THE STRUCTURE OF POWERS OF NON-NEGATIVE MATRICES

A. L. DULMAGE AND N. S. MENDELSOHN

1. Introduction. The theory of non-negative matrices was initiated by Perron **(7)** and Frobenius **(4)**. Wielandt in **(11)** gives an elegant exposition of the subject.

It is well known to workers in the field that if a matrix A has all its entries non-negative real numbers, then the pattern of zeros and non-zeros of A completely determines the pattern of zeros and non-zeros in every power of A. Ptak in **(8)** and Ptak and Sedlacek in **(9)** describe this behaviour in terms of some combinatorial constructs.

The main object of this paper is to discuss the structure of powers of a reducible non-negative matrix in terms of properties of the directed graph of the matrix. The constituents of a reducible matrix are irreducible, and the structure of powers of irreducible matrices has been considered in **(3, 4, 8, 9, 11)**. In this paper there is a discussion of the structure of the subdiagonal blocks in powers of reducible matrices.

2. Definitions, notation, and background. A *finite directed graph* D has a *vertex* set $V = \{1, 2, \ldots, n\}$ and a set of *edges* each of which is an ordered pair (i, j) of vertices. We say that the edge (i, j) *joins* vertex i to vertex j. We say that i is joined to j by a *path* of length m in D if D has a set of m edges $(i, k_1)(k_1, k_2)(k_2, k_3) \ldots (k_{m-1}, j)$. A path of length m from vertex i to vertex i is called a *cycle* of length m. An edge (i, i) is a cycle of length 1. Such a cycle is called a *loop*. If (i, i) is an edge, the vertex i is a *loop* vertex. If the edges of a cycle are such that each vertex which occurs appears exactly once as the first member of an edge, the cycle is called a *circuit*. A directed graph D is *strongly connected* if, for any ordered pair of vertices i and j with $i \neq j$, there is a path in D from vertex i to vertex j. Thus a graph with one vertex and no edges is strongly connected.

The remarks in this section are well known. In some cases the proofs which are given are simpler than those found elsewhere.

REMARK 1. *In a directed graph D, the greatest common divisor of the lengths of all the cycles is equal to the greatest common divisor of the lengths of all the circuits.*

Proof. Since a circuit is a cycle, the g.c.d. of the cycle lengths divides the length of every circuit.

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Since a cycle is a union of circuits, the g.c.d. of the circuit lengths divides the length of every cycle.

If D is strongly connected, the g.c.d. of the cycle lengths is denoted by d and is called the *index of imprimitivity* of D.

REMARK 2. *Let D be a strongly connected directed graph. If d_i is the g.c.d. of the lengths of the cycles through vertex i, then d_i is equal to the index of imprimitivity of D.*

Proof. Let a cycle through vertex j have length m_j. Since D is strongly connected, there is a path from i to j and a path from j to i. Let such paths have lengths u and v respectively. Combining these two paths, we have a cycle of length $u + v$ through vertex i. Combining this cycle with the cycle of length m_j through vertex j we have a cycle of length $u + v + m_j$ through i. It follows that d_i divides both $u + v$ and $u + v + m_j$. Hence d_i divides m_j and thus d_i divides d_j. Similarly d_j divides d_i. Thus $d_i = d_j$ and Remark 2 follows.

REMARK 3. *Let D be a strongly connected directed graph with index of imprimitivity d. If two paths from vertex i to vertex j have lengths u and v respectively, then* $u \equiv v \pmod{d}$.

Proof. Let a path from j to i have length t. There are cycles through i of lengths $u + t$ and $v + t$ respectively. Thus $u + t \equiv v + t \equiv 0 \pmod{d}$.

Remark 3 enables us to partition the vertex set V of a strongly connected directed graph into disjoint non-null *sets of imprimitivity*. Select an arbitrary vertex, say vertex 1 for definiteness. For $k = 1, 2, 3, \ldots, d$, define a set I_k to consist of all vertices i such that the lengths of all the paths from vertex 1 to vertex i are congruent to $k \pmod{d}$. Vertex 1 is in I_d. The following remark is immediate.

REMARK 4. *If the strongly connected directed graph D has sets of imprimitivity $I_1, I_2, \ldots, I_d$, then the length of any path from vertex $i \in I_r$ to vertex $j \in I_s$ is congruent to $s - r$* $\pmod{d}$.

The theorem stated as Remark 5 is due to Schur. A proof is given in (**6**, pp. 6f).

REMARK 5. *A set of positive integers which is closed under addition contains all but a finite number of multiples of its greatest common divisor.*

REMARK 6. *Let D be a strongly connected directed graph with index d of imprimitivity. Then there exists an integer N such that if the vertices i and j belong respectively to the imprimitivity sets I_r and I_s, then there are paths from i to j of length $s - r + td$ for all $t \geqslant N$.*

Proof. There is a path from i to j and thus there exists a non-negative integer v_{ij} such that the length of this path is $s - r + v_{ij} d$. Now apply

Remark 5, considering the set of lengths of cycles through vertex j which is closed under addition. There exists N_j such that there are cycles of length vd through vertex j for all $v \geqslant N_j$. Taking N to be the maximum of $\{v_{ij} + N_j\}$ taken over all ordered pairs of vertices, the result follows.

In a directed graph D, two collections of sets of paths from vertex i to vertex j are said to be *equivalent* if the union of the sets of each collection contains all but a finite number of the paths of the union of the sets of the other collection. Let b be an integer and let u be a positive integer. A set of paths in a directed graph D is said to be a *(b, u)-sequence of paths from vertex i to vertex j* if and only if (1) for every path in the set there exists an integer t such that the length of the path is $b + tu$ and (2) there exists an integer N such that for every integer $t \geqslant N$ there is a path in the set of length $b + tu$. Thus a (b_1, u)-sequence is equivalent to a (b_2, u)-sequence if and only if $b_1 \equiv b_2 \pmod{u}$. In Remark 5 the set of paths from i to j is an $(s - r, d)$-sequence.

Let A be an n by n matrix whose entries a_{ij} are all real numbers. A is said to be *reducible* if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} C & O \\ E & D \end{bmatrix},$$

where O represents a zero matrix and C and D are square matrices. Otherwise, A is said to be *irreducible*. Thus an n by n matrix of zeros is irreducible if $n = 1$ and reducible if $n > 1$. If A is reducible, there is a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} C_1 & & & & & \\ C_{21} & C_2 & & & & \\ C_{31} & C_{32} & C_3 & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ C_{m1} & C_{m2} & C_{m3} & \cdot & \cdot\ \cdot & C_m \end{bmatrix},$$

where each submatrix C_p $(p = 1, 2, \ldots, m)$ is square and irreducible. The matrices C_p are called the *constituents* of A. The uniqueness of the constituents, apart from transformation by a permutation matrix, follows from the fact that the strong components of a directed graph are uniquely determined. An algorithm for finding these constituents has been given in (**2**) and in (**5**). The matrix $P^{-1}AP$ will be called a *canonical transform* of the reducible matrix A. If we partition the matrix $P^{-1}AP$ by means of the constituents, the submatrices above the diagonal of constituents are zero. The submatrices below the diagonal, denoted by C_{rs}, $r > s$, are called the *subdiagonal blocks* of A. Let the rows and columns of A be indexed $1, 2, \ldots, n$ from top to bottom and left to right in the usual manner. Let F_p, $p = 1, 2, \ldots, m$, denote the set of those rows or columns of A which are the rows or columns of C_p. Thus the row set of C_{rs} is F_r and the column set is F_s. Let $P^{-1}A^kP = (a_{ij}^{(k)})$ and let

$$P^{-1}A^kP = \begin{bmatrix} C_1^{(k)} & & & & & & \\ C_{21}^{(k)} & C_2^{(k)} & & & & & \\ C_{31}^{(k)} & C_{32}^{(k)} & C_3^{(k)} & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & \cdot & & & \cdot & \\ C_{m1}^{(k)} & C_{m2}^{(k)} & \cdot & \cdot & \cdot & & C_m^{(k)} \end{bmatrix}.$$

We have $C_p^{(k)} = C_p^k$.

A matrix A is *non-negative* if all its entries are zero or positive. A matrix A is *positive* if all its entries are positive and this is written $A > 0$. If a matrix A is non-negative and irreducible, then A is said to be *primitive* if and only if there exists a positive integer t such that $A^t > 0$. It is well known **(4, 7, 11)** that if A is non-negative and irreducible, then A is primitive if and only if A has a unique characteristic root of maximum modulus and if this root has multiplicity 1. A non-negative irreducible matrix A which is not primitive is called *imprimitive.*

The *directed graph* D_A of an n by n matrix A of reals has vertex set $V = (1, 2, \ldots, n)$. The ordered pair (i, j) is an edge of D_A if and only if $a_{ij} \neq 0$.

The following remark is well known **(1, 5, 10)**.

REMARK 7. *The matrix A is irreducible if and only if D_A is strongly connected.*

Let A be an irreducible matrix and let $I_1, I_2, \ldots, I_d$, $d > 1$, be the sets of imprimitivity of D_A. Clearly, there exists a permutation matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} O_1 & B_1 & & & & \\ \cdot & O_2 & B_2 & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & O_{d-1} & B_{d-1} \\ B_d & & & & & O_d \end{bmatrix}.$$

Here $O_1, O_2, \ldots, O_d$ are square zero matrices and, for $k = 1, 2, \ldots, d$, the set I_k, considered as a set of integers, is the row and column set of O_k in A. The matrices $B_1, B_2, \ldots, B_d$ are rectangular non-zero matrices. If A is an irreducible matrix for which $d > 1$, the matrix $Q^{-1}AQ$ will be called a *canonical transform* of the irreducible matrix A. In **(3)**, these matrices have been called the *cyclic components* of A with respect to Q. Every entry of $Q^{-1}AQ$ which is not in $B_1, B_2, \ldots$, or B_d is zero. The sets $I_1, I_2, \ldots, I_d$ may be called the sets of imprimitivity of the matrix A (as well as of D_A) and d may be called the *index of imprimitivity of* A. Clearly, if A is non-negative, d is the smallest power of $Q^{-1}AQ$ which has the form $\operatorname{diag}(A_1, A_2, \ldots, A_d)$. The matrices $A_1, A_2, \ldots, A_d$ have been called the *diagonal components* of A^d with respect to Q **(3)**. If f is a divisor of d, there exists a permutation matrix R such that $R^{-1}A^fR$ has the form $\operatorname{diag}(A_1^*, A_2^*, \ldots, A_f^*)$ **(3, 10)**.

REMARK 8. *If A is non-negative, the graph D_A has a path of length t from vertex i to vertex j if and only if $a_{ij}^{(t)} > 0$.*

REMARK 9. *If A is imprimitive with index of imprimitivity d and if $A_1, A_2, \ldots, A_d$ are the diagonal components of A^d with respect to Q, then A_p is primitive for $p = 1, 2, \ldots, d$.*

Proof. By Remark 6, if $i \in I_p$ and $j \in I_p$ we have paths from i to j in D of lengths td for all $t \geqslant N_p$. By Remark 6, $a_{ij}^{(td)} > 0$ for i and j in I_p and $t \geqslant N_p$. Thus $A_p^t > 0$ for $t \geqslant N_p$.

Remark 9 has been generalized in **(3)**.

In a directed graph D each vertex belongs to exactly one maximal strongly connected subgraph. These subgraphs are called the *strong components* of D. We denote them by $D_1, D_2, \ldots, D_m$; let F_p be the vertex set of D_p. The directed graph D^* induced by the partitioning $V = F_1 + F_2 + \ldots + F_m$ is defined as follows. The vertices of D^* are $F_1, F_2, \ldots, F_m$ and an ordered pair (F_r, F_s) is an edge of D^* if and only if there exists $i \in F_r, j \in F_s$ such that (i, j) is an edge of D. It is well known **(1, 5)** that $F_1, F_2, \ldots, F_m$ may be so indexed that (F_r, F_s) is an edge of D^* only if $r \geqslant s$. If $m > 1$, D^* is not strongly connected, for with this indexing there is no path from F_1 to any other vertex. There are m strong components of D^* each consisting of a single vertex.

Now let A be a reducible matrix with constituents $C_1, C_2 \ldots, C_m$ and let D_A be the directed graph of A. The strong components of D_A are the subgraphs D_{C_p} and the indexing may be arranged so that F_p is the vertex set of D_{C_p}, $p = 1, 2, \ldots, m$. The set F_p can be thought of either as the vertex set of D_{C_p}, or as the row or column set of C_p. If D_A^* is the graph induced from D_A by the partitioning $V = F_1 + F_2 + \ldots + F_p$, it follows that (F_r, F_s), $r > s$, is an edge of D_A^* if and only if $C_{rs} \neq 0$ and that (F_r, F_r) is an edge if and only if $C_r \neq 0$.

If $C_p \neq 0$, let d_p be the index of imprimitivity of C_p. If $C_p = 0$, d_p is undefined. It is worth noting, in this connection, that if $C_p = 0$, then C_p is a 1 by 1 matrix, since, if $n > 1$, an n by n zero matrix is reducible.

3. The structure of powers of a non-negative reducible matrix. Let A be a reducible matrix with constituents $C_1, C_2, \ldots, C_m$ and subdiagonal blocks C_{rs}, $r > s$. The matrix $C_p^{(k)}$ is equal to the kth power of C_p and C_p is irreducible. The structure of powers of irreducible matrices has been discussed in **(3, 8, 11)**. In this section, the structure of subdiagonal blocks $C_{rs}^{(k)}$ which occur in the kth power of $P^{-1}AP$ is discussed.

THEOREM 1. *Let A be a reducible non-negative matrix with constituents $C_1, C_2, \ldots, C_m$ and let $F_1, F_2, \ldots, F_m$ be the row (or column) sets of these constituents. Let r and s be a pair of integers $1 \leqslant r < s \leqslant m$. Then exactly one of the following four alternatives holds:*

(1) $C_{rs}^{(k)} = 0$ *for all* k.

(2) *For some* k, $C_{rs}^{(k)} \neq 0$, *but there exists* N *such that* $C_{rs}^{(k)} = 0$ *for* $k > N$.

(3) *There exists* N *such that* $C_{rs}^{(k)} > 0$ *for* $k > N$.

(4) *Corresponding to every integer* N *and to every pair* i, j, *with* $i \in F_r$ *and* $j \in F_s$ *there exists* $k_1 > N$, $k_2 > N$ *such that* $a_{ij}^{(k_1)} = 0$ *and* $a_{ij}^{(k_2)} > 0$.

Proof. The four alternatives are mutually exclusive. It remains only to show that they are exhaustive. To this end we show that:

(a) if, for $i \in F_r, j \in F_s$ there exists N such that $a_{ij}^{(k)} > 0$ for $k > N$, then for every pair of vertices $i_0 \in F_r$ and $j_0 \in F_s$ there exists M such that $a_{i_0 j_0}^{(k)} > 0$ for $k > M$, and

(b) if, for $i \in F_r, j \in F_s$ there exists N such that $a_{ij}^{(k)} = 0$ for $k > N$, then for every pair of vertices $i_0 \in F_r$ and $j_0 \in F_s$ there exists M such that $a_{i_0 j_0}^{(k)} = 0$ for $k > M$.

To prove (a) note, since D_{C_r} is strongly connected, that if $i \neq i_0$, there is a path in D_{C_r} from i_0 to i. Now define $u(i_0, i)$ as follows. If $i \neq i_0$, $u(i_0, i)$ is the length of the shortest path from i_0 to i and if $i = i_0$, $u(i_0, i) = 0$. Similarly, define $v(j, j_0)$ to be the length of the shortest path from j to j_0 if $j \neq j_0$ and to be zero if $j = j_0$. Thus for $k > N + u(i_0, i) + v(j, j_0)$ there is a path from i_0 to j_0 of length k. Thus, if $M = N + u(i_0, i) + v(j, j_0)$ we have $a_{i_0 j_0}^{(k)} > 0$ for $k > M$. (b) follows at once, also.

If none of (1), (2), or (3) holds, then at least one pair of vertices i and j, $i \in F_r$, $j \in F_s$, has the property that to every N there corresponds $k_1 > N$ and $k_2 > N$ such that $a_{ij}^{(k_1)} = 0$ and $a_{ij}^{(k_2)} > 0$. From (a) and (b) it follows that every pair of vertices i and j, $i \in F_r$, $j \in F_s$, has this property. Thus alternative (4) holds.

In the induced graph $D_A{}^*$, F_s is not a loop vertex if and only if C_s is a 1 by 1 zero matrix. Thus F_s is a loop vertex if and only if d_s is defined. The following theorem concerns a path in $D_A{}^*$ none of the edges of which is a loop and at least one vertex of which is a loop vertex.

THEOREM 2. *If* $(F_{p_1}, F_{p_2}), (F_{p_2}, F_{p_3}), \ldots (F_{p_{t-1}}, F_{p_t})$ *is a path in* $D_A{}^*$, $p_1 > p_2 > \ldots > p_t$, *and if* $C_{p_q} \neq 0$ *for at least one* q, *and if* u *is the greatest common divisor of those* d_{p_q} *which are defined, then, for every pair* i, j, $i \in F_{p_1}$, $j \in F_{p_t}$, *there exist integers* b *and* N *such that* $a_{ij}^{(k)} > 0$ *for* $k = b + wu$ *and* $w > N$. *If* $u = 1$, *then alternative* (3) *of Theorem* 1 *holds.*

Proof. There exists an entry $a_{i_1 i_2}$ of $C_{p_1 p_2}$ which is >0, $a_{i_3 i_4} > 0$ in $C_{p_2 p_3}$, $a_{i_5 i_6} > 0$ in $C_{p_3 p_4}, \ldots$, and $a_{i_{2t-3} i_{2t-2}} > 0$ in $C_{p_{t-1} p_t}$. If $C_{p_1} = 0$, then $i = i_1$ and if $C_{p_1} \neq 0$ and $i \neq i_1$, then since $D_{C_{p_1}}$ is strongly connected, there exists a path in $D_{C_{p_1}}$ from i to i_1. If $i_2 \neq i_3$, then $C_{p_2} \neq 0$ and there exists a path in $D_{C_{p_2}}$ from i_2 to i_3. Continuing, finally we have $i_{2t-2} = j$ or a path in $D_{C_{p_t}}$ from i_{2t-2} to j. Let the length of the resulting path from i to j be b.

By Remarks 3 and 5, for each $q = 1, 2, \ldots, t$ such that $C_{p_q} \neq 0$ there exists N_q such that for $w_q > N_q$, $D_{C_{p_q}}$ has a cycle of length $w_q d_q$ through i_{2q-2} for $q = 2, 3, \ldots, t$ and through i_1 for $q = 1$. Thus we have paths from i to j of length $b + \sum w_q d_{p_q}$ for all $w_q > N_q$ where the summation is over all $q = 1, 2, \ldots, t$ for which $C_{p_q} \neq 0$. By Remark 5, there exists a (b, u)-sequence of paths from i to j.

The following example is interesting. It illustrates the fact that alternative (3) of Theorem 1 may hold when the number u of Theorem 2 is >1:

$$A = \left[\begin{array}{cccccc:cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right] = \begin{bmatrix} C_1 & O \\ C_{21} & C_2 \end{bmatrix}.$$

We can apply Theorem 2, since we have the path (F_2, F_1) consisting of a single edge in $D_A{}^*$. Moreover, C_2 and C_1 are $\neq 0$ with $d_2 = 4$ and $d_1 = 6$. Suppose we choose $i = 9$ and $j = 4$ as the pair of vertices respectively in F_2 and F_1. We can take $a_{i_1 i_2}$ to be $a_{10,2}$ or $a_{10,3}$. In the first case we have the path (9, 10) (10, 2) (2, 3) (3, 4) of length $b = 4$ from vertex 9 to vertex 4; in the second case we have the path (9, 10) (10, 3) (3, 4) of length $b = 3$. These paths may be augmented by adding cycles of length $d_1 = 6$ in D_{C_1} and $d_2 = 4$ in D_{C_2}. Thus from vertex 9 to vertex 4 in D_A we have paths of length $4 + 6w_1 + 4w_2$ and $3 + 6w_1 + 4w_2$ for all w_1 and w_2. The number u of Theorem 2 is the g.c.d. of 4 and 6. Thus $u = 2$, and we have a (4, 2)-sequence and a (3, 2)-sequence of paths from 9 to 6. Since 4 and 3 are congruent to 0 and 1 (mod 2), the collection consisting of these two sequences is equivalent to a (0, 1) sequence of paths from 9 to 6. Thus $a_{96}{}^{(k)} > 0$ for $k >$ some integer N. By Theorem 1, alternative (3) holds, and we have $C_{21}{}^{(k)} > 0$ for all sufficiently large k.

We now describe criteria for deciding whether alternative (1), (2), (3), or (4) of Theorem 1 hold.

Criterion (a). *If $i \in F_r$ and $j \in F_s$, then alternative* (1) *holds if and only if there is no path in D_A from vertex i to vertex j.*

Criterion (b). *If $i \in F_r$, and $j \in F_s$, alternative* (2) *holds if and only if there* *is at least one path in D_A from vertex i to vertex j but no (b, u)-sequence of paths from i to j. Given a path in D_A from i to j, the corresponding induced path in $D_A{}^*$ must be such that if F_p is a vertex of the induced path, then C_p is a 1 by 1 zero matrix. This means that no vertex of the induced path can be a loop vertex in $D_A{}^*$.*

CRITERION (c). *If $i \in F_r$ and $j \in F_s$, then alternative (3) holds if and only if there is a $(0, 1)$-sequence of paths of D_A from vertex i to vertex j.*

Any path of length b in D_A from $i \in F_r$ to $j \in F_s$ induces a path in $D_A{}^*$. The path of length b in D_A gives rise to a (b, u)-sequence of paths in D_A, if and only if at least one of the vertices F_p of the induced path is such that $C_p \neq 0$. Now suppose that at least one path from i to j gives rise to a (b, u)-sequence. Let g be the maximum number of mutually non-equivalent sequences which have the same second member u. Denoting these sequences by (b_1, u), $(b_2, u), \ldots, (b_g, u)$ we see that no two b's are in the same residue class (mod u). Thus $g \leqslant u$. If $g = u$, then the collection of g distinct sequences is equivalent to a $(0, 1)$-sequence.

CRITERION (d). *If $i \in F_r$ and $j \in F_s$ and if there exist u mutually non-equivalent sequences of paths from i to j each of which has the same second member u, then alternative (3) holds.*

Two different paths from vertex i to vertex j in D_A may induce different paths from F_r to F_s in $D_A{}^*$, and these may give rise to a (b_1, u_1) sequence and a (b_2, u_2)-sequence with $u_1 \neq u_2$. Since the number of paths from F_r to F_s in $D_A{}^*$ with no edges which are loops is finite, there must be at most a finite number M of such distinct u's. Now let

$$\begin{array}{ll} (b_{1_x}, u_1) & x = 1, 2, \ldots, v_1, \\ (b_{2_x}, u_2) & x = 1, 2, \ldots, v_2, \\ \vdots & \vdots \\ (b_{M_x}, u_M) & x = 1, 2, \ldots, v_M, \end{array}$$

denote the totality of mutually non-equivalent sequences of paths from i to j. If $v_y = u_y$ for some $y = 1, 2, \ldots, M$, then the sequences (b_{y_x}, u_y), $x = 1, 2, \ldots, v_y$, are together equivalent to a $(0, 1)$-sequence and alternative (3) holds. In any case, let L be the least common multiple of $u_1, u_2, \ldots, u_M$. Each sequence (b_{y_x}, u_y) is equivalent to the following collection Σ_{yx} of mutually non-equivalent sequences:

$$(b_{y_x}, L), (b_{y_x} + u_y, L), (b_{y_x} + 2u_y, L), \ldots, (b_{y_x} + L - u_y, L).$$

There are L/u_y sequences in the collection Σ_{yx}. Now let Σ denote the union of the collections Σ_{yx} for $y = 1, 2, \ldots, M$ and $x = 1, 2, \ldots, v_y$. Two sequences of the collection Σ are equivalent if and only if their first members are in the same residue class (mod L).

CRITERION (e). *If $i \in F_r$ and $j \in F_s$ and if the collection Σ contains L mutually non-equivalent sequences of paths from vertex i to vertex j, then Σ is equivalent to a* $(0, 1)$*-sequence and alternative* (3) *holds.*

CRITERION (f). *If $i \in F_r$ and $j \in F_s$ and if the collection Σ contains at least one sequence but fewer than L mutually non-equivalent sequences of paths from vertex i to vertex j, then alternative* (4) *holds.*

Alternative (4) of Theorem 1 is the most interesting and complex of the four alternatives. We conclude this section with a decomposition of the sub-diagonal blocks which yields a more rounded characterization in this case.

Suppose that $C_r \neq 0$ and $C_s \neq 0$ and let d_r and d_s be their indices of imprimitivity. Let d_{rs} be the greatest common divisor of d_r and d_s. For (b, u)-sequences of paths from vertex $i \in F_r$ to vertex $j \in F_s$, the second members $u_1, u_2, \ldots, u_m$ are divisors of d_r and d_s and hence are divisors of d_{rs}. Thus the least common multiple of $u_1, u_2, \ldots, u_r$ is a divisor of d_{rs}, and hence a (b, L)-sequence of paths is equivalent to d_{rs}/L mutually non-equivalent sequences with second member d_{rs}. The permutation matrix P used to achieve a canonical transform of the reducible matrix A can be so chosen that it achieves a canonical transform of any constituent C_p for which $d_p > 1$. Let $I_1{}^r, I_2{}^r, \ldots, I_{d_r}{}^r$ be the sets of imprimitivity of D_{C_r} and let $I_1{}^s, I_2{}^s, \ldots, I_{d_s}{}^s$ be the sets of imprimitivity of D_{C_s}. The set F_r is the union of $I_1{}^r, I_2{}^r, \ldots, I_{d_r}{}^r$ and the set F_s is the union of $I_1{}^s, I_2{}^s, \ldots, I_{d_s}{}^s$. Now let C_{rs} be partitioned

$$C_{rs} = \begin{bmatrix} C_{11}^{rs} & C_{12}^{rs} & \cdots & C_{1d_s}^{rs} \\ C_{21}^{rs} & C_{22}^{rs} & \cdots & C_{2d_s}^{rs} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ C_{d_r1}^{rs} & C_{d_r2}^{rs} & \cdots & C_{d_rd_s}^{rs} \end{bmatrix},$$

so that C_{pq}^{rs}, $1 \leqslant p \leqslant d_r$, $1 \leqslant q \leqslant d_s$, is the submatrix with row set $I_p{}^r$ and column set $I_q{}^s$. Let $C_{pq}^{rs(k)}$ denote the submatrix of $C_{rs}{}^{(k)}$ which has row set $I_p{}^r$ and column set $I_q{}^s$. In this setting, we have the following theorem.

THEOREM 3. *If $C_r \neq 0$, $C_s \neq 0$, and if*

$$i_1 \in I_{p_1}{}^r, \qquad j_1 \in I_{q_1}{}^s, \qquad i_2 \in I_{p_2}{}^r, \qquad j_2 \in I_{q_2}{}^s$$

and if $b_1 + p_1 - q_1 \equiv b_2 + p_2 - q_2 \pmod{d_{rs}}$, then there exists a (b_1, d_{rs})-sequence of paths in D_A from i_1 to j_1 if and only if there exists a (b_2, d_{rs})-sequence of paths from i_2 to j_2.

Proof. Suppose there exists a (b_1, d_{rs})-sequence of paths in D_A from i_1 to j_1. Since D_{C_r} has index of imprimitivity d_r, there exists a path from i_2 to i_1 in D_{C_r} of length $p_1 - p_2 + ed_r$ for some integer e. Also, there exists a path in D_{C_s} from j_1 to j_2 of length $q_2 - q_1 + fd_r$, where f is an integer. Since

$b_1 + p_1 - p_2 + q_2 - q_1 \equiv b_2 \pmod{d_{rs}}$, these paths combine with the (b_1, d_{rs}) sequence from i_1 to j_1 to yield a (b_2, d_{rs}) sequence from i_2 to j_2.

Now let $i \in F_r, j \in F_s$ and let π be a path from i to j which has no subgraph which is a cycle. Let $s(\pi)$ be the length of π. Let $F_r, F_{p1}, F_{p2}, \ldots F_{pt}, F_s$, be the vertices of $D_A{}^*$ in the path induced by π. Select a vertex on π in each of $F_r, F_{p1}, \ldots, F_{pt}, F_s$ and let $V(\pi)$ be the resulting set of vertices. Let $H(\pi)$ be the set of lengths of cycles through these vertices and let $u(\pi)$ be the greatest common divisor of the set $H(\pi)$. Thus $u(\pi)$ divides d_{rs}. Let $F(\pi)$ be the largest multiple of $u(\pi)$ which is not expressible as a non-negative linear combination of the numbers in $H(\pi)$. $F(\pi)$ exists by Remark 5. Let $N(\pi) = s(\pi) + F(\pi)$ and let $N_{rs} = \max\{N(\pi)\}$ taken over all acyclic paths from a vertex $i \in F_r$ to a vertex $j \in F_s$ for all choices of $V(\pi)$. N_{rs} exists since the cardinality of the set $\{N(\pi)\}$ is finite. In this setting we have the following theorem.

THEOREM 4. *If* $C_r \neq 0$, $C_s \neq 0$, *if*

$$i_1 \in I_{p1}{}^r, \qquad j_1 \in I_{q1}{}^s, \qquad i_2 \in I_{p2}{}^r, \qquad j_2 \in I_{q2}{}^s,$$

if $k_1 + p_1 - q_1 \equiv k_2 + p_2 - q_2 \pmod{d_{rs}}$, *if* $a_{i_1 j_1}{}^{(k_1)} > 0$ *and if* $k_2 > N_{rs}$, *then* $a_{i_2 j_2}{}^{(k_2)} > 0$.

Proof. Since $a_{i_1 j_1}{}^{(k_1)} > 0$, there is a path of length k_1 from i_1 to j_1. Let π be an acyclic subpath from i_1 to j_1. Consider a path which results by adjoining to π a path from i_2 to i_1 in D_{C_r} and a path from j_1 to j_2 in D_{C_s}, and let α be a path from i_2 to j_2 which is an acyclic subpath of this path. We have $u(\alpha) = u(\pi)$. Also

$$\begin{aligned} s(\alpha) &\equiv s(\pi) + p_1 - p_2 + q_2 - q_1 \pmod{d_{rs}}, \\ k_2 &\equiv k_1 + p_1 - p_2 + q_2 - q_1 \pmod{d_{rs}}, \\ k_1 &\equiv s(\pi) \pmod{u(\pi)}. \end{aligned}$$

Thus

$$k_2 \equiv s(\alpha) \pmod{u(\alpha)}.$$

Also $k_2 > N(\alpha) = s(\alpha) + F(\alpha)$. It follows that $k_2 = s(\alpha)$, where f is a positive integer. Hence $fu(\alpha) > F(\alpha)$. But since $fu(\alpha)$ is a multiple of $u(\alpha)$ which is greater than $F(\alpha)$, it follows that $fu(\alpha)$ is a non-negative linear combination of the numbers in $H(\alpha)$. Thus we have a path from i_2 to j_2 of length k_2 and $a_{i_2 j_2}{}^{(k_2)} > 0$.

There are a number of useful corollaries to this theorem.

COROLLARY 1. *If* $C_r \neq 0$, $C_s \neq 0$, *if*

$$i_1 \in I_{p1}{}^r, \qquad j_1 \in I_{q1}{}^s, \qquad i_2 \in I_{p2}{}^r, \qquad j_2 \in I_{q2}{}^s,$$

if $k_1 + p_1 - q_1 \equiv k_2 + p_2 - q_2 \pmod{d_{rs}}$, *if* $k_1 > N_{rs}$ *and* $k_2 > N_{rs}$, *then* $a_{i_1 j_1}{}^{(k_1)} > 0$ *if and only if* $a_{i_2 j_2}{}^{(k_2)} > 0$.

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Putting $p_1 = p_2$ and $q_1 = q_2 = q$ in Corollary 1, we have Corollary 2.

COROLLARY 2. *If $C_r \neq 0$, $C_s \neq 0$, i_1 and $i_2 \in I_p^r$, j_1 and $j_2 \in I_q^s$, $k_1 \equiv k_2$ (mod d_{rs}) and if $k_1 > N_{rs}$, $k_2 > N_{rs}$, then $a_{i_1 j_1}{}^{(k_1)} > 0$ if and only if $a_{i_2 j_2}{}^{(k_2)} > 0$.*

Putting $k_1 = k_2$ in Corollary 2, we get Corollary 3.

COROLLARY 3. *If $C_r \neq 0$, $C_s \neq 0$ and if $k > N_{rs}$, then either $C_{pq}^{rs(k)} > 0$ or $C_{pq}^{rs(k)} = 0$ for every such k.*

COROLLARY 4. *If $C_r \neq 0$, $C_s \neq 0$, if $k_1 > N_{rs}$, $k_2 > N_{rs}$ and if*

$$k_1 + p_1 - q_1 \equiv k_2 + p_2 - q_2 \pmod{d_{rs}},$$

then $C_{p_1 q_1}^{rs(k)} > 0$ if and only if $C_{p_2 q_2}^{rs(k)} > 0$.

COROLLARY 5. *If $C_r \neq 0$, $C_s \neq 0$, if $k > N_{rs}$ and if $p_1 - q_1 \equiv p_2 - q_2$ (mod d_{rs}), then $C_{p_1 q_1}^{rs(k)} > 0$ if and only if $C_{p_2 q_2}^{rs(k)} > 0$.*

There are obvious modifications of Theorems 3 and 4 when exactly one of C_r and C_s is zero, with d_{rs} being replaced by whichever one of d_r and d_s is defined.

When $C_r = C_s = 0$, then C_r and C_s are 1 by 1 zero matrices and C_{rs} is 1 by 1. Whether or not $C_{rs}{}^{(k)}$ is zero is decided by criterion (e) or criterion (f).

As an example consider the 22 by 22 reducible matrix shown below. In this

0	1	0																			
1	0	1																			
1	1	0																			
			0	1	1	0	0														
			0	0	0	1	0														
			0	0	0	1	0														
			0	0	0	0	1														
1			1	0	0	0	0														
			△	▽	▽	△	▽	0	1												
			∇_1	△	△	▽	△	1	0												
*	*	*	*	*	*	*	*	*	*	0											
*	*	*	*	*	*	*	*	*	*	1	1										
*	*	*	*	*	*	*	*	*	*	$*_1$	*	0									
			△	▽	▽	△	▽	△	∇_1		1	*	0	1	1	0	0	0	0	0	0
			▽	△	△	▽	△	▽	△			*	0	0	0	1	1	1	0	0	0
			▽	△	△	▽	△	▽	△			*	0	0	0	1	1	0	0	0	0
			△	▽	▽	△	▽	△	▽			*	0	0	0	0	0	0	1	0	0
			△	▽	▽	△	▽	△	▽			*	0	0	0	0	0	0	1	0	0
			△	▽	▽	△	▽	△	▽			*	0	0	0	0	0	0	1	0	0
			▽	△	△	▽	△	▽	△			*	0	0	0	0	0	0	0	1	0
			△	▽	▽	△	▽	△	▽			*	0	0	0	0	0	0	0	0	1
			▽	△	△	▽	△	▽	△			*	1	0	0	0	0	0	0	0	0

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matrix we have $m = 7$ with $d_1 = 1$, $d_2 = 4$, $d_3 = 2$, d_4 undefined, $d_5 = 1$, d_6 undefined, and $d_7 = 6$. The subdiagonal blocks C_{rs} are bounded by solid lines. Within these blocks the submatrices C_{pq}^{rs} are bounded by dotted lines.

For the places in the matrix below the diagonal blocks there exists N such that the positive or zero state of the entry in each place can be described for all powers of $t > N$ according to the following legend:

* indicates that the entry is zero,

$\triangle$ indicates that the entry is zero if t is odd and positive if t is even,

∇ indicates that the entry is positive if t is odd and zero if t is even.

An entry in a place not identified by *, $\triangle$, or ∇ is positive.

In **(3)** the authors have made an observation relative to Markov chains. Specifically, if A is the transition matrix of an ergodic Markov chain which is irregular and if the index of imprimitivity of A is d $(d > 1)$, then there are d limiting forms for A^t as $t \to \infty$ and these are expressible in terms of the fixed vector of one of the diagonal components of A^d and the cyclic components of A. If the Markov chain is not ergodic, then the transition matrix is reducible. For such a matrix, the results of **(3)** may be combined with the results of this paper to assist in finding the various limiting forms of the powers of the matrix.

Postscript (June 2, 1964). At a matrix conference held in Gatlinburg, Tennessee, April 13–18, 1964, M. S. Lynn and O. Taussky called our attention to the following reference: D. Rosenblatt, *On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices*, Naval Res. Logistic Quart., *4* (1957), 151–67.

This paper contains several fundamental results on the role of graph theory in the study of non-negative matrices. Two such results which the authors of this paper have used in much of their work are the following: (1) A non-negative matrix is irreducible if and only if the corresponding directed graph is strongly connected, and (2) a non-negative matrix is primitive if and only if the corresponding directed graph is strongly connected and the circuit lengths are relatively prime.

Had the authors been aware of this paper before, they would have included it as a reference in several of their previous papers.

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University of Alberta and
University of Manitoba