$ in $\Delta(ZA)$ such that $a_{K_n}a_{K_{n-1}} = a_{K_{n-1}}$ and $\|a_{K_n}\|_B \geq n$ for all $n \in \mathbb{N}$. Then $B$ is not approximately amenable. ∎

**Proof of Theorem 1** Note that for every SIN group $G$, $L^1(G)$ has a central bounded approximate identity. Moreover, for each Segal algebra $S^1(G)$, $ZS^1(G)$ is dense in $ZL^1(G)$, [9, Theorem 2]. On the other hand, [10, Theorem 1.8] implies that $ZL^1(G)$ is a semisimple regular commutative Tauberian algebra. So Proposition 3 can be applied to finish the proof. ∎

***Question*** Applying some results about structure of locally compact groups, Kotzmann and Rindler [9] showed that for every Segal algebra $S^1(G)$, $ZS^1(G)$ is dense in $ZL^1(G)$. The group structure in their proof is essential. It seems that there is no immediate argument to generalize this proof for a wider class of abstract Segal algebras. It would be of interest if one could generalize this result to abstract Segal algebras. In other words, is there an abstract Segal algebra whose centre is not dense in its ancestor?

***Question*** Is no proper Segal algebra of a locally compact group approximately amenable?

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