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THE MULTIDIMENSIONAL FUNDAMENTAL THEOREM OF CALCULUS

WASHEK F. PFEFFER

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Abstract

On compact oriented differentiable manifolds, we define a well behaved Riemann type integral which coincides with the Lebesgue integral on nonnegative functions, and such that the exterior derivative of a differentiable (not necessarily continuously) exterior form is always integrable and the Stokes formula holds.

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1. Introduction

The philosophical basis for the present work is a well established maxim: if we can calculate the value of an integral of a function, then such a function ought to be integrable. In the past, an application of this maxim to the bounded convergence theorem for the Riemann integral led to the Lebesgue integral, and an application to the one dimensional fundamental theorem of calculus led to the Denjoy-Perron integral. More recently, J. Mawhin and the author (see [12], [13], and [16], [17]) applied it to the divergence theorem, which is the *local* multidimensional fundamental theorem of calculus. While the resulting integral is well adapted to the local situation, it is coordinate bound and cannot be lifted to manifolds. In this paper, we apply the maxim to the *global* fundamental theorem of calculus, that is, to the Stokes theorem.

THEOREM. *Let M be an oriented compact differentiable manifold with boundary ∂M, and let ω be a continuous exterior form on M which is differentiable in $M - \partial M$. Then $\int_{\partial M} \omega = \int_M d\omega$ whenever the integral on the right exists.*

If ω is continuously differentiable in M, then the Lebesgue integral $\int_M d\omega$ exists. However, in general, $d\omega$ is not Lebesgue integrable even in dimension one. On the other hand, the Lebesgue integral $\int_{\partial M} \omega$ exists and its value depends only on $d\omega$. In accordance with the maxim, under these circumstances the nonexistence of the Lebesgue integral $\int_M d\omega$ is not a reflection of an inherent pathology of $d\omega$, but rather a shortcoming of the Lebesgue integral. Indeed, we show that there is a well behaved integral which coincides with the Lebesgue integral on nonnegative functions, and for which the integral $\int_M d\omega$ exists and has the correct value.

Initially, we define the integral in an m-dimensional Euclidean space by substantially modifying the basic idea of Henstock and Kurzweil (see [3] and [6]). We integrate over sets from a family invariant with respect to diffeomorphisms (to achieve the coordinate independence) and employ partitions satisfying a strong Vitali condition (to prove the divergence theorem). As it is not possible to satisfy the Vitali condition along the boundary of the integration domain, we use only partitions which lie in the interior, and bridge the resulting gap by a suitable continuity requirement imposed on the indefinite integral. Using standard techniques, we lift the integral to differentiable manifolds, and obtain the Stokes theorem from the divergence theorem by means of triangulations.

This approach is quite different from that employed in [4] and [5], and it yields the Stokes theorem (Theorem 7.3) which is considerably more general.

The paper is divided into eight sections whose content is sufficiently indicated by their titles.

2. Preliminaries

By $\mathbf{R}$ and $\mathbf{R}_+$ we denote the sets of all real and all positive real numbers, respectively. All functions in this paper are real-valued, and often the same letter is used to denote a function on a set A as well as its restriction to a set $B \subset A$. The algebraic operations, partial order, and convergence among functions on the same set are defined pointwise.

Throughout, $m \geqslant 1$ is a fixed integer, and $\mathbf{R}^m$ denotes the m-dimensional Euclidean space equipped with the usual inner product $x \cdot y$ and the corresponding norm $|x| = (x \cdot x)^{1/2}$. The distance between a point $x \in \mathbf{R}^m$ and a set $E \subset \mathbf{R}^m$ is denoted by $\operatorname{dist}(x, E)$. If $E \subset \mathbf{R}^m$, then E^-, E°, $E^\bullet$, and $d(E)$ denote, respectively, the closure, interior, boundary, and diameter of E.

For an integer $k \geqslant 1$, we denote by λ_k the outer Lebesgue measure in $\mathbf{R}^k$. We write λ instead of λ_1, and $|E|$ instead of $\lambda_m(E)$ whenever $E \subset \mathbf{R}^m$. Unless specified otherwise, the words "outer measure", "measure", and "measurable", as well as the expression "almost all", always refer to λ_m.

In Section 3, we shall define a new Riemann type integral which will be denoted by the usual integral sign $\int$. Thus to avoid confusion, throughout, we denote the Lebesgue integral by the sign $(L)\int$.

For $E \subset \mathbf{R}^m$ and $\eta > 0$, we let $U(E,\eta) = \{x \in \mathbf{R}^m : \operatorname{dist}(x, E) < \eta\}$. We say that a set $E \subset \mathbf{R}^m$ is, respectively, *thin* or *slight* if it is compact and $|(U(E,\eta)| = O(\eta)$ or $|U(e,\eta)| = o(\eta)$ as η approaches zero.

Clearly, each slight set is thin, and each thin set is of measure zero. Moreover, the families of all thin and slight sets are closed with respect to finite unions, and also with respect to intersections by compact sets. For heuristic purposes it is convenient, though inaccurate, to think of thin and slight sets as compact rectifiable pieces of $(m-1)$- and $(m-2)$-dimensional submanifolds of $\mathbf{R}^m$, respectively. More precisely, the $(m-1)$-dimensional *upper Minkowski content* of a thin set is finite, and that of a slight set is zero (see [1, Section 3.2.37]).

A bounded subset of $\mathbf{R}^m$ whose boundary is thin is called *admissible*. In view of the previous paragraph, the family $\mathscr{A}$ of all admissible sets is a ring of measurable subsets of $\mathbf{R}^m$, and $A \in \mathscr{A}$ whenever A^- is compact and $A^\cdot$ is thin. For $E \subset \mathbf{R}^m$, we set $\mathscr{A}(E) = \{A \in \mathscr{A} : A \subset E\}$.

Let $A \in \mathscr{A}$. By [9, Theorems 42, 16, and 18], on $A^\cdot$ there is a unique finite Borel measure σ_A, called the *surface measure*, and a σ_A-almost everywhere unique unit Borel vector field n_A, called the *unit exterior normal field*, such that

$$(L)\int_A \nabla \cdot v \, d\lambda_m = (L)\int_{A^\cdot} v \cdot n_A \, d\sigma_A$$

for each vector field $v = (f_1, \dots, f_m)$ continuously differentiable in an open set containing A^-; here $\nabla \cdot v = \sum_{i=1}^m \partial f_i / \partial x_i$ is the divergence of v. It follows from [9, Theorems 4 and 33] that the number $\|A\| = \sigma_A(A^\cdot)$, called the *surface area* of A, is positive if and only if $A^\circ \neq \varnothing$. Thus we can define the *regularity* $r(A)$ of A by setting $r(A) = |A|/d(A)\|A\|$ if $A^\circ \neq \varnothing$, and $r(A) = 0$ if $A^\circ \neq \varnothing$. This notion of regularity controls both the shape and surface area of A.

The traditional shape indicator of a set $E \subset \mathbf{R}^m$ is the number $r^*(E)$ defined by $r^*(E) = |E|/[d(E)]^m$ if E contains at least two points, and $r^*(E) = 0$ otherwise (see [19, Chapter IV, Section 2, p. 106]). For an admissible set A, the relationship between $r(A)$ and $r^*(A)$ takes a form of an isoperimetric inequality.

2.1. **Proposition.** *If $A \in \mathscr{A}$, then $[2r(A)]^m \leqslant r^*(A)$.*

Proof. Let $A \in \mathscr{A}$ and $A^\circ \neq \varnothing$. If A_i, $i = 1, \dots, m$, is the projection of A° to the hyperplane perpendicular to the ith coordinate axis, then it follows from [9,

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Theorems 4 and 33] that $2\lambda_{m-1}(A_i) \leqslant \|A\|$. By [8],

$$|A|^{m-1} \leqslant \prod_{i=1}^{m} \lambda_{m-1}(A_i) \leqslant 2^{-m}\|A\|^m,$$

and hence

$$[2r(A)]^m = \frac{2^m|A|^m}{[d(A)]^m\|A\|^m} \leqslant \frac{|A|}{[d(A)]^m} = r^*(A).$$

As $r(A) = r^*(A) = 0$ if $A^\circ = \varnothing$, the proposition is proved.

2.2. Remark. If A is an *interval*, then $\|A\| \leqslant 2m[d(A)]^{m-1}$, and we have $r^*(A) \leqslant 2m\, r(A)$. On the other hand, setting

$$A_n = \left(\bigcup_{k=1}^{n} \left[\frac{1}{2k}, \frac{1}{2k-1}\right]\right) \times [0,1]^{m-1}$$

for $n = 1, 2, \ldots,$ we obtain $r^*(A_n) \geqslant 1/2m^{1/2}$ and $r(A_n) \leqslant 1/2n$.

A *division* of an admissible set A is a finite disjoint family $\mathscr{D} \subset \mathscr{A}$ with $\bigcup\mathscr{D} = A$. An *additive function* in an admissible set A is a function F on $\mathscr{A}(A)$ such that $F(A) = \sum_{D\in\mathscr{D}} F(D)$ for each division $\mathscr{D}$ of A.

2.3. Definition. An additive function F in an admissible set A is called *continuous* whenever we can find a slight set $S_F \subset A^-$ so that the following conditions are satisfied

(i) Given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in \mathscr{A}[A \cap U(S_F, \eta)]$ with $\|B\| < \eta$.

(ii) Given $\varepsilon > 0$ and a thin set $T \subset A^- - S_F$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in \mathscr{A}[A \cap U(T, \eta)]$ with $\|B\| < 1/\varepsilon$.

The family of all continuous additive functions in an admissible set A is a real vector space denoted by $\mathscr{C}(A)$. If $F \in \mathscr{C}(A)$ and $B \in \mathscr{A}(A)$, then the restriction $G = F \upharpoonright \mathscr{A}(B)$ belongs to $\mathscr{C}(B)$; indeed, it suffices to let $S_G = S_F \cap B^-$.

2.4. Proposition. *If $A \in \mathscr{A}$ and $F \in \mathscr{C}(A)$, then $F(B) = 0$ for each $B \subset A$ for which B^- is thin.*

Proof. Let $B \subset A$ be such that B^- is thin, and let $\varepsilon > 0$. Then $B \in \mathscr{A}$ and $\|B\| = 0$. Hence there is an $\eta > 0$ such that $|F[B \cap U(S_F, \eta)]| < \varepsilon$. As $[B - U(S_F, \eta)]^-$ is a thin subset of A^- disjoint from S_F, we see that

$$|F[B - U(S_F, \eta)]| < \varepsilon.$$

Thus $|F(B)| < 2\varepsilon$ by the additivity of F, and the proposition follows.

Let $E \subset \mathbf{R}^m$. A *partition* in E is a set $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are disjoint admissible sets, $x_i \in A_i^-$, and $A_i^- \subset E^0$, $i = 1, \dots, p$. If, in addition, $r(A_i) \geqslant \varepsilon > 0$ for $i = 1, \dots, p$, then P is called an *ε-partition* in E. Given $\delta : E \to \mathbf{R}_+$, we say that P is *δ-fine* whenever $d(A_i) < \delta(x_i)$, $i = 1, \dots, p$. The family of all δ-fine ε-partitions in E is denoted by $\mathscr{P}(E, \varepsilon; \delta)$. If $E^* \subset E$, $\delta^* \leqslant \delta \restriction E^*$, and $\varepsilon^* \geqslant \varepsilon$, then clearly $\mathscr{P}(E^*, \varepsilon^*; \delta^*) \subset \mathscr{P}(E, \varepsilon; \delta)$.

A *half-open interval* is an interval of the form $\prod_{i=1}^m [a_i, b_i)$ where $a_i, b_i \in \mathbf{R}$ and $a_i < b_i$, $i = 1, \dots, m$. The following existence result is sometimes referred to as Cousin's lemma. For its proof, which is a simple compactness argument, and some historical discussion we refer to [11].

2.5. Proposition. *Let $E \subset \mathbf{R}^m$, $\delta : E \to \mathbf{R}_+$, and let A be a half-open interval with $A^- \subset E^\circ$. Then there is a δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in E such that $\bigcup_{i=1}^p A_i = A$ and each A_i is similar to A.*

3. Definition and basic properties of the integral

We begin with the definition of integrability.

3.1. Definition. Let f be a function on an admissible set A. We say that f is *integrable* in A if there is a continuous additive function F in A satisfying the following condition: we can find a thin set $T_F \subset A^-$ so that given $\varepsilon > 0$, there is a $\delta : A \to \mathbf{R}_+$ such that

$$\sum_{i=1}^{p} |f(x_i)\,|A_i| - F(A_i)| < \varepsilon$$

for each δ-fine ε-partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in $A - T_F$.

The family of all functions integrable in an admissible set A is denoted by $\mathscr{I}(A)$.

Let $A \in \mathscr{A}$ and $f \in \mathscr{I}(A)$. Any $F \in \mathscr{C}(A)$ satisfying the condition of Definition 3.1 is called an *indefinite integral* of f. If F is an indefinite integral of f and $B \in \mathscr{A}(A)$, then $G = F \restriction \mathscr{A}(B)$ is an indefinite integral of $f \restriction B$; indeed, it suffices to let $T_G = T_F \cap B^-$. In particular, we have the following proposition.

3.2. Proposition. *A function integrable in an admissible set A is integrable in each admissible subset of A.*

Our first goal is to show that each integrable function has *only one* indefinite integral.

3.3. LEMMA. *Let $A \in \mathscr{A}$, $f \in \mathscr{I}(A)$, and let F_1 and F_2 be indefinite integrals of f. Then for each $\varepsilon > 0$, there is a partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A such that*

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - F_j(A) \right| < \varepsilon$$

for $j = 1, 2$.

PROOF. (a) For $n = 1, 2, \ldots,$ we denote by $\mathscr{K}_n$ the family of all cubes $\prod_{i=1}^{m}[k_i 2^{-n}, (k_i + 1)2^{-n})$ where $k_1, \ldots, k_m$ are integers. Throughout this proof $j = 1, 2$. Let $T_j = T_{F_j}$ and $S_j = S_{F_j}$ be thin and slight sets associated to F_j by Definitions 3.1 and 2.3, respectively. Choose an $\varepsilon > 0$ and let $\alpha = \min(\varepsilon/4, 1/2m^{3/2})$. In three steps we shall construct a set A_0 which is a finite disjoint union of cubes, and for which $A_0^- \subset [A - (T_1 \cup T_2)]^-$ and $|F_j(A - A_0)| < 3\alpha$.

(b) Let $S_0 = S_1 \cap S_2$, and choose η_0 so that $|F_j(B)| < \alpha$ for each $B \in \mathscr{A}[A \cap U(S_0, \eta_0)]$ with $\|B\| < \eta_0$. Set $\mathscr{K}_{n,0} = \{K \in \mathscr{K}_n : S_0 \cap K^- \neq \varnothing\}$ and $K_{n,0} = \bigcup K_{n,0}$. For $t > 0$, let $\beta(t) = |U(S_0, t)|/t$. If k_0 is the number of cubes in $\mathscr{K}_{n,0}$, then

$$k_0 2^{-nm} = |K_{n,0}| \leqslant \left|U\left(S_0, 2^{-n}m^{1/2}\right)\right| = 2^{-n}m^{1/2}\beta(2^{-n}m^{1/2}),$$

and hence

$$\|K_{n,0}\| \leqslant \sum \left\{ \|K\| : K \in \mathscr{K}_{n,0} \right\} = k_0(2m\, 2^{-n(m-1)}) \leqslant 2m^{3/2}\beta(2^{-n}m^{1/2}).$$

Since S_0 is slight, we see that $\lim_{n \to \infty}\|K_{n,0}\| = 0$. By [14, Section 3], there is an integer n_0 such that $2^{-n_0}m^{1/2} < \eta_0$ and $\|A \cap K_{n,0}\| < \eta_0$. Thus letting $K_0 = K_{n,0}$, we have $S_0 \subset K_0^\circ$ and $|F_j(A \cap K_0)| < \alpha$.

(c) Choose $\eta_1 > 0$ so that $|F_j(B)| < \alpha/2$ for each

$$B \in \mathscr{A}\left[A \cap U(S_1 - K_0^\circ, \eta_1)\right]$$

with $\|B\| < \eta_1$. Such a choice is clearly possible for $j = 1$; it is also possible for $j = 2$ because $S_1 - K_0^\circ$ is a thin subset of $A^- - S_2$. Analogously, choose $\eta_2 > 0$ so that $|F_j(B)| < \alpha/2$ for each $B \in \mathscr{A}[A \cap U(S_2 - K_0^\circ, \eta_2)]$ with $\|B\| < \eta_2$. Let $S = S_1 \cup S_2$. Since $S_j - K_0^\circ$ are disjoint compact sets, there is a positive $\eta_S \leqslant \min(\eta_1, \eta_2)$ such that $U(S - K_0^\circ, \eta_S)$ is a disjoint union of $U_j = U(S_j - K_0^\circ, \eta_S)$. Now if $B \in \mathscr{A}[A \cap U(S - K_0^\circ, \eta_S)]$ and $\|B\| < \eta_S$, then $\|B \cap U_j\| \leqslant \|B\| < \eta_j$, and we have

$$\left|F_j(B)\right| \leqslant \left|F_j(B \cap U_1)\right| + \left|F_j(B \cap U_2)\right| < \alpha.$$

For $n \geqslant n_0$, set

$$K_{n,S} = \bigcup \left\{ K \in \mathscr{K}_n : S \cap K^- \neq \varnothing \text{ and } K_0 \cap K = \varnothing \right\}.$$

Arguing similarly as in (b), we find an integer $n_S \geqslant n_0$ such that $2^{-n_S}m^{1/2} < \eta_S$ and $\|A \cap K_{n_S,S}\| < \eta_S$. Thus letting $K_S = K_{n_S,S}$, we have $S \subset (K_0 \cap K_S)^\circ$ and $|F_j(A \cap K_S)| < \alpha$.

(d) Since $T = A^{\cdot} \cup T_1 \cup T_2 - (K_0 \cup K_S)^\circ$ is a thin subset of $A^- - S$, there is a positive $\eta_T \leqslant 1$ such that $|F_j(B)| < \alpha$ for each $B \in \mathscr{A}[A \cap U(T, \eta_T)]$ with $\|B\| \leqslant \|A\| m^{1/2} + 2m^{3/2}\gamma$ where

$$\gamma = \sup\left\{ \frac{|U(T,t)|}{t} : 0 < t < 1 \right\}.$$

Find $n_T \geqslant n_S$ so that $2^{-n_T}m^{1/2} < \eta_T$, and set

$$\mathscr{K}_T = \left\{ K \in \mathscr{K}_{n_T} : T \cap K^- \neq \varnothing \text{ and } (K_0 \cup K_S) \cap K = \varnothing \right\}.$$

If $K_T = \bigcup \mathscr{K}_T$, then $A^{\cdot} \cup T_1 \cup T_2 \subset (K_0 \cup K_S \cup K_T)^\circ$. Denoting by k_T the number of cubes in $\mathscr{K}_T$, we have

$$k_T 2^{-n_T m} = |K_T| \leqslant \left| U(T, 2^{-n_T} m^{1/2}) \right| \leqslant 2^{-n_T} m^{1/2} \gamma.$$

From this and [9, Theorem 9(c) and Lemma 36] we obtain

$$\|A \cap K_T\| \leqslant \sum \left\{ \|A \cap K\| : K \in \mathscr{K}_T \right\} \leqslant \|A\| m^{1/2} + \sum \left\{ \|K\| : K \in \mathscr{K}_T \right\}$$
$$= \|A\| m^{1/2} + k_T(2m 2^{-n_T(m-1)}) \leqslant \|A\| m^{1/2} + 2m^{3/2}\gamma,$$

and hence $|F_j(A \cap K_T)| < \alpha$. Now if $A_0 = A - (K_0 \cup K_S \cup K_T)$, then $A_0^- \subset [A - (T_1 \cup T_2)]^\circ$ and $A_0 = \bigcup\{ K \in \mathscr{K}_{n_T} : K \subset A_0 \}$. Moreover,

$$\left| F_j(A - A_0) \right| \leqslant \left| F_j(A \cap K_0) \right| + \left| F_j(A \cap K_S) \right| + \left| F_j(A \cap K_T) \right| < 3\alpha.$$

(e) Find $\delta : A \to \mathbf{R}_+$ so that

$$\sum_{i=1}^{q} \left| f(y_i)|B_i| - F_j(B_i) \right| < \alpha$$

for each δ-fine α-partition $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ in $A - (T_1 \cup T_2)$. As A_0 is a finite disjoint union of cubes which are half-open intervals, it follows from Proposition 2.5 that there is a δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A - (T_1 \cup T_2)$ such that $A_1, \ldots, A_p$ are cubes and $\bigcup_{i=1}^{p} A_i = A_0$. The regularity of a cube equals $1/2m^{3/2}$, and so P is an α-partition by our choice of α. Thus

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - F_j(A) \right| \leqslant \sum_{i=1}^{p} \left| f(x_i)|A_i| - F_j(A_i) \right| + \left| F_j(A - A_0) \right|$$
$$< 4\alpha < \varepsilon.$$

3.4. PROPOSITION. *All indefinite integrals of an integrable function are equal.*

PROOF. Let $A \in \mathscr{A}$ and $f \in \mathscr{I}(A)$, and assume that F_1 and F_2 are indefinite integrals of f such that $F_1(B) \neq F_2(B)$ for some $B \in \mathscr{A}(A)$. Since $F_1 \restriction \mathscr{A}(B)$ and $F_2 \restriction \mathscr{A}(B)$ are indefinite integrals of $f \restriction B$, a direct application of Lemma 3.3 yields a contradiction.

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Let $A \in \mathscr{A}$ and $f \in \mathscr{I}(A)$. In view of Proposition 3.4, we can talk about *the* indefinite integral of f denoted by $\int f$. Its value at $B \in \mathscr{A}(A)$ is called the *integral* of f over B denoted by $\int_B f$.

3.5. PROPOSITION. *If $A \in \mathscr{A}$, then $\mathscr{I}(A)$ is a linear space, and the map $f \mapsto \int_A f$ is a nonnegative linear functional on $\mathscr{I}(A)$.*

Lemma 3.3 shows that $\int_A f \geqslant 0$ for each nonnegative function $f \in \mathscr{I}(A)$. The rest of the proposition follows directly from Definition 3.1.

3.6. PROPOSITION. *Let f be a function on $A \in \mathscr{A}$, and let $\mathscr{D}$ be a division of A. If $f \in \mathscr{I}(D)$ for each $D \in \mathscr{I}$, then $f \in \mathscr{I}(A)$ and $\int_A f = \sum_{D \in \mathscr{D}} \int_D f$.*

PROOF. For $D \in \mathscr{D}$, let $F_D = \int(f \upharpoonright D)$, and let $T_D = T_{F_D}$ be a thin set associated to F_D by Definition 3.1. Setting

$$F(B) = \sum_{D \in \mathscr{D}} F_D(B \cap D)$$

for each $B \in \mathscr{A}(A)$, it is easy to see that $F \in \mathscr{C}(A)$. Choose $\varepsilon > 0$, and denote by k the number of elements in $\mathscr{D}$. Given $D \in \mathscr{D}$, find a $\delta_D : D \to \mathbf{R}_+$ so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F_D(A_i) \right| < \varepsilon/k$$

for every $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $\mathscr{P}(D - T_D, \varepsilon; \delta_D)$. Let $T = \bigcup_{D \in \mathscr{D}}(D^{\boldsymbol{\cdot}} \cup T_D)$, and for $x \in A$ set

$$\delta(x) = \min\left[\delta_D(x), \operatorname{dist}(x, D^{\boldsymbol{\cdot}} \cup T_D)\right]$$

if $x \in D^\circ - T_D$ and $D \in \mathscr{D}$, and $\delta(x) = 1$ otherwise. Now if $P \in \mathscr{P}(A - T, \varepsilon; \delta)$, then $P_D = \{(B, x) \in P \colon x \in D\}$ belongs to $\mathscr{P}(D - T_D, \varepsilon; \delta_D)$ for each $D \in \mathscr{D}$, and P is a disjoint union of the P_D's. Consequently,

$$\sum_{(B,x) \in P} \left| f(x)|B| - F(B) \right| = \sum_{D \in \mathscr{D}} \sum_{(B,x) \in P_D} \left| f(x)|B| - F_D(B) \right| < k\frac{\varepsilon}{k} = \varepsilon,$$

and we see that $F = \int f$. The proposition follows.

4. The relationship to the Lebesgue integral, and its consequences

If $E \subset \mathbf{R}^m$ is a measurable set, we denote by $\mathscr{L}(E)$ the family of all functions f on E for which the *finite* Lebesgue integral $(L)\int_E f\, d\lambda_m$ exists. When no misunderstanding is possible, we write $(L)\int_E f$ instead of $(L)\int_E f\, d\lambda_m$.

4.1. PROPOSITION. *If $A \in \mathscr{A}$, then $\mathscr{L}(A) \subset \mathscr{I}(A)$ and $\int_A f = (L)\int_A f$ for each $f \in \mathscr{L}(A)$.*

PROOF. Let $f \in \mathscr{L}(A)$, and set $F(B) = (L)\int_B f$ for each $B \in \mathscr{A}(A)$. As thin sets are compact and of measure zero, it is easy to conclude from the absolute continuity of the Lebesgue integral that $F \in \mathscr{C}(A)$. We now show that $F = \int f$.

Given $\varepsilon > 0$, there are functions g and h on A which are, respectively, upper and lower semicontinuous, and such that $g \leqslant f \leqslant h$ and $(L)\int_A (h - g) < \varepsilon$. Find δ: $A \to R_+$ so that $g(y) < g(x) + \varepsilon$ and $h(y) > h(x) - \varepsilon$ for each $x, y \in A$ with $|x - y| < \delta(x)$. Now let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a δ-fine partition in A. By the choice of δ,

$$(L)\int_{A_i} g - \varepsilon|A_i| \leqslant g(x_i)|A_i| < f(x_i)|A_i| \leqslant h(x_i)|A_i| \leqslant (L)\int_{A_i} h + \varepsilon|A_i|,$$

and since $(L)\int_{A_i} g \leqslant F(A_i) \leqslant (L)\int_{A_i} h$, we conclude

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| \leqslant \sum_{i=1}^{p} \left[(L)\int_{A_i} (h - g) + \varepsilon|A_i| \right]$$

$$= (L)\int_A (h - g) + \varepsilon|A| < \varepsilon(1 + |A|).$$

Let $A \in \mathscr{A}$, $x \in A^-$, and let F be a function on $\mathscr{A}(A)$. We say that F is *derivable* at x if a finite $\lim[F(B_n)/|B_n|]$ exists for each sequence $\{B_n\}$ in $\mathscr{A}(A)$ such that $x \in B_n^-$, $\lim d(B_n) = 0$, and $\inf r(B_n) > 0$. If all these limits exist, then they have the same value, which is denoted by $F'(x)$ and called the *derivative* of F at x. In view of Proposition 2.1, it is clear that for each $x \in A^\circ$, the existence of the derivative $F'(x)$ implies the existence of the *ordinary derivative* of F at x defined in [19, Chapter IV, Section 2, p. 106]; both derivatives have the same value.

4.2. PROPOSITION. *Let $A \in \mathscr{A}$, $f \in \mathscr{I}(A)$, and let $F = \int f$. Then for almost all $x \in A$ the function F is derivable at x and $F'(x) = f(x)$.*

PROOF. Let T be a thin set associated to F by Definition 3.1, and let E be the set of all $x \in A^\circ - T$ such that either F is not derivable at x, or $F'(x) \neq f(x)$. Then given $x \in E$, we can find an $\alpha(x) > 0$ so that for each $\beta > 0$ there is a closed set $B \in \mathscr{A}(A)$ with $x \in B$, $d(B) < \beta$, $r(B) \geqslant \alpha(x)$, and $|[F(B)/|B|] - f(x)| \geqslant \alpha(x)$; for by Proposition 2.4, $F(B^-) = F(B)$ for each $B \in \mathscr{A}(A)$ with $B^- \subset A$. Fix an integer $n \geqslant 1$, and let $E_n = \{x \in E : \alpha(x) \geqslant 1/n\}$. Choose a positive $\varepsilon \leqslant 1/n$, and find a δ: $A \to R_+$ such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \frac{\varepsilon}{n}$$

for each $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $\mathscr{P}(A - T, \varepsilon; \delta)$. Let $\mathscr{B}$ be the family of all closed sets $B \in \mathscr{A}(A^\circ - T)$ such that $r(B) \geqslant \varepsilon$, $d(B) < \delta(x_B)$ for some $x_B \in B$, and $|f(x_B)|B| - F(B)| \geqslant |B|/n$. Using Proposition 2.1, it is easy to see that $\mathscr{B}$ covers E_n in the sense of Vitali. Thus by [19, Chapter IV, Theorem (3.1)], there are disjoint sets $B_1, B_2, \ldots$ in $\mathscr{B}$ such that $|E_n - \bigcup_{i=1}^\infty B_i| = 0$. As $\{(B_1, x_{B_1}), \ldots, (B_k, x_{B_k})\}$ belongs to $\mathscr{P}(A - T, \varepsilon; \delta)$, we have

$$\sum_{i=1}^{k} |B_i| \leqslant n \sum_{i=1}^{k} \left| f(x_{B_i})|B_i| - F(B_i) \right| < \varepsilon$$

for $k = 1, 2, \ldots$, and consequently

$$|E_n| \leqslant \left| \bigcup_{i=1}^{\infty} B_i \right| \leqslant \sum_{i=1}^{\infty} |B_i| \leqslant \varepsilon.$$

As ε can be arbitrarily small, $|E_n| = 0$, and as $E = \bigcup_{n=1}^\infty E_n$, also $|E| = 0$. Since $|A^\bullet \cup T| = 0$, the proposition is proved.

4.3. Corollary. *Each function integrable on an admissible set is measurable.*

The corollary follows immediately from Proposition 4.2 and [19, Chapter IV, Theorem (4.2)].

4.4. Proposition. *Let f be a function on an admissible set A. Then f belongs to $\mathscr{L}(A)$ if and only if both f and $|f|$ belong to $\mathscr{I}(A)$.*

Proof. As the converse is given by Proposition 4.1, assume that f and $|f|$ are integrable, and let $g_n = \min(|f|, n)$, $n = 1, 2, \ldots$. Since $|A| < +\infty$, it follows from Corollary 4.3 that $g_n \in \mathscr{L}(A)$. By Propositions 4.1 and 3.5, we have $g_n \in \mathscr{I}(A)$ and

$$(L)\int_A |f| = \lim (L)\int_A g_n = \lim \int_A g_n \leqslant \int_A |f| < +\infty.$$

Thus $f \in \mathscr{L}(A)$ by Corollary 4.3.

4.5. Corollary. *Let f be a function on an admissible set A. Then $f = 0$ almost everywhere if and only if $f \in \mathscr{I}(A)$ and $\int f = 0$.*

Indeed, if $f \in \mathscr{I}(A)$ and $\int f = 0$, it follows directly from Definition 3.1 that $|f| \in \mathscr{I}(A)$ and $\int |f| = 0$.

4.6. COROLLARY. *Let g, h, and f_n, $n = 1, 2, \ldots$, be integrable on an admissible set A, and let $f = \lim f_n$. Suppose that either of the following conditions holds:*
(i) $f_n \leqslant f_{n+1}$, $n = 1, 2, \ldots$, *and* $\lim \int_A f_n < +\infty$;
(ii) $g \leqslant f_n \leqslant h$, $n = 1, 2, \ldots$.
Then $f \in \mathscr{I}(A)$ *and* $\int_A f = \lim \int_A f_n$.

Indeed, in view of Propositions 3.5, 4.1, and 4.4, it suffices to apply the monotone and dominated convergence theorems for the Lebesgue integral to the sequences $\{f_n - f_1\}$ and $\{f_n - g\}$, respectively.

NOTE. If $m \geqslant 2$, then a countable union of slight sets need not be a thin set. Thus it is unclear how to obtain Corollary 4.6 directly from Definition 3.1.

5. The divergence theorem

We say that sets $A, B \subset \mathbf{R}$ are *λ-equivalent* if $\lambda[(A - B) \cup (B - A)] = 0$. For $z = (\zeta_1, \ldots, \zeta_{m-1})$ in $\mathbf{R}^{m-1}$ and t in $\mathbf{R}$, we write (z, t) instead of $(\zeta_1, \ldots, \zeta_{m-1}, t)$. Given $E \subset \mathbf{R}^m$, we let $E^z = \{t \in \mathbf{R} : (z, t) \in E\}$ for each $z \in \mathbf{R}^{m-1}$, and we set $E^\wedge = \{z \in \mathbf{R}^{m-1} : E^z \neq \varnothing\}^-$. If $A \in \mathscr{A}$, we denote by $\bar{\sigma}_A$ the *completion* of the surface measure σ_A.

We begin by summarizing some results of J. Mařík.

5.1. LEMMA. *Let $A \in \mathscr{A}$ and let $n_A = (\nu_1, \ldots, \nu_m)$. There is a set $A^* \subset A^\wedge$ such that the following conditions are satisfied.*
(i) $\lambda_{m-1}(A^\wedge - A^*) = 0$.
(ii) *For each $z \in A^*$, there is an integer $k(z) \geqslant 0$ and real numbers $a_1^z < b_1^z < \cdots < a_{k(z)}^z < b_{k(z)}^z$ such that A^z and $\bigcup_{j=1}^{k(z)}(a_j^z, b_j^z)$ are λ-equivalent. In particular, the points (z, a_j^z) and (z, b_j^z), $j = 1, \ldots, k(z)$, belong to $A^\bullet$.*
(iii) $\nu_m(z, a_j^z) < 0 < \nu_m(z, b_j^z)$ *for each $z \in A^*$ and $j = 1, \ldots, k(z)$.*
(iv) *If f is a bounded $\bar{\sigma}_A$-measurable function on $A^\bullet$, and*

$$f^*(z) = \sum_{j=1}^{k(z)} \left[f(z, b_j^z) - f(z, a_j^z) \right]$$

for each $z \in A^*$, then f^* is Lebesgue integrable on A^* and

$$(L)\int_{A^*} f^* \, d\lambda_{m-1} = (L)\int_{A^\bullet} f \nu_m \, d\bar{\sigma}_A.$$

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In particular,

$$(L)\int_{A^{\cdot}} |\nu_m|\, d\sigma_A = (L)\int_{A^*} 2k\, d\lambda_{m-1}$$

where k denotes the function $z \mapsto k(z)$.

PROOF. The existence of $A^* \subset A^{\wedge}$ which satisfies (i) and (ii) follows directly from [9, Theorem 33]. Moreover,

$$(L)\int_{A^{\cdot}} |\nu_m|\, d\sigma_A = (L)\int_{A^*} 2k\, d\lambda_{m-1}$$

by [9, Remark 2 to Theorem 18]. Using [9, Theorem 20], we see that (iv) is satisfied whenever f is a bounded Borel function on $A^{\cdot}$. In particular,

$$(L)\int_{A^*} (\operatorname{sign} \nu_m)^*\, d\lambda_{m-1} = (L)\int_{A^{\cdot}} |\nu_m|\, d\sigma_A = (L)\int_{A^*} 2k\, d\lambda_{m-1}.$$

Since $(\operatorname{sign} \nu_m)^* \leqslant 2k$ in A^*, replacing A^* by $\{z \in A^* : (\operatorname{sign} \nu_m)^*(z) = 2k(z)\}$ proves (iii). Now let f be a bounded function on $A^{\cdot}$ which is zero $\bar{\sigma}_A$-almost everywhere. Then there is a nonnegative bounded Borel function g on $A^{\cdot}$ which is zero σ_A-almost everywhere, and for which $|f| \leqslant g$. By (iii), we have $|f^*| \leqslant (g \operatorname{sign} \nu_m)^*$, and so

$$(L)\int_{A^*} |f^*|\, d\lambda_{m-1} \leqslant (L)\int_{A^*} (g \operatorname{sign} \nu_m)^* d\lambda_{m-1} = (L)\int_{A^{\cdot}} g|\nu_m|\, d\sigma_A = 0.$$

As the map $f \mapsto f^*$ is linear, it is easy to see that (iv) holds in general.

5.2. PROPOSITION. *Let $A \in \mathscr{A}$, and let $B \subset A^{\cdot}$ be $\bar{\sigma}_A$-measurable. Then $\bar{\sigma}_A(B) \leqslant (m/2) \liminf_{n \to \infty} n|U(B, 1/n)|$.*

PROOF. Let $n_A = (\nu_1, \ldots, \nu_m)$, and let χ be the characteristic function of B. Then

$$\bar{\sigma}_A(B) = (L)\int_{A^{\cdot}} \chi\, d\bar{\sigma}_A = \sum_{i=1}^{m} (L)\int_{A^{\cdot}} \chi \nu_i^2\, d\bar{\sigma}_A.$$

In the notation of Lemma 5.1, fix a $z \in A^*$ and observe that

$$(\chi \nu_m)^*(z) \leqslant \sum_{j=1}^{k(z)} \left[\chi(z, b_j^z) + \chi(z, a_j^z)\right].$$

If $U_n = U(B, 1/n)$, $n = 1, 2, \ldots$, then U_n^z contains intervals $(a_j^z - 1/n, a_j^z + 1/n)$ and $(b_j^z - 1/n,\ b_j^z + 1/n)$ whenever $\chi(z, a_j^z) = 1$ and $\chi(z, b_j^z) = 1$, respectively. Thus

$$\frac{2}{n} \sum_{j=1}^{k(z)} \left[\chi(z, b_j^z) + \chi(z, a_j^z)\right] \leqslant \lambda(U_n^z)$$

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for all sufficiently large n. It follows that $2(\chi\nu_m)^*(z) \leqslant \liminf_{n\to\infty} n\lambda(U_n^z)$, and so by Lemma 5.1, (iv), the Fatou lemma and Fubini theorem,

$$2(L)\int_{A^\cdot}\chi\nu_m^2\,d\bar{\sigma}_A = 2(L)\int_{A^*}(\chi\nu_m)^*\,d\lambda_{m-1} \leqslant (L)\int_{A^*}\liminf_{n\to\infty} n\lambda(U_n^z)\,d\lambda_{m-1}$$

$$\leqslant \liminf_{n\to\infty} n(L)\int_{A^*}\lambda(U_n^z)\,d\lambda_{m-1} = \liminf_{n\to\infty} n|U_n|.$$

From this the proposition follows by symmetry.

5.3. Corollary. *If $A \in \mathscr{A}$, then $\sigma_A(S) = 0$ for each slight set $S \subset A$.*

5.4. Lemma. *Let $A \in \mathscr{A}$, let S be a slight subset of A^-, and let v be a bounded vector field on A^- which is continuous in $A^- - S$. Then $v \restriction B^\cdot$ is $\bar{\sigma}_B$-measurable for each $B \in \mathscr{A}(A)$, and the map $B \mapsto (L)\int_{B^\cdot} v \cdot n_B\,d\bar{\sigma}_B$ is a continuous additive function in A.*

Proof. The $\bar{\sigma}_B$-measurability of $v \restriction B^\cdot$ for each $B \in \mathscr{A}(A)$ follows from Corollary 5.3. Thus letting $F(B) = (L)\int_{B^\cdot} v \cdot n_B\,d\bar{\sigma}_B$ for $B \in \mathscr{A}(A)$, we see from [9, Remark 2 to Theorem 14] that F is an additive function in A. Furthermore, if $\alpha = \sup\{|v(x)|: x \in A^-\}$ then $|F(B)| \leqslant \alpha\|B\|$ for each $B \in \mathscr{A}(A)$.

Let $T \subset A^- - S$ be a thin set and let

$$\beta = \sup\left\{\frac{|U(T,\eta)|}{\eta}: 0 < \eta < 1\right\}.$$

Choose a positive $\varepsilon \leqslant 1/2\beta m$, and find a positive $\eta_0 < 1$ so that $C = [U(T,\eta_0)]^-$ does not meet S. Then v is uniformly continuous in $A^- \cap C$, and hence there is a positive $h < \eta_0/m^{1/2}$ such that $|v(x) - v(y)| < \varepsilon^2/2m^{1/2}$ for each $x, y \in A^- \cap C$ with $|x - y| < hm^{1/2}$. Among all cubes $\prod_{i=1}^m [k_i h, (k_i + 1)h]$ where $k_1, \ldots, k_m$ are integers, let $K_1, \ldots, K_n$ be those which meet T. Then

$$nh^m = \left|\bigcup_{j=1}^n K_j\right| \leqslant |U(T, hm^{1/2})| \leqslant \beta h m^{1/2}$$

and so

$$\sum_{j=1}^n \|K_j\| = n(2mh^{m-1}) \leqslant 2\beta m^{3/2}.$$

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Moreover, $\bigcup_{j=1}^{n} K_j \subset U(T, \eta_0)$ and $T \subset (\bigcup_{j=1}^{n} K_j)^\circ$. Find $\eta > 0$ so that $U(T, \eta) \subset \bigcup_{j=1}^{n} K_j$, and for $j = 1, \dots, n$ select an $x_j \in K_j \cap T$. Now if $B \in \mathscr{A}[A \cap U(T, \eta)]$ and $\|B\| < 1/\varepsilon$, then by [9, Theorem 36],

$$\begin{aligned} |F(B)| &\leqslant \sum_{j=1}^{n} |F(B \cap K_j)| \\ &= \sum_{j=1}^{n} \left| (L) \int_{(B \cap K_j)^\cdot} [v(x) - v(x_j)] \cdot n_{B \cap K_j} \, d\bar{\sigma}_{B \cap K_j}(x) \right| \\ &\leqslant \frac{\varepsilon^2}{2m^{1/2}} \sum_{j=1}^{n} \|B \cap K_j\| \leqslant \frac{\varepsilon^2}{2m^{1/2}} \left(\|B\| m^{1/2} + \sum_{j=1}^{n} \|K_j\| \right) \\ &< \frac{\varepsilon}{2} + \varepsilon^2 \beta m \leqslant \varepsilon. \end{aligned}$$

It follows that F is continuous.

Throughout this paper, the word *differentiable* is used in the usual sense (see, e.g. [18, Definition 9.11]). Thus differentiability of a function implies its continuity and the existence of partial derivatives which *need not* be continuous. The ith partial derivative of a differentiable function f is denoted by $\partial_i f$, $i = 1, \dots, m$. If $v = (f_1, \dots, f_m)$ is a vector field on a set $E \subset \mathbf{R}^m$, we call a *divergence* of v any function g on E such that $g(x) = \sum_{i=1}^{m} \partial_i f(x)$ for each $x \in E^\circ$ at which v is differentiable. Each divergence of a vector field v is denoted by $\nabla \cdot v$.

5.5. LEMMA. *Let v be a bounded vector field in an open set $U \subset \mathbf{R}^m$, and let $x \in U$. Suppose that v is differentiable at x, and that $v \restriction B^\cdot$ is $\bar{\sigma}_B$-measurable for each $B \in \mathscr{A}$ with $B^- \subset U$. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| \nabla \cdot v(x)|B| - (L) \int_{B^\cdot} v \cdot n_B \, d\bar{\sigma}_B \right| < \varepsilon |B|$$

for each $B \in \mathscr{A}$ with $B^- \subset U$, $x \in B^-$, $d(B) < \delta$, and $r(B) \geqslant \varepsilon$.

PROOF. It suffices to prove the lemma for $v = (0, \dots, 0, f)$. Let $x = (\xi_1, \dots, \xi_m)$, and let $y = (\eta_1, \dots, \eta_m)$ be in U. By our assumption, there is a function α in U such that

$$f(y) - f(x) = \sum_{i=1}^{m} \partial_i f(x)(\eta_i - \xi_i) + |y - x| \alpha(y)$$

and $\lim_{y\to x}\alpha(y)=0$. Given $\varepsilon>0$, choose a $\delta>0$ so that $|\alpha(y)|<\varepsilon^2$ whenever $|y-x|<\delta$. Select $B\in\mathscr{A}$ with $B^-\subset U$, $x\in B^-$, $d(B)<\delta$, and $r(B)\geqslant\varepsilon$. In the notation of Lemma 5.1, for $z\in B^*$ and $j=1,\dots,k(z)$ we have

$$\begin{aligned}
f(z,b_j^z)-f(z,a_j^z) &= \left[f(z,b_j^z)-f(x)\right]-\left[f(z,a_j^z)-f(x)\right]\\
&= \sum_{i=1}^{m-1}\partial_i f(x)(\zeta_i-\xi_i)+\partial_m f(x)(b_j^z-\xi_m)+\left|(z,b_j^z)-x\right|\alpha(z,a_j^z)\\
&\quad -\sum_{i=1}^{m-1}\partial_i f(x)(\zeta_i-\xi_i)-\partial_m f(x)(a_j^z-\xi_m)-\left|(z,a_j^z)-x\right|\alpha(z,a_j^z)\\
&= \partial_m f(x)(b_j^z-a_j^z)+\left|(z,b_j^z)-x\right|\alpha(z,b_j^z)-\left|(z,a_j^z)-x\right|\alpha(z,a_j^z),
\end{aligned}$$

and hence

$$\left|f^*(z)-\partial_m f(x)\lambda(B^z)\right|<2k(z)\varepsilon^2 d(B).$$

By Lemma 5.1(iv) and the Fubini theorem, we obtain

$$\begin{aligned}
\left|\nabla\cdot v(x)|B|-(L)\int_{B^\cdot}v\cdot n_B\,d\bar\sigma_B\right| &= \left|\partial_m f(x)|B|-(L)\int_{B^\cdot}f\nu_m\,d\bar\sigma_m\right|\\
&= \left|\partial_m f(x)(L)\int_{B^*}\lambda(B^z)\,d\lambda_{m-1}(z)-(L)\int_{B^*}f^*(z)\,d\lambda_{m-1}(z)\right|\\
&\leqslant (L)\int_{B^*}\left|\partial_m f(x)\lambda(B^z)-f^*(z)\right|d\lambda_{m-1}(z)\\
&< \varepsilon^2 d(B)(L)\int_{B^*}2k(z)\,d\lambda_{m-1}(z)\\
&= \varepsilon^2 d(B)(L)\int_{B^\cdot}|\nu_m|\,d\sigma_B\leqslant\varepsilon^2 d(B)\|B\|=\varepsilon^2\frac{|B|}{r(B)}\leqslant\varepsilon|B|.
\end{aligned}$$

Next we prove the *divergence theorem.*

5.6. THEOREM. *Let $A\in\mathscr{A}$, and let C, S, and T be, respectively, a countable, slight, and thin subset of A^-. Let v be a bounded vector field on A^- which is continuous in $A^- - S$ and differentiable in $A^\circ - C\cup T$. Then $v\restriction A^\cdot$ is $\bar\sigma_A$-measurable, $\nabla\cdot v$ is integrable in A, and $\int_A\nabla\cdot v=(L)\int_{A^\cdot}v\cdot n_A\,d\bar\sigma_A$.*

PROOF. By Lemma 5.4, $v\restriction B^\cdot$ is $\bar\sigma_B$-measurable for each $B\in\mathscr{A}(A)$, and if $F(B)=(L)\int_{B^\cdot}v\cdot n_B\,d\bar\sigma_B$ for $B\in\mathscr{A}(A)$, then F is a continuous additive function in A. Choose an $\varepsilon>0$, and enumerate $C-S$ as $\{z_1,z_2,\dots\}$. For each z_n

find a positive

$$\delta(z_n) \leqslant \min\left(1, \frac{2^{-n-2}\varepsilon}{|\nabla \cdot v(z_n)| + 1}\right)$$

so that $|v(x) - v(z_n)| < 2^{-n-2}\varepsilon^2$ whenever $x \in A^-$ and $|x - z_n| < \delta(z_n)$. If $B \in \mathscr{A}(A)$, $z_n \in B^-$, $d = d(B) < \delta(z_n)$, and $r(B) \geqslant \varepsilon$, then

$$\begin{aligned}
&\left|\nabla \cdot v(z_n)|B| - F(B)\right| \\
&\quad \leqslant |\nabla \cdot v(z_n)| \cdot |B| + \left|(L)\int_{B^\cdot} [v(x) - v(z_n)] \cdot n_B \, d\bar{\sigma}_B(x)\right| \\
&\quad \leqslant |\nabla \cdot v(z_n)| d^m + 2^{-n-2}\varepsilon^2 \|B\| < |\nabla \cdot v(z_n)| \delta(z_n) + 2^{-n-2}\frac{\varepsilon^2}{r(B)} \cdot \frac{|B|}{d} \\
&\quad \leqslant 2^{-n-2}\varepsilon + 2^{-n-2}\varepsilon d^{m-1} \leqslant 2^{-n-1}\varepsilon.
\end{aligned}$$

Given $x \in A^\circ - C \cup T$, use Lemma 5.5 to find a $\delta(x) > 0$ such that $|\nabla \cdot v(x)|B| - F(B)| < \varepsilon|B|/2(|A| + 1)$ for each $B \in \mathscr{A}(A)$ with $x \in B^-$, $d(B) < \delta(x)$, and $r(B) \geqslant \varepsilon$. Finally, set $\delta(x) = 1$ whenever $x \in A - [(A^\circ - C \cup T) \cup (C - S)]$. Now let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be in $\mathscr{P}(A - S \cup T, \varepsilon; \delta)$. Then each x_i, $i = 1, \ldots, p$, belongs to $(A - S \cup T)^\circ = A^\circ - S \cup T$, and hence either to $C - S$ or to $A^\circ - C \cup T$. It follows that

$$\sum_{i=1}^{p} \left|\nabla \cdot v(x_i)|A_i| - F(A_i)\right| < \varepsilon \sum_{n=1}^{\infty} 2^{-n-1} + \frac{\varepsilon}{2(|A| + 1)} \sum_{i=1}^{p} |A_i| \leqslant \varepsilon.$$

Since $S \cup T$ is a thin set, $F = \int \nabla \cdot v$ and the theorem is proved.

Letting $C = S = T = \varnothing$ in Theorem 5.6 yields the following corollary.

5.7. COROLLARY. *Let $A \in \mathscr{A}$, and let v be a continuous vector field in A^- which is differentiable in A°. Then $\nabla \cdot v$ is integrable in A and $\int_A \nabla \cdot v = (L)\int_{A^\cdot} v \cdot n_A \, d\sigma_A$.*

The estimates in the proof of Theorem 5.5 suggest that the boundedness of the vector field v may not be essential. Specifically, it appears that at countably many points $z \in Z^-$ a mere growth condition such as $v(x) = o(|x - z|^{1-m})$ when x approaches z should be sufficient (see [17, Theorem 5.4]). However, the next example shows that such a generalization is not possible for the integral defined in this paper (see Remark 7.5).

5.8. EXAMPLE. Let $x_0 = (0, 0)$, and set $v(x_0) = x_0$ and $v(x) = x|x|^{-3/2}\cos \pi|x|^{-1/2}$ for each $x \in \mathbf{R}^2 - \{x_0\}$. Then v is a vector field in $\mathbf{R}^2$

which is continuously differentiable in $\mathbf{R}^2 - \{x_0\}$, and $|v(x) - v(x_0)| = O(|x - x_0|^{-1/2})$ as x approaches x_0. If $A_k = \{x \in \mathbf{R}^2 : (2k+1)^{-2} \leqslant |x| \leqslant (2k)^{-2}\}$, $k = 1, 2, \ldots$, then it is easy to see that $A = \bigcup_{k=1}^{\infty} A_k$ belongs to $\mathscr{A}$. For $x \in \mathbf{R}^2 - \{x_0\}$, we have

$$\nabla \cdot v(x) = \frac{1}{2}|x|^{-3/2}\cos\pi|x|^{-1/2} + \frac{\pi}{2}|x|^{-2}\sin\pi|x|^{-1/2}.$$

Using the polar coordinates, we see that the finite integral

$$(L)\int_A |x|^{-3/2}\cos\pi|x|^{-1/2}\,d\lambda_2(x)$$

exists, and that

$$(L)\int_A |x|^{-2}\sin\pi|x|^{-1/2}\,d\lambda_2(x) = +\infty.$$

Since $\sin\pi|x|^{-1/2} \geqslant 0$ for each $x \in A$, it follows from Propositions 3.5 and 4.4 that $\int_A \nabla \cdot v$ does not exist. An easy calculation also reveals that $(L)\int_{A^{\cdot}} v \cdot n_A\, d\sigma_A = +\infty$.

6. Change of variable

Let $E \subset \mathbf{R}^m$ and $\Phi : E \to \mathbf{R}^m$. We say that Φ is a *regular* map of E if it can be extended to a C^1-diffeomorphism (also denoted by Φ) of an open set $U \subset \mathbf{R}^m$ containing E^-. For a regular map Φ, we denote by $\det\Phi$ the determinant of its Jacobi matrix. If Φ is regular, then Φ and $\det\Phi$ are defined uniquely in E^- and $(E^\circ)^-$, respectively, and they both extend continuously to a neighborhood of E^-.

6.1. LEMMA. *Let Φ be a regular map of $E \subset \mathbf{R}^m$. If E is, respectively, thin or slight, then so is $\Phi(E)$.*

PROOF. Let E be compact. There is an open set $U \subset \mathbf{R}^m$ containing E, and positive numbers α and β such that $|x - y| \leqslant \alpha|\Phi(x) - \Phi(y)|$ for each $x, y \in U$ and $|\Phi(B)| \leqslant \beta|B|$ for each $B \subset U$. Since $\Phi(E)$ is a compact subset of the open set $\Phi(U)$, there is an $\eta_0 > 0$ with $U(\Phi(E), \eta_0) \subset \Phi(U)$. Now if $0 < \eta \leqslant \eta_0$, then $U(\Phi(E), \eta) \subset \Phi[U(E, \alpha\eta)]$ and consequently $|U(\Phi(E), \eta)| \leqslant \beta|U(E, \alpha\eta)|$. Thus as η approaches zero, we have $|U(\Phi(E), \eta)| \leqslant \beta\, O(\alpha\eta)$ or $|U(\Phi(E), \eta)| \leqslant \beta\, o(\alpha\eta)$ according to whether E is thin or slight, respectively. The lemma follows.

6.2. Corollary. *If Φ is a regular map of an admissible set A, then $\Phi(A)$ is admissible.*

Indeed, since $[\Phi(A)]^{\cdot} = \Phi(A^{\cdot})$.

6.3. Theorem. *Let Φ be a regular map of an admissible set A, and let f be an integrable function in $\Phi(A)$. Then $f \circ \Phi \cdot |\det \Phi|$ is integrable in A and $\int_A f \circ \Phi \cdot |\det \Phi| = \int_{\Phi(A)} f$.*

Proof. There is an open set $U \subset \mathbf{R}^m$ containing A^- and positive numbers α, β, β^*, and γ such that the following conditions are satisfied:

(i) $\gamma \geqslant 1$ and $\beta^*/\alpha\gamma \leqslant 1/2$;

(ii) $|\Phi(x) - \Phi(y)| \leqslant \alpha|x - y|$ for each $x, y \in U$;

(iii) $\beta^*|B| \leqslant |\Phi(B)| \leqslant \beta|B|$ for each $B \subset U$;

(iv) $\|\Phi(B)\| \leqslant \gamma\|B\|$ for each $B \in \mathscr{A}(A)$.

Here condition (iv) follows from [9, Theorem 52]. Let $F = \int f$, and let T_F and S_F be the thin and slight sets associated to F by Definitions 3.1 and 2.3, respectively. Applying Lemma 6.1 to Φ^{-1}, we see that the sets $T = \Phi^{-1}(T_F)$ and $S = \Phi^{-1}(S_F)$ are thin and slight subsets of A^-, respectively. We define an additive function G in A by setting $G(B) = F[\Phi(B)]$ for each $B \in \mathscr{A}(A)$.

Choose $\varepsilon_S > 0$, and find an $\eta_S > 0$ so that $|F(C)| < \varepsilon_S$ whenever $C \in \mathscr{A}[\Phi(A) \cap U(S_F, \eta_S)]$ and $\|C\| < \eta_S$. Letting $\eta_S^* = \eta_S/\max(\alpha, \gamma)$ and using (ii) and (iv), it is easy to verify that $|G(B)| < \varepsilon_S$ for each $B \in \mathscr{A}[A \cap U(S, \eta_S^*)]$ with $\|B\| < \eta_S^*$. Now let $E \subset A^- - S$ be a thin set, and let $\varepsilon_E > 0$. By Lemma 6.1, $\Phi(E)$ is a thin subset of $[\Phi(A)]^- - S_F$, and hence there is an $\eta_E > 0$ such that $|F(C)| < \varepsilon_E/\gamma$ for each $C \in \mathscr{A}[\Phi(A) \cap U(\Phi(E), \alpha\eta_E)]$ with $\|C\| < \gamma/\varepsilon_E$. From this, (i), (ii), and (iv) it is easy to see that $|G(B)| < \varepsilon_E$ for each $B \in \mathscr{A}[A \cap U(E, \eta_E)]$ with $\|B\| < 1/\varepsilon_E$. It follows that G is continuous, and we show next that $G = \int f \circ \Phi \cdot |\det \Phi|$.

Choose $\varepsilon > 0$, and for each $x \in A$ find $\eta(x) > 0$ so that

$$\big||\Phi(B)| - |\det \Phi(x)| \cdot |B|\big| < \frac{\varepsilon|B|}{2(f[\Phi(x)] + 1)(|A| + 1)}$$

for each measurable $B \subset A$ with $x \in B^-$ and $d(B) < \eta(x)$. This is possible, for $\det \Phi$ is continuous and $|\Phi(B)| = (L)\int_B |\det \Phi|$. Letting $\varepsilon^* = \varepsilon\beta^*/\alpha\gamma$, there is a $\delta^* : \Phi(A) \to R_+$ such that $\sum_{i=1}^p |f(y_i)|C_i| - F(C_i)| < \varepsilon^*$ for each $\{(C_1, y_1), \ldots, (C_p, y_p)\}$ in $\mathscr{P}(\Phi(A) - T_F, \varepsilon^*; \delta^*)$. Now set $\delta(x) = \min(\eta(x), \delta^*[\Phi(x)]/\alpha)$ for each $x \in A$, and choose a partition

$\{(B_1, x_1), \ldots, (B_p, x_p)\}$ in $\mathscr{P}(A - T, \varepsilon; \delta)$. If $C_i = \Phi(B_i)$ and $y_i = \Phi(x_i)$, $i = 1, \ldots, p$, then $\{(C_1, y_1), \ldots, (C_p, y_p)\}$ belongs to $\mathscr{P}(\Phi(A) - T_F, \varepsilon^*; \delta^*)$; for $T_F = \Phi(T)$, $d(C_i) \leqslant \alpha d(B_i) < \alpha\delta(x_i) \leqslant \delta^*(y_i)$, and $r(C_i) \geqslant (\beta^*/\alpha\gamma)r(B_i) \geqslant \varepsilon^*$, $i = 1, \ldots, p$. As $d(B_i) < \eta(x_i)$ for $i = 1, \ldots, p$, we have

$$\sum_{i=1}^{p} \left| f[\Phi(x_i)] \cdot |\det \Phi(x_i)| \cdot |B_i| - G(B_i) \right|$$

$$\leqslant \sum_{i=1}^{p} |f[\Phi(x_i)]| \cdot \left| |\det \Phi(x_i)| \cdot |B_i| - |\Phi(B_i)| \right|$$

$$+ \sum_{i=1}^{p} \left| f(y_i)|C_i| - F(C_i) \right|$$

$$< \frac{\varepsilon}{2(|A| + 1)} \sum_{i=1}^{p} |B_i| + \varepsilon^* < \frac{\varepsilon}{2} + \varepsilon^*.$$

Since $\varepsilon^* \leqslant \varepsilon/2$ by (i), the theorem is proved.

7. Integration on manifolds and the Stokes theorem

A certain familiarity with manifolds and exterior forms is assumed in this section. For the notation and terminology, we refer to [20] and [2].

Throughout, $\mathbf{R}^m$ and $\mathbf{R}^{m-1}$ are oriented by the standard volume elements $\mu^m = d\xi_1 \wedge \cdots \wedge d\xi_m$ and $\mu^{m-1} = d\xi_1 \wedge \cdots \wedge d\xi_{m-1}$, respectively. By a manifold we mean a paracompact C^1 manifold M with boundary ∂M, which may be empty. If a manifold M is oriented, then ∂M is always given the induced orientation. For a subset E of a manifold M, we give the symbols E^-, E°, and $E^\bullet$ the obvious meaning, and we let $\partial E = E^\bullet \cup (E^- \cap \partial M)$.

Let M be an m-dimensional manifold. We say that $E \subset M$ is an *elementary admissible set* if there is a chart (U, Φ) such that $E^- \subset U$ and $\Phi(E) \in \mathscr{A}$. Using Corollary 6.2, it is easy to show that the family $\mathscr{E}^M$ of all elementary admissible subsets of M is closed with respect to differences. A finite union of elements of $\mathscr{E}^M$ is called an *admissible* subset of M. *Slight* and *thin* subsets of M are defined similarly. The family $\mathscr{A}^M$ of all admissible subsets of M is a ring which contains all compact submanifolds of M. It is not difficult to see that $A \in \mathscr{A}^M$ if and only if A^- is compact and ∂A is a thin subset of M.

Let A be an admissible subset of an *oriented* manifold M, and let θ be an m-form on A. If $A \in \mathscr{E}^M$, then there is a positively oriented chart (U, Φ) with $A^- \subset U$, and a unique function g on $\Phi(A)$ such that $(\Phi^{-1})^*\theta = g\mu^m$. We let $\int_A \theta = \int_{\Phi(A)} g$ whenever the integral on the right exists. In general, A is a disjoint union of elementary admissible sets $A_1, \dots, A_k$, and we let $\int_A \theta = \sum_{i=1}^k \int_{A_i} \theta$ whenever the integrals on the right exist. It follows from Theorem 6.3 and Proposition 3.6 that the value of $\int_A \theta$ does not depend on the choice of (U, Φ) and $A_1, \dots, A_k$. We say that θ is *integrable* in A whenever $\int_A \theta$ is defined.

Let N be an oriented $(m-1)$-dimensional manifold. In complete analogy to the previous paragraphs, we define families $\mathscr{M}^N$ and $\mathscr{N}^N$ of *measurable* and *null subsets* of N, respectively, and $(L)\int_A \omega$ for a *bounded* $\mathscr{M}^N$*-measurable* $(m-1)$-form ω on $A \in \mathscr{M}^N$. We note that according to our definition the family $\mathscr{M}^N$ is a ring, which becomes a σ-algebra whenever N is compact. Similarly, the family $\mathscr{N}^N$ is an ideal in $\mathscr{M}^N$, which becomes a σ-ideal whenever N is compact. We omit further details as this is a well-known process of Lebesgue integration on manifolds.

Let M be an m-dimensional manifold, and let ω be an $(m-1)$-form on a set $E \subset M$. We denote by $d\omega$ any m-form on E such that $d\omega(x)$ is the *exterior derivative* of ω at each $x \in E^\circ$ at which ω is differentiable.

For $k = m-1, m$, let

$$\Delta^k = \left\{ (t_1, \dots, t_k) \in \mathbf{R}^k : \sum_{i=1}^k t_i \leqslant 1 \text{ and } t_i \geqslant 0,\ i = 1, \dots, k \right\}.$$

To simplify the notation, we write μ and Δ instead of μ^m and Δ^m, respectively. We define maps $\varepsilon_j : \mathbf{R}^{m-1} \to \mathbf{R}^m$ by setting

$$\varepsilon_0(\zeta_1, \dots, \zeta_{m-1}) = \left(1 - \sum_{i=1}^{m-1} \zeta_i, \zeta_1, \dots, \zeta_{m-1} \right)$$

and

$$\varepsilon_j(\zeta_1, \dots, \zeta_{m-1}) = (\zeta_1, \dots, \zeta_{j-1}, 0, \zeta_j, \dots, \zeta_{m-1})$$

for $j = 1, \dots, m$. If $\Delta_j = \varepsilon_j(\Delta^{m-1})$, then $\Delta^\bullet = \bigcup_{j=0}^m \Delta_j$. Let $n_0 = (1/m^{1/2}, \dots, 1/m^{1/2})$, and for $j = 1, \dots, m$, let $n_j = (0, \dots, -1, \dots, 0)$ where -1 is the jth coordinate. Clearly $\Delta \in \mathscr{A}$, and using Lemma 5.1 and the symmetry of Δ, it is not difficult to see that $n_\Delta(x) = n_j$ for σ_Δ-almost all $x \in \Delta_j$, $j = 0, \dots, m$.

Given a vector field v on a set $E \subset \mathbf{R}^m$, we define an $(m-1)$-form μ_v on E by setting $\langle \mu_v(x) \mid v_1, \dots, v_{m-1} \rangle = \langle \mu \mid v(x), v_1, \dots, v_{m-1} \rangle$ for each $x \in E$ and each $v_1, \dots, v_{m-1}$ in $\mathbf{R}^m$. If $v = (a_1, \dots, a_m)$, then

$$\mu_v = \sum_{i=1}^m (-1)^{i+1} a_i \, d\xi_1 \wedge \dots \widehat{d\xi_i} \dots \wedge d\xi_m$$

where $d\xi_1 \wedge \ldots \widehat{d\xi_i} \ldots \wedge d\xi_m$ stands for $d\xi_1 \wedge \ldots \wedge d\xi_{i-1} \wedge d\xi_{i+1} \wedge \ldots \wedge d\xi_m$. Thus for each $(m-1)$-form ω on $E \subset \mathbf{R}^m$ there is a unique vector field v on E such that $\omega = \mu_v$. Clearly, $\omega = \mu_v$ is, respectively, bounded, continuous, or differentiable if and only if v is.

7.1. LEMMA. *Let n be a unit vector in $\mathbf{R}^m$, and let $N = \{x \in \mathbf{R}^m : x \cdot n = 0\}$. If $v \in \mathbf{R}^m$, then $\mu_v = (v \cdot n)\mu_n$ on N.*

PROOF. Let $n = (\nu_1, \ldots, \nu_m)$ and $v = (a_1, \ldots, a_m)$. On N, we have $\sum_{i=1}^m \nu_i \xi_i = 0$, and by symmetry, we may assume that $\nu_1 \neq 0$. Then

$$\begin{aligned}
\mu_v &= \sum_{i=1}^m (-1)^{i+1} a_i \, d\xi_1 \wedge \ldots \widehat{d\xi_i} \ldots \wedge d\xi_m \\
&= a_1 \, d\xi_2 \wedge \ldots \wedge d\xi_m + \sum_{i=2}^m (-1)^{i+1} a_i \left(- \sum_{j=2}^m \frac{\nu_j}{\nu_1} d\xi_j \right) \\
&\quad \wedge d\xi_2 \wedge \ldots \widehat{d\xi_i} \ldots \wedge d\xi_m \\
&= \frac{a_1 \nu_1}{\nu_1} d\xi_2 \wedge \ldots \wedge d\xi_m + \sum_{i=2}^m (-1)^i \frac{a_i \nu_i}{\nu_1} d\xi_i \wedge d\xi_2 \wedge \ldots \widehat{d\xi_i} \ldots \wedge d\xi_m \\
&= \frac{v \cdot n}{\nu_1} d\xi_2 \wedge \cdots \wedge d\xi_m,
\end{aligned}$$

and similarly, $\mu_n = (1/\nu_1)\, d\xi_2 \wedge \ldots \wedge d\xi_m$. The lemma follows.

Throughout, we orient the linear submanifold $N_j = \varepsilon_j(\mathbf{R}^{m-1})$ of $\mathbf{R}^m$ by the form μ_{n_j}, $j = 0, \ldots, m$.

7.2. LEMMA. *Let C, S, and T be, respectively, a countable, slight, and thin subset of Δ, and let ω be a bounded $(m-1)$-form on Δ which is continuous in $\Delta - S$ and differentiable in $\Delta^\circ - C \cup T$. Then $\omega \restriction \Delta_j$ is $\mathscr{M}^{N_j}$-measurable for $j = 0, \ldots, m$, $d\omega$ is integrable in Δ, and*

$$\int_\Delta d\omega = \sum_{j=0}^m (L) \int_{\Delta_j} \omega.$$

PROOF. Let $\omega = \mu_v$ where $v = (a_1, \ldots, a_m)$. Then

$$d\omega(x) = \sum_{i=1}^m (-1)^{i+1} \partial_i a_i(x) \, d\xi_i \wedge d\xi_1 \wedge \ldots \widehat{d\xi_i} \ldots \wedge d\xi_m = [\nabla \cdot v(x)]\mu$$

for each $x \in \Delta^\circ - C \cup T$. It follows from Theorem 5.6 that $v \restriction \Delta^\bullet$ is $\bar{\sigma}_\Delta$-measurable, $d\omega$ is integrable in Δ, and $\int_\Delta d\omega = (L)\int_{\Delta^\bullet} v \cdot n_\Delta \, d\bar{\sigma}_\Delta$.

It is easy to check that $\varepsilon_0^*\mu_{n_0} = m^{1/2}\mu^{m-1}$ and $\varepsilon_j^*\mu_{n_j} = (-1)^j\mu^{m-1}$ for $j = 1,\dots,m$. Thus $(N_j, (-1)^j\varepsilon_j^{-1})$ is a positively oriented chart of N_j, $j = 0,\dots,m$. If χ is a characteristic function of Δ_0, then in the notation of Lemma 5.1, we have

$$a_i\left(\zeta_1,\dots,\zeta_{m-1}, 1 - \sum_{k=1}^{m-1}\zeta_k\right) = (a_i\chi)^*(\zeta_1,\dots,\zeta_{m-1}), \qquad i = 1,\dots,m.$$

It follows from Lemma 5.1(iv) that $v \circ \varepsilon_0$ is λ_{m-1}-measurable in Δ^{m-1}, and similarly we establish that so are $v \circ \varepsilon_1, \dots, v \circ \varepsilon_m$. Thus $\omega \restriction \Delta_j$ is $\mathscr{M}^{N_j}$-measurable for $j = 0,\dots,m$.

By Lemma 7.1,

$$\begin{aligned}\left\langle \varepsilon_j^*[\omega(x)] \mid u_1,\dots,u_{m-1}\right\rangle &= \left\langle \mu_v(x) \mid \varepsilon_{j*}u_1,\dots,\varepsilon_{j*}u_{m-1}\right\rangle \\ &= \left[v(x)\cdot n_j\right]\cdot\left\langle \mu_{n_j} \mid \varepsilon_{j*}u_1,\dots,\varepsilon_{j*}u_{m-1}\right\rangle \\ &= \left[v(x)\cdot n_j\right]\cdot\left\langle \varepsilon_j^*\mu_n \mid u_1,\dots,u_{m-1}\right\rangle\end{aligned}$$

for each $x \in \Delta_j$, $j = 0,\dots,m$, and each $u_1,\dots,u_{m-1}$ in $\mathbf{R}^{m-1}$. Thus on Δ^{m-1}, we have $\varepsilon_0^*\omega = m^{1/2}(v\circ\varepsilon_0\cdot n_0)\mu^{m-1}$, and $(-1)^j\varepsilon_j^*\omega = (v\circ\varepsilon_j\cdot n_j)\mu^{m-1}$ for $j = 1,\dots,m$. Consequently,

$$\begin{aligned}\sum_{j=0}^{m}(L)\int_{\Delta_j}\omega &= m^{1/2}(L)\int_{\Delta^{m-1}} v\circ\varepsilon_0\cdot n_0\,d\lambda_{m-1} + \sum_{i=1}^{m}(L)\int_{\Delta^{m-1}} v\circ\varepsilon_i\cdot n_i\,d\lambda_{m-1} \\ &= \sum_{i=1}^{m}(L)\int_{\Delta^{m-1}}(a_i\circ\varepsilon_0 - a_i\circ\varepsilon_i)\,d\lambda_{m-1}.\end{aligned}$$

If $n_\Delta = (\nu_1,\dots,\nu_m)$, then by Lemma 5.1(iv),

$$\begin{aligned}&(L)\int_{\Delta^{m-1}}(a_m\circ\varepsilon_0 - a_m\circ\varepsilon_m)\,d\lambda_{m-1} \\ &\qquad= (L)\int_{\Delta^{m-1}} a_m\left(1 - \sum_{k=1}^{m-1}\zeta_k, \zeta_1,\dots,\zeta_{m-1}\right)d\lambda_{m-1}(\zeta_1,\dots,\zeta_{m-1}) \\ &\qquad\quad -(L)\int_{\Delta^{m-1}} a_m(z,0)\,d\lambda_{m-1}(z) \\ &\qquad= (L)\int_{\Delta^{m-1}} a_m\left(\zeta_1,\dots,\zeta_{m-1}, 1 - \sum_{k=1}^{m-1}\zeta_k\right)d\lambda_{m-1}(\zeta_1,\dots,\zeta_{m-1}) \\ &\qquad\quad -(L)\int_{\Delta^{m-1}} a_m(z,0)\,d\lambda_{m-1}(z) \\ &\qquad= (L)\int_{\Delta^*}(a_m)^*\,d\lambda_{m-1} \\ &\qquad= (L)\int_{\Delta^{\cdot}} a_m\nu_m\,d\bar{\sigma}_\Delta;\end{aligned}$$

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here Δ^* and $(a_m)^*$ are the symbols introduced in Lemma 5.1. Now by symmetry,

$$(L)\int_{\Delta^{m-1}} (a_i \circ \varepsilon_0 - a_i \circ \varepsilon_i)\, d\lambda_{m-1} = (L)\int_{\Delta^{\cdot}} a_i \nu_i \, d\bar{\sigma}_\Delta$$

for $i = 1, \dots, m-1$. It follows that

$$\sum_{j=0}^{m} (L)\int_{\Delta_j} \omega = (L)\int_{\Delta^{\cdot}} v \cdot n_\Delta \, d\bar{\sigma}_\Delta$$

and the lemma is proved.

Now we can prove the *Stokes theorem*.

7.3. THEOREM. *Let M be an m-dimensional compact oriented manifold, and let C, S, and T be, respectively, a countable, slight and thin subset of M. Let ω be a bounded $(m-1)$-form on M which is continuous in $M - S$ and differentiable in $M - (\partial M \cup C \cup T)$. Then $\omega \restriction \partial M$ is $\mathscr{M}^{\partial M}$-measurable, $d\omega$ is integrable in M, and $\int_M d\omega = (L)\int_{\partial M} \omega$.*

PROOF. By [21, Chapter IV, Theorem 12A], the manifold M has a finite C^1 triangulation $\mathscr{T}$. For each $A \in \mathscr{T}$, there is a positively oriented chart (U_A, Φ_A) such that $A \subset U_A$ and $\Phi_A(A) = \Delta$. Given $A \in \mathscr{T}$ and $j = 0, \dots, m$, we denote by A_j the $(m-1)$-dimensional submanifold $\Phi_A^{-1}(\Delta_j^\circ)$ of M oriented by the form $\Phi_A^* \mu_{n_j}$. In particular, if $A_j \subset \partial M$ the the orientation of A_j is induced by that of ∂M. If $A \in \mathscr{T}$, then A is an elementary admissible set, and it follows from Lemma 7.2 that $\omega \restriction A_j$ is $\mathscr{M}^{A_j}$-measurable, $d\omega$ is integrable in A, and

$$\int_A d\omega = \int_\Delta (\Phi_A^{-1})^* d\omega = \int_\Delta d(\Phi_A^{-1})^* \omega = \sum_{j=0}^{m} (L)\int_{\Delta_j} (\Phi_A^{-1})^* \omega = \sum_{j=0}^{m} (L)\int_{A_j} \omega.$$

If A and B are in $\mathscr{T}$ and $\Phi_A^{-1}(\Delta_j) = \Phi_B^{-1}(\Delta_i)$ for some integers j and i, then $(L)\int_{A_j} \omega = -(L)\int_{B_i} \omega$; for the oriented manifolds A_j and B_i have opposite orientations. Since $M = \bigcup \mathscr{T}$ and $\partial M = \bigcup\{A_j^- : A \in \mathscr{T},\ A_j \subset \partial M\}$, we see that $d\omega$ is integrable in M and $\omega \restriction \partial M$ is $\mathscr{M}^{\partial M}$-measurable. As the integrals of $d\omega$ and ω vanish on the overlaps, we have

$$\int_M d\omega = \sum_{A \in \mathscr{T}} \int_A d\omega = \sum_{A \in \mathscr{T}} \sum_{j=0}^{m} (L)\int_{A_j} \omega = \sum_{A \in \mathscr{T}} \sum_{A_j \subset \partial M} (L)\int_{A_j} \omega = (L)\int_{\partial M} \omega,$$

and the theorem is proved.

Letting $C = S = T = \varnothing$ in Theorem 7.3, we obtain the following corollary.

7.4. COROLLARY. *Let M be an m-dimensional compact oriented manifold, and let ω be a continuous $(m-1)$-form on M which is differentiable in $M - \partial M$. Then $d\omega$ is integrable in M and $\int_M d\omega = (L)\int_{\partial M} \omega$.*

7.5. REMARK. At this point it appears appropriate to mention that our choice of admissible sets is only one of many. For example, replacing $\mathscr{A}$ by a smaller family of all diffeomorphic images of simplices or convex linear cells, leads to an integration process very similar to ours. Whether there is an optimal family of admissible sets is unclear.

8. The one-dimensional case

Throughout this section, we assume that $m = 1$. Our aim is to compare the integral of Definition 3.1 to the classical Denjoy-Perron integral (see [19, Chapter VI, Section 6, and Chapter VIII]), or equivalently to the Henstock-Kurzweil integral defined in [10]. For a simple proof of the fact that the Denjoy-Perron and Henstock-Kurzweil integrals are indeed equivalent we refer to [7] or [15, Appendix B].

8.1. LEMMA. *Slight sets are empty, and thin sets are finite.*

PROOF. The first part of the lemma is obvious, because $|U(E, \eta)| \geqslant 2\eta$ for each nonempty set $E \subset \mathbf{R}$ and each $\eta > 0$. Let $E \subset \mathbf{R}$ be infinite, and let $\{x_1, x_2, \dots\}$ be an infinite sequence of distinct points from E. Given an integer $n \geqslant 1$, find $\eta > 0$ so that $2\eta < |x_i - x_j|$ for all distinct $i, j = 1, \dots, n$. Then $|U(E, \eta)| \geqslant 2n\eta$, and it follows that E is not thin.

8.2. COROLLARY. *Modulo finite sets, each admissible set is a finite disjoint union of nondegenerate compact intervals.*

For $A, B \subset R$, we write $A \sim B$ whenever the symmetric difference $(A - B) \cup (B - A)$ is finite.

Let $A \in \mathscr{A}$. By Corollary 8.2, there is an integer $k > 0$ and real numbers $a_1 < b_1 < \cdots < a_k < b_k$ such that $A \sim \bigcup_{i=1}^{k}(a_i, b_i)$. It follows from Lemma 5.1 that σ_A is the counting measure concentrated on $\{a_1, \dots, a_k, b_1, \dots, b_k\}$, and that $n_A(b_i) = -n_A(a_i) = 1$ for $i = 1, \dots, k$. If F is a function on $A^{\bullet}$, then extending the notation of Lemma 5.1, we let $F^*(A) = \sum_{i=1}^{k}[F(b_i) - F(a_i)]$. As F is also a vector field on $A^{\bullet}$, we have $F^*(A) = (L)\int_{A^{\bullet}} F \cdot n_A \, d\sigma_A$. Now if F is a function on A^-, then the map F^* given by $B \mapsto F^*(B)$ for $B \in \mathscr{A}(A)$ is an additive function in A. Moreover, F^* is continuous if and only if F is; for the continuity of F^* clearly implies that of F, and the converse follows from Lemma 5.4.

If f is an integrable function in $[a, b]$, we write $\int_a^b f$ instead of $\int_{[a,b]} f$. From Theorem 5.6, we obtain immediately the following proposition.

8.3. **Proposition.** *Let f and F be functions on $[a, b]$ such that F is continuous and $F'(x) = f(x)$ for all but countably many $x \in (a, b)$. Then f is integrable in $[a, b]$ and $\int_a^b f = F(b) - F(a)$.*

Next we show that the integral is closed with respect to the formation of *improper integrals.*

8.4. **Proposition.** *Let f be a function on $[a, b]$ which is integrable in $[a, x]$ for each $x \in [a, b)$, and let a finite $\lim_{x \to b-} \int_a^x f = I$ exist. Then f is integrable in $[a, b]$ and $\int_a^b f = I$.*

Proof. If $F(x) = \int_a^b f$ for $x \in [a, b)$ and $F(b) = I$, then F is continuous in $[a,b]$, and by Lemma 5.4, so is F^*. Choose an $\varepsilon > 0$ and a sequence $\{b_n\}$ such that $a = b_1 < b_2 < \cdots < b$ and $\lim b_n = b$. Now $F^* = \int f$ in each $[b_n, b_{n+1}]$, $n = 1, 2, \ldots$. By extending the sequence $\{b_n\}$, we may assume that $\{b_n, b_{n+1}\}$ is a thin subset of $[b_n, b_{n+1}]$ associated to F^* by Definition 3.1. Choose an $\varepsilon > 0$, and for $n = 1, 2, \ldots$, find $\delta_n : [b_n, b_{n+1}] \to \mathbf{R}_+$ so that $\sum_{i=1}^q |f(y_i)|B_i| - F^*(B_i)| < \varepsilon/2^{n+1}$ for each $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ in $\mathscr{P}([b_n, b_{n+1}], \varepsilon; \delta_n)$. There is a $\delta : [a, b] \to \mathbf{R}_+$ such that for $n = 1, 2, \ldots$ the following conditions are satisfied:

(i) $\delta(x) = \min(\delta_n(x), x - b_n, b_{n+1} - x)$ for each $x \in (b_n, b_{n+1})$;

(ii) $|f(b_n)|\delta(b_n) < \varepsilon/3 \cdot 2^{n+1}$, and $|F^*(B)| < \varepsilon/3 \cdot 2^{n+1}$ for each admissible set $B \subset [a, b] \cap U[\{b_n\}, \delta(b_n)]$ with $\|B\| < 1/\varepsilon$.

Let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be in $\mathscr{P}([a, b], \varepsilon; \delta)$. By (i), we see that $\{(A_1, x_i) \in P : x_i \in (b_n, b_{n+1})\}$ belongs to $\mathscr{P}([b_n, b_{n+1}], \varepsilon; \delta_n)$, $n = 1, 2, \ldots$. Since

$$\|A_i\| = \frac{|A_i|}{d(A_i) r(A_i)} \leqslant \frac{1}{\varepsilon},$$

we obtain from (ii) that $|f(x_i)|A_i| - F^*(A_i)| < \varepsilon/3 \cdot 2^n$ whenever $x_i = b_n$ for some $n = 2, 3, \ldots$; of course, no x_i equals $b_1 = a$. As the map $i \mapsto x_i$ is at most three-to-one, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \left| f(x_i)|A_i| - F^*(A_i) \right| \\
&\quad = \sum_{n=2}^{\infty} \left[\sum_{x_i = b_n} \left| f(x_i)|B_i| - F(B_i) \right| + \sum_{x_i \in (b_{n-1}, b_n)} \left| f(x_i)|A_i| - F^*(A_i) \right| \right] \\
&\quad < \varepsilon \sum_{n=2}^{\infty} 2^{-n+1} = \varepsilon,
\end{aligned}$$

and the proposition is proved.

The *Denjoy-Perron integral* over $[a, b]$ of a Denjoy-Perron integrable function f in $[a, b]$ is denoted by $(DP)\int_a^b f$.

8.5. Proposition. *If f is an integrable function in $[a, b]$, then f is Denjoy-Perron integrable in $[a, b]$ and $(DP)\int_a^b f = \int_a^b f$.*

Proof. Let f be an integrable function in $[a, b]$. We may assume that $\{a, b\}$ is a thin set T_F associated to $F = \int f$ by Definition 3.1; for if $T_F \subset \{z_0, \ldots, z_n\}$ where $a = z_0 < \cdots < z_n = b$, we can repeat the argument for each interval $[z_{i-1}, z_i]$, $i = 1, \ldots, n$. Given $\varepsilon > 0$, there is a $\delta : [a, b] \to \mathbf{R}_+$ satisfying the following conditions:

(i) $\delta(x) \leqslant \min(x - a, b - x)$ for each $x \in (a, b)$;

(ii) $\delta(a) = \delta(b) = \eta$ where $\max(|f(a)|, |f(b)|) < \varepsilon/8\eta$;

(iii) $|F(B)| < \varepsilon/8$ for each admissible set $B \subset [a, b] \cap U(\{a, b\}, \eta)$ with $\|B\| \leqslant 2$;

(iv) $\sum_{i=1}^p |f(x_i)|A_i| - F(A_i)| < \varepsilon/2$ for each $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $\mathscr{P}([a, b], 1/2; \delta)$.

Now let $a = t_0 < \cdots < t_k = b$ and $x_1, \ldots, x_k$ be such that $t_{i-1} \leqslant x_i \leqslant t_i$ and $t_i - t_{i-1} < \delta(x_i)$, $i = 1, \ldots, k$. Set $A_i = [t_{i-1}, t_i)$ for $i = 1, \ldots, k - 1$, and $A_k = [t_{k-1}, t_k]$. By (i), $x_1 = a$, $x_k = b$, and $\{(A_2, x_2), \ldots, (A_{k-1}, x_{k-1})\}$ belongs to $\mathscr{P}([a, b], 1/2; \delta)$. Thus by (ii)–(iv), we have

$$\left| \sum_{i=1}^{k} f(x_i)(t_i - t_{i-1}) - \int_a^b f \right| \leqslant |f(a)| \cdot |A_1| + |f(b)| \cdot |A_k| + |F(A_1)|$$
$$+ |F(A_2)| + \sum_{i=2}^{k-1} \bigl| f(x_i)|A_i| - F(A_i) \bigr| < \varepsilon,$$

and the proposition follows from the aforementioned equivalence of the Henstock-Kurzweil and Denjoy-Perron integrals.

The next example shows that the converse of Proposition 8.5 is false. In view of Propositions 4.1, 8.3, and 8.4, this is somewhat surprising.

8.6. Example. If $J = (a, b) \subset \mathbf{R}$ is nonempty, let $J_+^n = (a + 2^{-2n}|J|, a + 2^{-2n+1}|J|)$, $J_-^n = (a + 2^{-2n+1}|J|, a + 2^{-2n+2}|J|)$, $n = 1, 2, \ldots$. For $x \in \mathbf{R}$, set $f_J(x) = 2^{2n}/n$ if $x \in J_+^n$, $f_J(x) = 2^{2n-1}/n$ if $x \in J_-^n$, and $f_J(x) = 0$ otherwise. Since $\int_{J_+^n} f_J = -\int_{J_-^n} f_J = |J|/n$, $n = 1, 2, \ldots$, it follows from Proposition 8.4 that f_J is integrable in J with $\int_a^b f_J = 0$, and that

$$\sup\left\{ \left| \int_\alpha^\beta f_J \right| : a \leqslant \alpha \leqslant \beta \leqslant b \right\} = |J|.$$

Let D be the Cantor ternary set in $[0, 1]$, and let $\mathscr{J}$ be the collection of all components of $[0, 1] - D$. Letting $f = \sum_{J \in \mathscr{J}} f_J$, it is easy to see from [19, Chapter VIII, Theorem (5.1)] that f is Denjoy-Perron integrable in $[0, 1]$ with $(DP)\int_0^1 f = 0$. We show next that the integral $\int_0^1 f$ does not exist.

Proceeding towards contradiction, suppose that f is integrable in $[0,1]$, and choose a $\delta:[0,1]\to\mathbf{R}_+$. Using the Baire category theorem in D, we find an open set $U\subset\mathbf{R}$ with $D\cap U\neq\varnothing$, and an $\eta>0$ such that the set $E=\{x\in D\cap U:\delta(x)\geqslant\eta\}$ is dense in $D\cap U$. Select a $J=(a,b)$ in $\mathscr{J}$ with $J^-\subset U$, and construct a sequence $\{x_n\}$ in E so that $x_1<x_2<\cdots<a$, and $a-x_n<2^{-2n}|J|$ for $n=1,2,\ldots$. If $A_n=J_+^n\cup\{x_n\}$, then $\|A_n\|=2$, $|A_n|=2^{-2n}|J|$, and $d(A_n)\leqslant 3\cdot 2^{-2n}|J|$, $n=1,2,\ldots$. There is an n_0 with $d(A_{n_0})<\eta$, and we see that $\{(A_{n_0},x_{n_0}),\ldots,(A_{n_0+p},x_{n_0+p})\}$ belongs to $\mathscr{P}([0,1],1/6;\eta)$ for each $p=0,1,\ldots$. Since thin sets are finite and

$$\sum_{k=n_0}^{n_0+p}\left|f(x_k)|A_k|-\int_{A_k}f\right|=\sum_{k=n_0}^{n_0+p}\int_{J_+^n}f_J=|J|\sum_{k=n_0}^{n_0+p}\frac{1}{k},$$

the contradiction follows.

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Department of Mathematics
University of California
Davis, California 95616
U.S.A.