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Fractional time differential equations as a singular limit of the Kobayashi–Warren–Carter system

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This paper is concerned with a singular limit of the Kobayashi–Warren–Carter system, a phase field system modelling the evolutions of structures of grains. Under a

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suitable scaling, the limit system is formally derived when the interface thickness parameter tends to zero. Different from many other problems, it turns out that the limit system is a system involving fractional time derivatives, although the original system is a simple gradient flow. A rigorous derivation is given when the problem is reduced to a gradient flow of a single-well Modica–Mortola functional in a one-dimensional setting.

Keywords: fractional time derivative; gradient flow; Kobayashi–Warren–Carter system; singular limit

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1. Introduction

We consider the Kobayashi–Warren–Carter system, introduced in [14, 15, 29], to model evolutions of structures in a multi-grain problem. It is a kind of phase-field system in a domain Ω in $\mathbf{R}^n$, formally a gradient flow of the energy

$$E^{\varepsilon}_{\mathrm{KWC}}(u,v) := \int_{\Omega} \alpha_0(v)|\nabla u| + E^{\varepsilon}_{\mathrm{sMM}}(v), \tag{1.1}$$

$$E^{\varepsilon}_{\mathrm{sMM}}(v) := \int_{\Omega} \frac{\varepsilon}{2}|\nabla v|^2\,dx + \int_{\Omega} \frac{1}{2\varepsilon}F(v)\,dx. \tag{1.2}$$

Here, $\alpha_0(v) \geq 0$ is a given function, typically $\alpha_0(v) = rv^2$, with a constant $r > 0$, and $F(v)$ is a single-well potential, typically $F(v) = a^2(v-1)^2$, with a constant $a > 0$. The functional $E^{\varepsilon}_{\mathrm{sMM}}$ is often called a single-well Modica–Mortola functional. The Kobayashi–Warren–Carter system is regarded as a gradient flow with respect to L^2-inner product

$$((u_1,v_1),(u_2,v_2)) = \int_{\Omega} \alpha_w u_1 u_2\,dx + \tau \int_{\Omega} v_1 v_2\,dx, \qquad (u_i,v_i) \in L^2(\Omega) \times L^2(\Omega), \quad i = 1,2,$$

where $\alpha_w \geq 0$ and $\tau > 0$ are weights. The function $\alpha_w = \alpha_w(v)$ is given, but it may depend on $v \in L^2(\Omega)$, so the above inner product is a Riemann metric on the tangent bundle $TL^2(\Omega)$. A typical form of $\alpha_w(v)$ equals $\alpha_w(v) = \tau_0 v^2$, where τ_0 is a positive constant. We consider the gradient flow of $E^{\varepsilon}_{\mathrm{KWC}}$ under this metric, and its explicit form is

$$\begin{cases} \tau v_t = \varepsilon \Delta v - \frac{1}{2\varepsilon}F'(v) - \alpha_0'(v)|\nabla u|, & (1.3) \\ \alpha_w(v) u_t = \operatorname{div}\left(\alpha_0(v)\frac{\nabla u}{|\nabla u|}\right). & (1.4) \end{cases}$$

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An explicit form in [14] corresponds to the case when $s = \varepsilon$, $\nu = \varepsilon^2$, $\tau_1 = \tau\varepsilon$, $F(v) = (v-1)^2$, $\alpha_0(v) = v^2$, $\alpha_w(v) = \tau_0 v^2/\varepsilon$ with $\tau_0 > 0$. In other words,

$$\begin{cases} \tau_1 v_t = \nu\Delta v + (1-v) - 2sv|\nabla u|, & (1.5) \\ \tau_0 v^2 u_t = s \operatorname{div}\left(v^2 \frac{\nabla u}{|\nabla u|}\right). & (1.6) \end{cases}$$

The function v represents an order parameter, where $v=1$ corresponds to a grain region and where v away from 1 corresponds to grain boundaries. The function u represents a structure-like averaged angle in each grain.

We are interested in a singular limit problem for Eqs. (1.3) and (1.4) as $\varepsilon \downarrow 0$. It turns out that the correct scaling of time should be $\tau = \tau_1/\varepsilon$, while τ_1 is independent of ε. Since the system Eqs. (1.3)–(1.4) is regarded as a gradient flow of $E^{\varepsilon}_{\mathrm{KWC}}(u,v)$ of Eq. (1.1), we are tempted to expect that the limit flow is the gradient flow of its limit energy E^0_{KWC} which was obtained in our papers [8, 9]. Surprisingly, this conjecture is wrong. The limit flow contains a fractional time derivative. In this paper, we consider the problem in a one-dimensional setting. Moreover, we consider a special but typical case when the problem is essentially reduced to a single equation for v in Eq. (1.3) because handing Eq. (1.4) is technically involved since it is a total variation flow type equation. This reduced problem becomes a linear problem and is easy to discuss.

We consider Eqs. (1.3)–(1.4), where Ω is an interval $\Omega = (-L, L)$, and impose the Dirichlet boundary condition for u and the Neumann boundary condition for v. More precisely,

$$u(-L,t) = 0, \quad u(L,t) = b > 0, \tag{1.7}$$

while

$$v_x(\pm L, t) = 0 \quad \text{for} \quad t > 0. \tag{1.8}$$

We set

$$F(v) = a^2(v-1)^2 \quad \text{with} \quad a \geq 0, \quad \alpha_0(v) = v^2. \tag{1.9}$$

We expect that the function

$$u^b(x) = \begin{cases} b, & x > 0, \\ 0, & x < 0, \end{cases}$$

with $b > 0$, solves Eq. (1.4). Since Eq. (1.4) is of total variation flow type, the definition of a solution is not obvious. Fortunately, under a suitable assumption of v, say $v(0,t) \leq v(x,t)$ for all $x \in (-L,L)$, $t > 0$, the function u^b solves Eq. (1.4) under Eq. (1.7), as shown in the following lemma by setting $\beta = \alpha_0(v)$.

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LEMMA 1.1. (A stationary solution) *Assume that* $\beta \in C[-L, L]$ *satisfies*

$$\beta(0) \leq \beta(x) \quad \textit{for all} \quad x \in (-L, L).$$

Then, u^b *solves*

$$\left(\beta \frac{u_x}{|u_x|}\right)_x = 0 \quad \textit{in} \quad (-L, L)$$

under Eq. (1.7).

We stress that the notion of a solution of the equation for u in lemma 1.1 and also Eq. (1.4) under the Dirichlet condition Eq. (1.7) is not obvious and will be discussed in §4.

Problem Eqs. (1.3)–(1.4) are reduced to

$$\frac{\tau_1}{\varepsilon} v_t = \varepsilon v_{xx} - \frac{a^2(v-1)}{\varepsilon} - 2bv\partial_x(1_{x>0}) \quad \text{in} \quad \Omega \times (0, \infty) \tag{1.10}$$

under the boundary condition

$$v_x(\pm L, t) = 0, \quad t > 0, \tag{1.11}$$

and the initial condition

$$v(x, 0) = v_0^\varepsilon(x), \quad x \in \Omega, \tag{1.12}$$

where $1_{x>0}$ is a characteristic function of $(0, +\infty)$, i.e., the Heaviside function, so that its distributional derivative equals the Dirac δ function. We note that if we assume that v_0^ε is even and non-decreasing for $x > 0$, i.e., $v_0^\varepsilon(x) = v_0^\varepsilon(-x)$ and $v_{0x}^\varepsilon(x)x \geq 0$ for $x \in (-L, L)$, then u^b is indeed a stationary solution of Eq. (1.4) with Eq. (1.7). This follows from lemma 1.1 since we have $v^\varepsilon(0, t) \leq v^\varepsilon(x, t)$ for all $x \in (-L, L)$, $t > 0$ by the maximum principle. By a scaling transformation $y = x/\varepsilon$, Eq. (1.10) becomes

$$\tau_1 V_t = V_{yy} - a^2(V - 1) - 2bV\partial_y(1_{y>0}) \tag{1.13}$$

in $(-L/\varepsilon, L/\varepsilon) \times (0, \infty)$ for $V = V^\varepsilon(y, t) = v^\varepsilon(\varepsilon y, t)$, where v^ε is a solution of Eq. (1.10). Thus, we expect that this limit $V = \lim_{\varepsilon \to 0} V^\varepsilon$ solves Eq. (1.13) on $\mathbf{R}$ and is bounded. Since the solution v^ε of Eq. (1.10) is expected to converge to 1 except at $x = 0$, we are interested in the behaviour of $\xi^\varepsilon(t) = v^\varepsilon(0, t)$. More precisely, we would like to find the equation which $\xi = \lim_{\varepsilon \to 0} \xi^\varepsilon$ solves.

We set

$$E_{\text{sMM}}^{0,b}(\zeta) = b\zeta^2 + 2G(\zeta) \quad \text{where} \quad G(\zeta) = \left| \int_1^\zeta \sqrt{F(\rho)}\, d\rho \right|.$$

Since $F(v) = a^2(v-1)^2$ to get $G(\zeta) = a(\zeta - 1)^2/2$, we have

$$E_{\text{sMM}}^{0,b}(\zeta) = b\zeta^2 + a(\zeta - 1)^2.$$

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As proved in [9], this energy is obtained as a value at Ξ of Γ-limit of $E^{\varepsilon}_{\mathrm{sMM}}(v)$ in Eq. (1.2) as $\varepsilon \to 0$ if v converges to a set-valued function Ξ (under the graph convergence) of the form

$$\Xi(x) = \begin{cases} \{1\} & \text{for} \quad x \neq 0, \\ [\zeta, 1] & \text{for} \quad x = 0 \end{cases}$$

for $\zeta \in (0,1)$.

A key observation is to derive an equation for ξ. For $\alpha \in \mathbf{R}$, $\beta > 0$, we set

$$f^{\alpha}_{\beta}(t) := \frac{e^{-\alpha t} t^{\beta-1}}{\Gamma(\beta)},$$

where Γ denotes the gamma function. We consider well-prepared initial data in the sense that it is continuous and solves Eq. (1.13) outside $y = 0$.

LEMMA 1.2. (Limit equation) *Assume that $\tau_1 = 1$. Let V be the bounded solution of Eq. (1.13) in $\mathbf{R} \times (0,\infty)$ with initial data $V(y,0) = 1 - ce^{-a|y|}$ with some $c \in \mathbf{R}$. Then, $\xi(t) = V(0,t)$ solves*

$$\int_0^t m_a(t-s)\xi_s(s)\, ds = -\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\xi), \tag{1.14}$$

with

$$m_a(t) = 2\left\{ f^{a^2}_{1/2}(t) + a^2 \int_0^t f^{a^2}_{1/2}(s)\, ds - a \right\}.$$

Moreover, $m'_a(t) < 0$ and $m_a(t) > 0$, and $\lim_{t\to\infty} m_a(t) = 0$.

The assumption $\tau_1 = 1$ is just for the convenience of presentation. The formula for general τ_1 is obtained by rescaling the time variable t by $t' = t\tau_1$.

Note that in case $a = 0$, the left-hand side of Eq. (1.14) becomes the Caputo derivative $2\partial_t^{1/2}$. For even initial data, the solution V of Eq. (1.13) is even. Since V_y is odd in y, we see that the delta part of V_{yy} equals

$$\left(V_y(+0,t) - V_y(-0,t)\right)\delta(y) = -2V_y(-0,t)\delta(y),$$

where $V(\pm 0,t) = \lim_{\gamma\to\pm 0} V(\gamma,t)$ and δ denotes the Dirac delta function. Since V is smooth outside $y = 0$ and V_t has no delta part, Eq. (1.13) deduces that

$$2V_y(-0,t)\delta(y) + 2bV(0,t)\delta(y) = 0.$$

Thus, Eq. (1.13) in $\mathbf{R} \times (0,\infty)$ is reduced to the Robin boundary problem in $(-\infty,0) \times (0,\infty)$ with

$$\partial_y V(-0,t) + bV(0,t) = 0.$$

The Caputo derivative appears in the equation of the boundary value.

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COROLLARY 1.3. *Let $w = w(x,t)$ be the bounded solution of the heat equation*

$$w_t - w_{xx} = 0 \quad in \quad (-\infty, 0) \times (0, \infty),$$

with the Robin boundary condition

$$w_x(0,t) + bw(0,t) = 0 \quad for \quad t > 0.$$

Assume that $w(x,0) = -c$ for some $c \in \mathbf{R}$; the boundary value $\xi(t) = w(0,t)$ solves

$$\int_0^t f^0_{1/2}(t-s)\xi_s(s)\, ds = -b\xi(t), \quad t > 0.$$

In other words, $\partial_t^{1/2}\xi = -b\xi$, where $\partial_t^{1/2}$ is the Caputo half derivative.

It is well known that the fractional Laplace operator $(-\Delta)^{1/2}$ arises as the Dirichlet–Neumann map of the Laplace equation. Here, the Caputo derivative $\partial_t^{1/2}$ is obtained as the Dirichlet–Neumann map of the heat equation. Formally, it is easy to guess since the Robin boundary condition yields $\partial_t^{1/2}\xi + b\xi = 0$ by replacing ∂_x with $\partial_t^{1/2}$, which is natural since $\partial_t = \partial_x^2$ for w. In a seminal paper, Caffarelli and Silvestre [3] show that $(-\Delta)^\gamma$ $(0 < \gamma < 1)$ is obtained as the Dirichlet–Neumann map for the degenerate Laplace equation. We remark that ∂_t^γ is obtained as the Dirichlet–Neumann map of a degenerate heat equation $w_t - (-x)^\alpha w_{xx} = 0$ with $\gamma = 1/(2-\alpha)$ as in [3]; see remark 2.4 at the end of §2. We do not pursue this problem in this paper.

Since Eq. (1.13) is linear and of constant coefficients, we can use the Laplace transform to obtain the desired equation. As one expects, we are able to prove the convergence of V^ε.

LEMMA 1.4. (Convergence) *Let V be the bounded solution of Eq. (1.13) in $\mathbf{R} \times (0,\infty)$ with initial data V_0, which is bounded and uniformly continuous on $\boldsymbol{R}$. Let v^ε be the solution of Eq. (1.10) under Eqs. (1.11) and (1.12). Assume that $V_0^\varepsilon = V^\varepsilon|_{t=0} \in C\,[-L/\varepsilon, L/\varepsilon]$ converges to V_0 in the sense that*

$$\lim_{\varepsilon \downarrow 0} \sup_{|y| \le L/\varepsilon} |V_0^\varepsilon(y) - V_0(y)| = 0.$$

Then, V^ε converges to V locally uniformly in $\mathbf{R} \times [0,\infty)$. In particular, $\xi^\varepsilon(t) = v^\varepsilon(0,t) = V^\varepsilon(0,t)$ converges to $\xi(t) = V(0,t)$ locally uniformly in $[0,\infty)$. If we assume that

$$\lim_{|y| \to \infty} (V_0(y) - 1)\,|y| = 0,$$

then

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le t \le T} \|V^\varepsilon - V\|_{L^\infty(-L/\varepsilon, L/\varepsilon)} = 0$$

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for any $T > 0$*. Moreover,*

$$\lim_{|y|\to\infty} \sup_{0\le t\le T} (|V(y,t)-1|\,|y|) = 0.$$

Although the statement is for linear equations, we have to be a little bit careful since the domains where solutions are defined depend on ε. We extend the initial data V_0^ε to a whole space. If the solution W^ε to the whole space problem with the extended data $\overline{V_0^\varepsilon}$ solves the Neumann problem in $(-L/\varepsilon, L/\varepsilon)$, it is very convenient. We take this strategy. However, $\overline{V_0^\varepsilon}$ may not be close to V_0 outside $(-L/\varepsilon, L/\varepsilon)$, so the maximum principle does not imply the convergence of the solution W^ε starting from $\overline{V_0^\varepsilon}$ to V even locally. We regularize initial data and prove that $\{W^\varepsilon\}$ is bounded and equi-continuous in $\mathbf{R}\times[0,T]$, $0<T<\infty$. By Arzelà–Ascoli theorem and a diagonal argument, it converges to some function by taking a subsequence locally uniformly in $\mathbf{R}\times[0,T]$. The limit is identified with a solution V, so it is a full convergence. This eventually yields that V^ε converges to V locally uniformly in $\mathbf{R}\times[0,\infty)$, so it yields the convergence of ξ^ε. Because of the term $V\partial_y(1_{y>0})$, the actual proof is more involved. We divide the initial data into even part and odd part. For the odd part, the problem is reduced to the problem without the $V\partial_y(1_{y>0})$ term. For the even part, the problem is reduced to the Robin boundary problem and the proof is more involved. Note that we do not assume behaviour of $V_0(y)$ so $|y|\to\infty$ to derive locally uniform convergence.

To obtain the uniform convergence of V^ε to V in $(-L/\varepsilon, L/\varepsilon)\times(0,T)$, it seems that the behaviour of $V_0(y)$ as $|y|\to\infty$ should be controlled. We observe that our decay assumption for $V_0(y)-1$ is preserved not only for V but also for $\partial_y V$ for $t>\delta>0$. This allows us to compare V^ε with V globally in space. All proofs are quite elementary, but we give full proofs for the reader's convenience.

Applying lemmas 1.2 and 1.4, we can obtain a characterization of the limit equation.

THEOREM 1.5 *Assume that* $\tau_1 = 1$*. Let* v^ε *be the solution of Eq. (1.10) under Eqs. (1.11) and (1.12). Assume that* v_0^ε *is well-prepared in the sense*

$$\sup_{x\in\Omega}\left|v_0^\varepsilon(x) - \left(1 - ce^{-a|x|/\varepsilon}\right)\right| \to 0,$$

as $\varepsilon\to 0$*, with some* c *independent of* ε*. Then,* $\xi^\varepsilon(t)$ *converges to* ξ *locally uniformly in* $[0,\infty)$*, and* ξ *solves Eq. (1.14). Moreover, the graph of* v_ε *converges to a set-valued function* Ξ *of the form*

$$\Xi(x,t) = \begin{cases} \{1\} \quad \textit{for} \quad x\neq 0, \\ \textit{closed interval between } \xi(t) \textit{ and } 1 \textit{ for } x = 0. \end{cases}$$

The convergence is in the sense of the Hausdorff distance of graphs over $[-L,L]\times[0,T]$ *for any* $T > 0$*.*

We also handle initial data not necessarily well-prepared. In this case, Eq. (1.14) is altered because there are a few lower-order terms; see Eq. (2.9). If $V(y,0) =$

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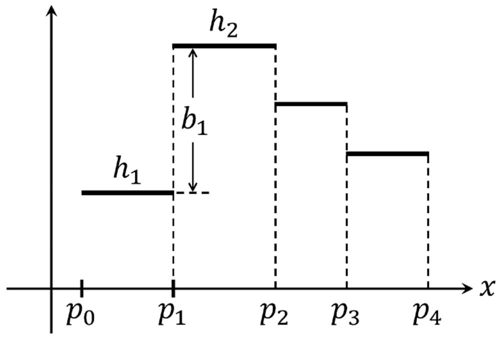

Figure 1. Graph of u with $\chi_1 = 1, \chi_2 = -1$ and $\chi_3 = -1$

$1 - ce^{-\mu|x|}$ with $\mu \neq a$, we still get an explicit form corresponding to Eq. (1.14); see Eq. (2.16). In these cases, the solution ξ is explicitly represented by using the error function. We shall discuss these extensions for non-well-prepared data and the proof of lemma 1.2 in §2. We also calculate numerically how the solution of the ε-problem converges by comparing it with the explicit solution of Eq. (1.14) and more general Eq. (2.16).

We now come back to the singular limit problem of the Kobayashi–Warren–Carter system Eqs. (1.3)–(1.4), with $\tau = \tau_1/\varepsilon$, $F(v) = a^2(v-1)^2$, and $\alpha_0(v) = v^2$ as $\varepsilon \to 0$. We give a purely formal argument. Eq. (1.4) is of the total variation flow type, and its well-posedness for given v is known when v is independent of time and $\inf \alpha_w(v) > 0$; in this case, Eq. (1.4) is the gradient flow of the weighted total variation $\int \alpha_0(v)|\nabla u|$ under $\alpha_w(v)$-weighted L^2 inner product if we impose the natural boundary condition like the Neumann boundary condition. As in [4], we consider Eq. (1.4) for piecewise constant functions. For an interval $I = (p, q)$ and its given division $p = p_0 < p_1 < \cdots < p_m = q$, we consider a piecewise constant function of form

$$u(x,t) = h_j(t), \quad p_{j-1} < x \le p_j \quad (j = 1, \ldots, m).$$

We interpret a solution by mimicking the notion of a solution when v is independent of time and $\inf \alpha_w(v) > 0$ (under the periodic boundary condition for simplicity). It is of the form

$$\begin{cases} \alpha_w(v)\partial_t u = \operatorname{div}(v^2 z), \quad |z| \le 1, \\ z(p_j) = \chi_j, \quad j = 0, 1, \ldots, m-1, \end{cases} \tag{1.15}$$

where $\chi_j = \operatorname{sgn}(h_{j+1} - h_j)$; we identify $p_m = p_0$ by periodicity. See figure 1. The function z is called a Cahn–Hoffman vector field.

For a given v, it is unclear whether the solution remains in a class of piecewise constant functions [4], known as a facet-splitting problem. If v is constant, it is well known that the solution remains in a class of piecewise constant functions.

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We postulate that our system Eq. (1.15) has a (spatially) piecewise constant solution. Integrating the first equation of Eq. (1.15) in (p_{j-1}, p_j) yields

$$\int_{p_{j-1}}^{p_j} \alpha_w \left(v(x,t)\right) \, dx \frac{d}{dt} h_j(t) = v^2(p_j, t)\chi_j - v^2(p_{j-1}, t)\chi_{j-1}.$$

If $v_\varepsilon \to \Xi$ in the graph topology as $\varepsilon \to 0$, we arrive at

$$\alpha_w(1) \frac{d}{dt} h_j = \frac{\xi_j^2 \chi_j - \xi_{j-1}^2 \chi_{j-1}}{p_j - p_{j-1}}, \tag{1.16}$$

where the set-valued function Ξ is defined as

$$\Xi(x,t) = \begin{cases} \{1\}, & x \notin \{p_1, \ldots, p_{m-1}\}, \\ [\xi_j(t), 1], & x = p_j. \end{cases}$$

Since jump $b_j = |h_{j+1} - h_j|$ determines the ξ_j-equation as indicated in theorem 1.5, the equation for ξ_j (assuming $\tau_1 = 1$) is expected to be

$$M_a \partial_t \xi_j = -\operatorname{grad} E_{\mathrm{sMM}}^{0,b_j}(\xi_j) \left(= -2 \left(\left(|h_{j+1} - h_j| + a \right) \xi_j - a \right) \right), \tag{1.17}$$

where $M_a f = \int_0^t m_a(t-s) f(s) \, ds$ for well-prepared initial data at least when $h_{j+1} - h_j$ is independent of time t. We postulate that Eq. (1.17) is still valid when h_j depends on time. Thus, the singular limit equation of the Kobayashi–Warren–Carter equations Eqs. (1.3)–(1.4) as $\varepsilon \to 0$ (under the periodic boundary condition) is expected to be the system Eqs. (1.16)–(1.17) for h_j and ξ_j. If we consider other boundary conditions, we always impose the homogeneous Neumann boundary condition for v, like Eq. (1.8). If we consider the Dirichlet boundary condition for u with time-independent data, the values of h_1 and h_m are prescribed. We can impose the Neumann condition for u; in this case, imposing $z(p_0) = z(p_m) = 0$ in Eq. (1.15) is natural. It is rather standard [23] to construct a unique local-in-time solution for a system of a fractional differential equation and an ordinary differential equation.

We note that the solvability of the initial value problem for the original Kobayashi–Warren–Carter system Eqs. (1.5)–(1.6) is still an open problem, even in a one-dimensional setting. If we write it in the form of Eqs. (1.3)–(1.4), the difficulty stems from the fact that the weights α_w and α_0 can vanish somewhere. In the literature, α_w is assumed to be away from zero. If α_0 is allowed to vanish, the Δu term is added in the right-hand side of Eq. (1.6). For example, instead of considering Eq. (1.6), we consider

$$\tau_0(v^2 + \delta) u_t = s \operatorname{div} \left((v^2 + \delta') \nabla u / |\nabla u| + \mu \nabla u \right), \tag{1.18}$$

with $\delta > 0$, $\delta' \geq 0$ and $\mu \geq 0$ such that $\delta' + \mu > 0$. The existence of a solution to Eqs. (1.5) and (1.18) with its large time behaviour is established in [10–13, 19, 20, 26, 27, 30] under several homogeneous boundary conditions. (The case $\delta = 0$ is included in [11, 30].) Unfortunately, the uniqueness of their solution is only known

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in a one-dimensional setting under the relaxation term $\mu > 0$ [10, Theorem 2.2]. The extension of these results to inhomogeneous boundary condition is not difficult. In [21], under non-homogeneous Dirichlet boundary conditions, structured patterns of stationary (i.e., time-independent) solutions were studied. In a one-dimensional setting, they thoroughly characterized all stationary solutions. Our Γ-convergence result in [9] gives the convergence of minimizer of $E^{\varepsilon}_{\mathrm{KWC}}$ to that of the limit energy as $\varepsilon \to 0$. We do not know the convergence of stationary solutions.

This paper is organized as follows. In §2, we derive the limit equation for both well-prepared and non-well-prepared initial data. We also give the large time behaviour of a solution. Most of the calculations are very explicit. In §3, we prove lemma 1.4. Section 4 gives a rigorous definition of the Dirichlet problem for Eq. (1.4), assuming that v is given and $\alpha_w(v) \equiv 1$. In §5, we give several numerical tests.

2. Derivation of equations with the fractional time derivative

The first goal of this section is to prove lemma 1.2. We then consider more general initial data. We in particular consider the case $V(y,0) = 1 - ce^{-\mu|y|}$ with $\mu > 0$ not necessarily equal to a and derive the equation corresponding to Eq. (1.14); see Eq. (2.16). The solution ξ can be written explicitly (see Eq. (2.18)), and it is of the form

$$\xi = 1 + \bar{\eta} + \eta_e(t), \quad \lim_{t\to\infty} \bar{\eta}(t) = -\frac{b}{b+a},$$

with a lower-order term

$$\lim_{t\to\infty} \eta_e(t) e^{a^2 t} = 0.$$

This asymptotic result is shown for general μ; see lemma 2.3 for a detailed statement. At the end of this section, we give remarks related to corollary 1.3. It turns out that corollary 1.3 follows from an expression of the Dirichlet–Neumann map of the heat equation. The Caputo derivatives can be naturally derived using the Dirichlet–Neumann map.

We begin with recalling several elemental properties of the Laplace transform

$$\mathcal{L}[g](\lambda) := \int_0^\infty e^{-\lambda s} g(s)\, ds, \quad \lambda > 0$$

for a locally integrable function g in $[0,\infty)$. By definition,

$$\mathcal{L}\left[e^{-\mu s} g(s)\right](\lambda) = \mathcal{L}[g](\lambda + \mu), \tag{2.1}$$

and by definition of the Gamma function, we see

$$\mathcal{L}[s^{\beta-1}](\lambda) = \Gamma(\beta)\lambda^{-\beta}.$$

We now arrive at a well-known formula

$$\mathcal{L}[f^{\alpha}_{\beta}](\lambda) = \mathcal{L}[e^{-\alpha s} f^{0}_{\beta}](\lambda) = (\lambda + \alpha)^{-\beta} \tag{2.2}$$

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for $\alpha, \beta > 0$.

LEMMA 2.1. *For $a > 0$, let $m_a(t)$ be*

$$m_a(t) = 2\left\{ f_{1/2}^{a^2}(t) + a^2 \int_0^t f_{1/2}^{a^2}(s)\, ds - a \right\}.$$

Then

$$\mathcal{L}[m_a](\lambda) = 2\left(\frac{\sqrt{\lambda + a^2} - a}{\lambda} \right).$$

Moreover, $m_a'(t) < 0$ and $m_a(t) > 0$ for $t > 0$, with $\lim_{t\to\infty} m_a(t) = 0$.

Proof. We recall that the Laplace transform of the convolution $(g_1 * g_2)(t) = \int_0^t g_1(t-s)g_2(s)\, ds$ is given by

$$\mathcal{L}[g_1 * g_2] = \mathcal{L}[g_1]\mathcal{L}[g_2]. \tag{2.3}$$

If we take $g_1 \equiv 1$, we see

$$\mathcal{L}\left[\int_0^t g_2(s)\, ds \right](\lambda) = \mathcal{L}[g_2](\lambda)\lambda^{-1} \tag{2.4}$$

since $\mathcal{L}[g_1](\lambda) = \lambda^{-1}$ by Eq. (2.2). Setting $\alpha = a^2$, we apply Eqs. (2.2) and (2.4) to get

$$\begin{aligned} \mathcal{L}\left[\frac{m_a}{2}\right] &= (\lambda+\alpha)^{-1/2} + \alpha(\lambda+\alpha)^{-1/2}\lambda^{-1} - a\lambda^{-1} \\ &= \left((\lambda+\alpha)^{1/2} - a \right) \lambda^{-1}, \end{aligned}$$

which yields the desired formula for $\mathcal{L}[m_a]$.

We differentiate $m_a/2$ to get

$$\begin{aligned} \frac{1}{2} m_a'(t) &= (-\alpha) f_{1/2}^{\alpha}(t) - \frac{1}{2t} f_{1/2}^{\alpha}(t) + \alpha f_{1/2}^{\alpha}(t) \\ &= -\frac{1}{2t} f_{1/2}^{\alpha}(t) < 0 \quad \text{for} \quad t > 0. \end{aligned}$$

We observe that

$$\lim_{t\to\infty} \int_0^t f_{1/2}^{\alpha}(s)\, ds = \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-\alpha s} s^{\frac{1}{2}-1}\, ds = \alpha^{-1/2}.$$

Thus, $\lim_{t\to\infty} m_a(t) = 0$ since $\lim_{t\to\infty} f_{1/2}^{\alpha}(t) = 0$, which implies that $m_a(t) > 0$ for $t > 0$ since $m_a'(t) < 0$. □

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Proof of lemma 1.2. Since the property of m_a has been proved in lemma 2.1, it suffices to derive Eq. (1.14). Studying the equation for $w = V - 1$ instead of Eq. (1.13) is more convenient. Eq. (1.13) for w (with $\tau_1 = 1$) becomes

$$w_t - w_{xx} + a^2 w + 2b(w+1)\delta = 0, \quad x \in \mathbf{R}, \quad t > 0. \tag{2.5}$$

The initial data $w(x,0)$ equal

$$w(x,0) = w_0(x),$$

with $w_0(x) = -ce^{-a|x|}$. Let $\widehat{w}(x,\lambda)$ be the Laplace transform of w in the t variable, i.e.,

$$\widehat{w}(x,\lambda) = \mathcal{L}[w](x,\lambda) = \int_0^\infty e^{-\lambda t} w(x,t)\, dt.$$

We note that

$$\widehat{w_t} = \lambda \widehat{w} - w_0, \quad w|_{t=0} = w_0.$$

Taking the Laplace transform of Eq. (2.5), we arrive at

$$\lambda \widehat{w} - w_0 - \widehat{w}_{xx} + a^2 \widehat{w} + 2b\left(\widehat{w} + \lambda^{-1}\right)\delta = 0, \tag{2.6}$$

a second-order linear ordinary differential equation with a jump of the derivatives $\widehat{w}_x$. Since the coefficients are constants, we can solve Eq. (2.6) explicitly.

A bounded solution satisfying

$$\lambda \widehat{w} - w_0 - \widehat{w}_{xx} + a^2 \widehat{w} = 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad x < 0$$

is of form

$$\widehat{w} = Ae^{-\sqrt{\lambda + a^2}|x|} - \frac{c}{\lambda} e^{-a|x|},$$

with $A \in \mathbf{R}$. One should determine A so that Eq. (2.6) holds. We observe

$$\widehat{w}_x = \left(-A\sqrt{\lambda + a^2} e^{-\sqrt{\lambda+a^2}|x|} + \frac{ca}{\lambda} e^{-a|x|}\right)(\operatorname{sgn} x)$$

so that

$$\widehat{w}_{xx} = \left(-2A\sqrt{\lambda + a^2} + \frac{2ca}{\lambda}\right)\delta + A(\lambda + a^2)e^{-\sqrt{\lambda+a^2}|x|} - \frac{ca^2}{\lambda} e^{-a|x|}$$

since $(\operatorname{sgn} x)' = 2\delta$. Eq. (2.6) imposes that

$$\text{the } \delta \text{ part of } \widehat{w}_{xx} = 2b(\widehat{w} + \lambda^{-1})\delta,$$

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which implies

$$-2A\sqrt{\lambda + a^2} + \frac{2ca}{\lambda} = 2b\left(\widehat{w}(0,\lambda) + \lambda^{-1}\right).$$

Since $\widehat{w}(0,\lambda) = A - c\lambda^{-1}$, we end up with

$$\left(\sqrt{\lambda + a^2} + b\right) A = -\frac{b(1-c)}{\lambda} + \frac{ca}{\lambda}.$$

We set $\eta(t) = w(0,t)$ and observe that

$$\widehat{\eta} = A - \frac{c}{\lambda} = \frac{1}{\sqrt{\lambda + a^2} + b}\frac{1}{\lambda}(ca + bc - b) - \frac{c}{\lambda}.$$

We next extract the time derivative of η. We note that

$$\widehat{\eta_t} = \lambda\widehat{\eta} - \eta(0)$$

so that $\widehat{\eta_t} = \lambda\widehat{\eta} + c$ since $\eta(0) = -c$. Then,

$$\widehat{\eta_t} = \frac{1}{\sqrt{\lambda + a^2} + b}(ca + bc - b).$$

Multiplying $\sqrt{\lambda + a^2} + b$ and subtracting $(a+b)\widehat{\eta_t}$ from both sides, we get

$$\begin{aligned}\left(\sqrt{\lambda + a^2} - a\right)\widehat{\eta_t} &= -(b+a)\widehat{\eta_t} + (ca + bc - b) \\ &= -(b+a)\lambda\widehat{\eta} - (b+a)c + (ca + bc - b) \\ &= \lambda\left(-(b+a)\widehat{\eta} - \frac{b}{\lambda}\right).\end{aligned}$$

The energy for ξ equals

$$E^{0,b}_{\mathrm{sMM}}(\xi) = b\xi^2 + a(\xi - 1)^2.$$

Since $\eta = \xi - 1$, we see $\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\xi) = 2(b+a)\eta + 2b$. Thus,

$$2\left(\frac{\sqrt{\lambda + a^2} - a}{\lambda}\right)\widehat{\eta_t} = -2(b+a)\widehat{\eta} - 2b\widehat{1}. \tag{2.7}$$

By lemma 2.1 and Eq. (2.3), we now conclude that $\xi = \eta + 1$ solves Eq. (1.14) by taking the inverse Laplace transform of Eq. (2.7). □

In lemma 1.2, we only considered well-prepared initial data, meaning that $V(y,t)$ is a stationary solution of Eq. (1.13) in $\mathbf{R}\backslash\{0\}$. We take this opportunity to write

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a general equation corresponding to Eq. (1.14) starting from general initial data. We set

$$(M_a f)(t) = m_a * f(t) = \int_0^t m_a(t-s) f(s)\, ds,$$

and Eq. (1.14) is

$$M_a \xi_s = -\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\xi). \tag{2.8}$$

For general initial data, our equation corresponding to Eq. (2.8) becomes more complicated than Eq. (2.8).

We set

$$G^a_\lambda(y) = \frac{e^{-\sqrt{\lambda + a^2}|y|}}{2\sqrt{\lambda + a^2}},$$

which is the Green function of $-\partial_y^2 + \sigma$ with $\sigma = \lambda + a^2$, i.e.,

$$-\partial_y^2 G^a_\lambda + (\lambda + a^2) G^a_\lambda = \delta.$$

For $w_0 \in L^\infty(\mathbf{R})$, we set

$$g^a(\lambda, w_0) = (G^a_\lambda *_x w_0)(0),$$

where $*_x$ denotes the convolution in space, i.e.,

$$(G^a_\lambda *_x w_0)(y) = \int_{-\infty}^{\infty} G^a_\lambda(y - z) w_0(z)\, dz.$$

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Lemma 2.2. *Let* $\tau_1 = 1$. *Let* V *be the bounded solution of Eq. (1.13) in* $\mathbf{R} \times (0, \infty)$ *with initial data* $V(y, 0) = 1 + w_0$, *where* w_0 *is bounded and Lipschitz continuous in* $\boldsymbol{R}$. *Then,* $\xi(t) = V(0, t)$ *solves*

$$M_a \xi_t + m_a (\xi(0) - 1) - \mathcal{L}^{-1} \left[2\sqrt{\lambda + a^2} g^a(\lambda, w_0) \right] = -\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\xi). \tag{2.9}$$

Proof. If w_0 is Lipschitz, then we easily see that $|w_t| t^{1/2}$ is bounded (near $t = 0$) so that the integrand of

$$\mathcal{L}[w_t](\lambda) = \int_0^\infty e^{-\lambda s} w_s(x, s)\, ds$$

is integrable for all $x \in \mathbf{R}$ near $s = 0$; see lemma 3.4.

We argue in the same way to prove lemma 1.2. We begin with Eq. (2.6), where $w = V - 1$. Its solution outside $x = 0$ is given as

$$\widehat{w} = A e^{-\sqrt{\lambda + a^2} |x|} + G^a_\lambda *_x w_0,$$

with $A \in \mathbf{R}$. As before, we determine A and obtain that $\eta = \xi - 1$ satisfies

$$\widehat{\eta}(\lambda) = \frac{-b}{\sqrt{\lambda + a^2} + b} \left(g^a(\lambda, w_0) + \frac{1}{\lambda} \right) + g^a(\lambda, w_0). \tag{2.10}$$

Since $\widehat{\eta_t} = \lambda \widehat{\eta} - \eta(0)$, we proceed with

$$\begin{aligned}
&\left(\sqrt{\lambda + a^2} - a \right) \widehat{\eta_t} \\
&\quad = -(b + a) \widehat{\eta_t} - \lambda b \left(g^a(\lambda, w_0) + \frac{1}{\lambda} \right) + \left(\sqrt{\lambda + a^2} + b \right) (\lambda g^a(\lambda, w_0) - \eta(0)) \\
&\quad = -(b + a) \lambda \widehat{\eta} + (b + a) \eta(0) - \lambda b \left(g^a(\lambda, w_0) + \frac{1}{\lambda} \right) \\
&\qquad + \left(\sqrt{\lambda + a^2} + b \right) (\lambda g^a(\lambda, w_0) - \eta(0)) \\
&\quad = -(b + a) \lambda \widehat{\eta} - \lambda \frac{b}{\lambda} + \left(a - \sqrt{\lambda + a^2} \right) \eta(0) + \sqrt{\lambda + a^2} \lambda g^a(\lambda, w_0).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{2\left(\sqrt{\lambda + a^2} - a \right)}{\lambda} \widehat{\eta_t} &= -2(b + a) \widehat{\eta} \\
&\qquad - 2\frac{b}{\lambda} - \frac{2\left(\sqrt{\lambda + a^2} - a \right)}{\lambda} \eta(0) + 2\sqrt{\lambda + a^2} g^a(\lambda, w_0).
\end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$M_a \eta_t = -\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\eta + 1) - m_a(t) \eta(0) + \mathcal{L}^{-1} \left(2\sqrt{\lambda + a^2} g^a(\lambda, w_0) \right),$$

the same as Eq. (2.9). □

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The term $\mathcal{L}^{-1}\left[2\sqrt{\lambda+a^2}g^a\right]$ has a more explicit form. Let $E(x,t)$ be the Gauss kernel, i.e.,

$$E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}.$$

We know that its Laplace transform (as a function of t) is

$$\mathcal{L}[E](\lambda) = \frac{1}{2\lambda^{1/2}} e^{-|x|\lambda^{1/2}}.$$

Let $E^a = e^{-a^2 t}E$, then

$$\mathcal{L}[E^a](\lambda) = \frac{1}{2\sqrt{\lambda+a^2}} e^{-\sqrt{\lambda+a^2}|x|} = G^a_\lambda(x).$$

Since $\mathcal{L}\left[f^{a^2}_{1/2}\right] = (\lambda+a^2)^{-1/2}$ and $\mathcal{L}[g_1 * g_2] = \mathcal{L}[g_1]\mathcal{L}[g_2]$, we end up with

$$\mathcal{L}\left[f^{a^2}_{1/2} * E^a\right](\lambda) = \frac{1}{2(\lambda+a^2)} e^{-\sqrt{\lambda+a^2}|x|}.$$

Thus,

$$\mathcal{L}\left[\partial_t\left(f^{a^2}_{1/2} * E^a\right) + a^2\left(f^{a^2}_{1/2} * E^a\right)\right] = \frac{1}{2} e^{-\sqrt{\lambda+a^2}|x|}.$$

Since $\sqrt{\lambda+a^2}g^a = \frac{1}{2}\left(e^{-\sqrt{\lambda+a^2}|x|} *_x w_0\right)(0,t)$, we conclude that

$$\mathcal{L}^{-1}\left[2\sqrt{\lambda+a^2}g^a\right] = 2\partial_t\left(f^{a^2}_{1/2} * (E^a *_x w_0)(0)\right) + 2a^2 f^{a^2}_{1/2} * (E^a *_x w_0)(0).$$

If $w_0 = -ce^{-\mu|x|}$, we observe that

$$\mathcal{L}^{-1}\left[2\sqrt{\lambda+a^2}g^a\right] = -2c\left(f^{a^2}_{1/2} - \mu e^{(\mu^2-a^2)t}\operatorname{erfc}\left(\mu\sqrt{t}\right)\right), \tag{2.11}$$

where erfc denotes the complementary error function, i.e.,

$$\operatorname{erfc}(s) = \frac{2}{\sqrt{\pi}}\int_s^\infty e^{-\tau^2}\,d\tau.$$

Indeed, we proceed with

$$\begin{aligned} g^a(\lambda) &= \left(G^a_\lambda *_x \left(-ce^{-\mu|x|}\right)\right)(0) = -c\frac{1}{2\sqrt{\lambda+a^2}}\int_{-\infty}^\infty e^{-\sqrt{\lambda+a^2}|0-y|}e^{-\mu|y|}\,dy \\ &= -\frac{c}{\sqrt{\lambda+a^2}}\int_0^\infty e^{-\left(\mu+\sqrt{\lambda+a^2}\right)y}\,dy = -\frac{c}{\sqrt{\lambda+a^2}\left(\mu+\sqrt{\lambda+a^2}\right)}. \end{aligned} \tag{2.12}$$

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By a direct calculation, we observe that

$$
\begin{aligned}
\mathcal{L}\left[\operatorname{erfc}\left(\mu\sqrt{t}\right)\right] &= \int_0^\infty e^{-\lambda t}\left(\frac{2}{\sqrt{\pi}}\int_{\mu\sqrt{t}}^\infty e^{-s^2}\,ds\right)dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty\left[\int_0^{s^2/\mu^2} e^{-s^2}e^{-\lambda t}\,dt\right]ds = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}\left[\int_0^{s^2/\mu^2} e^{-\lambda t}\,dt\right]ds \\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}\left(\frac{1-e^{\lambda s^2/\mu^2}}{\lambda}\right)ds = \frac{2}{\sqrt{\pi}}\frac{1}{\lambda}\left(\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\sqrt{\frac{\mu^2}{\mu^2+\lambda}}\right) \\
&= \frac{1}{\lambda}\left(1-\frac{\mu}{\sqrt{\lambda+\mu^2}}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{L}\left[f_{1/2}^0 - \mu e^{\mu^2 t}\operatorname{erfc}\left(\mu\sqrt{t}\right)\right] &= \frac{1}{\sqrt{\lambda}} - \frac{\mu}{\lambda-\mu^2}\left(1-\frac{\mu}{\sqrt{\lambda}}\right) = \frac{\lambda-\mu^2-\sqrt{\lambda}\mu+\mu^2}{\sqrt{\lambda}(\lambda-\mu^2)} \\
&= \frac{\sqrt{\lambda}-\mu}{\lambda-\mu^2} = \frac{1}{\sqrt{\lambda}+\mu},
\end{aligned}
\tag{2.13}
$$

where we invoked Eqs. (2.2) and (2.1). We set $q^\mu(t) = f_{1/2}^0(t) - \mu e^{\mu^2 t}\operatorname{erfc}\left(\mu\sqrt{t}\right)$ so that Eq. (2.13) becomes

$$
\mathcal{L}[q^\mu] = \frac{1}{\sqrt{\lambda}+\mu}. \tag{2.14}
$$

Using Eq. (2.1) again, we now obtain Eq. (2.11).

The formula (2.13) gives another representation of m_a. Indeed, since

$$
\frac{1}{2}\mathcal{L}[m_a](\lambda) = \frac{\sqrt{\lambda+a^2}-a}{\lambda} = \frac{1}{\sqrt{\lambda+a^2}+a},
$$

we see

$$
\frac{1}{2}m_a(t) = e^{-a^2 t}q^a(t) = f_{1/2}^{a^2}(t) - a\operatorname{erfc}\left(a\sqrt{t}\right), \tag{2.15}
$$

in particular, implies that $m_a(t) \le 2f_{1/2}^{a^2}(t)$ and

$$
\operatorname{erfc}\left(a\sqrt{t}\right) = 1 - a\int_0^t f_{1/2}^{a^2}(s)\,ds.
$$

Moreover, by Eq. (2.15), we see

$$
q^a(t) = e^{a^2 t}m_a(t)/2.
$$

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Therefore, the positivity of m_a (lemma 2.1) implies $q^a(t) > 0$ for $t > 0$. The formula Eq. (2.9) in lemma 2.2 becomes

$$M_a \xi_t + 2ca \operatorname{erfc}\left(a\sqrt{t}\right) - 2c\mu e^{(\mu^2 - a^2)t}\left(\operatorname{erfc}\left(\mu\sqrt{t}\right)\right) = -\operatorname{grad} E^{0,b}_{\mathrm{sMM}}(\xi) \tag{2.16}$$

if $w_0 = -ce^{-\mu|x|}$. If $\mu = a$, this is reduced to Eq. (1.14) (or Eq. (2.8)).

One can give an explicit form of a solution of Eq. (2.16) starting from $\xi(0) = 1 - c$. We substitute Eq. (2.12) into Eq. (2.10) to get

$$\begin{aligned}\widehat{\eta}(\lambda) &= \frac{\sqrt{\lambda + a^2}}{\sqrt{\lambda + a^2} + b} g^a(\lambda) + \frac{-b}{\lambda\left(\sqrt{\lambda + a^2} + b\right)} \\ &= \frac{1}{\sqrt{\lambda + a^2} + b}\left(\frac{-c}{\mu + \sqrt{\lambda + a^2}} - \frac{b}{\lambda}\right).\end{aligned} \tag{2.17}$$

Since

$$\mathcal{L}\left[e^{-a^2 t} q^{\mu}\right] = \frac{1}{\sqrt{\lambda + a^2} + \mu}$$

by Eqs. (2.13) and (2.1), we have

$$\begin{aligned}\eta &= -ce^{-a^2 t} q^b * e^{-a^2 t} q^{\mu} - b\int_0^t e^{-a^2 s} q^b(s)\, ds \\ &= -ce^{-a^2 t}(q^b * q^{\mu}) - b\int_0^t e^{-a^2 s} q^b(s)\, ds.\end{aligned}$$

However, the calculation of $q^b * q^{\mu}$ is quite involved, and it is easier to calculate $\widehat{\eta}$ in Eq. (2.17) more. We proceed

$$\begin{aligned}\widehat{\eta}(\lambda) &= \frac{1}{\sqrt{\lambda + a^2} + b}\left(-\frac{c\left(\sqrt{\lambda + a^2} - \mu\right)}{\lambda + a^2 - \mu^2} - \frac{b}{\lambda}\right) \\ &= \frac{1}{\sqrt{\lambda + a^2} + b}\left(\frac{-c\left(\left(\sqrt{\lambda + a^2} + b\right) - b - \mu\right)}{\lambda + (a^2 - \mu^2)} - \frac{b}{\lambda}\right) \\ &= \frac{1}{\sqrt{\lambda + a^2} + b}\frac{(b + \mu)c}{\lambda + (a^2 - \mu^2)} - \frac{c}{\lambda + (a^2 - \mu^2)} - \frac{1}{\sqrt{\lambda + a^2} + b}\frac{b}{\lambda} \\ &\equiv I + II + III.\end{aligned}$$

It is easy to see that

$$\mathcal{L}^{-1}(II) = -ce^{-(a^2 - \mu^2)t}.$$

As we already observed,

$$\mathcal{L}^{-1}(III) = -b\int_0^t e^{-a^2 s} q^b(s)\, ds.$$

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For the first term,

$$\mathcal{L}^{-1}(I) = c(b+\mu)e^{-(a^2-\mu^2)t} * e^{-a^2 t} q^b.$$

By definition,

$$\begin{aligned} e^{-(a^2-\mu^2)t} * e^{-c^2 t} q^b &= \int_0^t e^{-(a^2-\mu^2)(t-s)} e^{-a^2 s} q^b(s)\, ds \\ &= e^{-(a^2-\mu^2)t} \int_0^t e^{-\mu^2 s} q^b(s)\, ds. \end{aligned}$$

We thus conclude that

$$\eta(t) = c\left[(\mu+b)\int_0^t e^{-\mu^2 s} q^b(s)\, ds - 1\right] e^{-(a^2-\mu^2)t} - b\int_0^t e^{-a^2 s} q^b(s)\, ds. \tag{2.18}$$

Thus, $\xi = \eta + 1$ is the solution of Eq. (2.16) with $\xi(0) = 1 - c$.

From this solution formula, we can establish the solution's large-time behaviour. We set

$$\eta = \bar{\eta}(t) + \eta_e(t), \quad \bar{\eta}(t) = -b\int_0^t e^{-a^2 s} q^b(s)\, ds.$$

LEMMA 2.3.

(i) The function $\bar{\eta}$ is negative, monotonically decreasing, and converging to $-b/(b+a)$ as $t \to \infty$.

(ii) The estimate

$$\left|\frac{\eta_e}{c(\mu+b)}\right| \le e^{-a^2 t}\int_t^\infty q^b(s)\, ds \le e^{-a^2 t}\frac{1}{b}$$

holds for $t > 0$. In particular,

$$\lim_{t\to\infty} \eta_e e^{a^2 t} = 0.$$

Proof.

(i) Since $q^b \ge 0$, the monotonicity is clear. We observe that

$$\int_0^\infty e^{-a^2 s} q^b(s)\, ds = \mathcal{L}\left[q^b\right](0 + a^2) = \frac{1}{\sqrt{a^2+b}}$$

by Eq. (2.13). Thus, $\lim_{t\to\infty} \bar{\eta}(t) = -b/(b+a)$.

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(ii) Since

$$\int_0^\infty e^{-\mu^2 s} q^b(s)\, ds = \frac{1}{\mu + b},$$

we observe that

$$\begin{aligned}\eta_e &= -c(\mu + b) \int_t^\infty e^{-\mu^2 s} q^b(s)\, ds\, e^{-(a^2-\mu^2)t} \\ &= -c(\mu + b) \int_t^\infty e^{\mu^2(t-s)} q^b(s)\, ds\, e^{-a^2 t}.\end{aligned}$$

Since $e^{\mu^2(t-s)} \leq 1$ for $s \geq t$, this implies $|\eta_e / c(\mu + b)| \leq e^{-a^2 t} \int_t^\infty q^b(s)\, ds \leq e^{-a^2 t} \int_0^\infty q^b(s)\, ds = e^{-a^2 t}/b$. The proof is now complete.

□

We conclude this section by giving remarks on corollary 1.3. It turns out that corollary 1.3 can be derived using the expression of the Dirichlet–Neumann map.

Remark 2.4. We consider the initial-boundary value problem for

$$\begin{cases} w_t - (-x)^\alpha w_{xx} = 0 & \text{in} \quad (-\infty, 0) \times (0, \infty), \\ w(0,t) = \eta(t) & \text{for} \quad t > 0, \quad \eta(0) = 0, \\ \lim_{x \to -\infty} w(x,t) = 0 & \text{for} \quad t > 0, \\ w(x, 0) = 0 & \text{for} \quad x < 0. \end{cases}$$

Then, as in [3], we obtain that

$$w_x(0,t) = c_\gamma \partial_t^\gamma \eta,$$

where $\gamma = 1/(2 - \alpha)$ with some constant $c_\gamma > 0$, provided that $\alpha < 1$. Indeed, as in [3], let ψ be a solution of

$$\begin{cases} \psi - (-x)^\alpha \psi_{xx} = 0 & \text{in} \quad (-\infty, 0), \\ \psi(0) = 1, \\ \lim_{x \to -\infty} \psi(x) = 0. \end{cases}$$

Since the Laplace transform $\widehat{w}$ of w satisfies

$$\lambda \widehat{w} - (-x)^\alpha \widehat{w}_{xx} = 0, \quad \widehat{w}(0, \lambda) = \widehat{\eta}(\lambda),$$

we see, by scaling, that

$$\widehat{w}(x, \lambda) = \widehat{\eta}(\lambda) \psi\left(\lambda^{1/(2-\alpha)} x\right).$$

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Thus,

$$\begin{aligned}\partial_x \widehat{w}(0,\lambda) &= \lambda^{1/(2-\alpha)}\psi'(0)\widehat{\eta}(\lambda)\\ &= \psi'(0)\lambda^{\gamma-1}\widehat{\eta_t}(\lambda)\end{aligned}$$

since $\eta(0)=0$. Thus,

$$w_x(0,t) = \psi'(0) f^0_{1-\gamma} * \eta_t.$$

If $\gamma<1$, then $f^0_{1-\gamma}$ is integrable. As noted in [3], $\psi'(0)$ exists (even for the degenerate case, i.e., $\alpha>0$) and $\psi'(0)>0$. Thus,

$$w_x(0,t) = c_\gamma \partial_t^\gamma \eta \quad \text{with} \quad c_\gamma = \psi'(0)$$

at least for $\gamma = 1/(2-\alpha) < 1$, i.e., $\alpha<1$.

Corollary 1.3 is easily derived by this result since $w_x = \partial_t^{1/2} w$ and $w_x + bw = 0$; note that $c_{1/2}=1$.

Remark 2.5. The reader might be interested in how fractional partial differential equations like fractional diffusion equations are derived. We consider

$$\begin{cases} w_t - (-x)^\alpha w_{xx} = 0 & \text{in} \quad (-\infty,0)\times \mathbf{R}^{n-1}\times(0,\infty),\\ w_x - \Delta_y w = f & \text{on} \quad (-\infty,0)\times \mathbf{R}^{n-1}\times(0,\infty),\\ w(x,y,0) = 0 & \text{on} \quad (-\infty,0)\times \mathbf{R}^{n-1}, \end{cases}$$

where $f=f(y,t)$ is a given function. Then, by remark 2.4, the equation for $\eta(y,t) = w(0,y,t)$ is formally obtained as

$$c_\gamma \partial_t^\gamma \eta - \Delta_y \eta = f \quad \text{in} \quad \mathbf{R}^{n-1}\times(0,\infty). \tag{2.19}$$

This type of equation is a kind of fractional diffusion that has been well-studied; see [17, 32]. Here, we briefly recall only the well-posedness of its initial boundary value problem for Eq. (2.19) in a domain. In the framework of distributions, the well-posedness of its initial boundary value problems has been established in [25, 31] by using the Galerkin method. The unique existence of viscosity solutions for Eq. (2.19), including general nonlinear problems, has been established in [7, 22] and also in [28] for the whole space $\mathbf{R}^{n-1}$. The scope of equations these theories apply is different. However, it has been proved in [6] that two notions of solutions (viscosity solution and distributional solution) agree for Eq. (2.19) when we consider the Dirichlet problem in a smooth bounded domain.

3. Convergence

The goal of this section is to prove lemma 1.4. For this purpose, we shall study a homogeneous version of Eq. (1.13) in a whole space $\mathbf{R}\times(0,\infty)$ and estimate the derivative of a solution as $|y|\to\infty$. We begin with several estimates related to the

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heat semigroup in $\mathbf{R}$. Let $E(x,t)$ be the Gauss kernel, and write $E_t(x) = E(x,t)$, i.e.,

$$E_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}$$

and $E_t^a = e^{-a^2 t} E_t$. Then,

$$(e^{\Delta_a t} f)(x) := (E_t^a *_x f)(x) = \int_{-\infty}^{\infty} E_t^a(x-y) f(y)\, dy$$

solves

$$(\partial_t - \Delta_a)u = 0 \quad \text{in} \quad \mathbf{R} \times (0,\infty), \quad \Delta_a = \Delta - a^2, \quad a \geq 0,$$

with initial data $f \in L^\infty(\mathbf{R}) \cap C(\mathbf{R})$, i.e., f is bounded and continuous on $\mathbf{R}$.

Proposition 3.1. *Assume that $f \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$ satisfies a decay condition*

$$\lim_{|x|\to\infty} |f(x)x| = 0.$$

Then, for $T > \delta > 0$,

$$\lim_{|x|\to\infty} \sup_{\delta \leq t \leq T} |x|\, |\partial_x (E_t^a *_x f)(x)| = 0$$

and

$$\lim_{|x|\to\infty} \sup_{0 < t \leq T} |x|\, |(E_t^a *_x f)(x)| = 0.$$

Proof. We may assume $a = 0$. We notice that

$$|(\partial_x E_t)(x)| \leq \frac{C}{t^{1/2}} E_{2t}(x),$$

with some C independent of t and x, since $\partial_x E_t = -(x/2t)E_t$ and $\sup_{y>0} ye^{-y^2} < \infty$. Thus, it suffices to prove that

$$\lim_{|x|\to\infty} \sup_{0 \leq t \leq T} |x|\, |(E_t *_x f)(x)| = 0. \tag{3.1}$$

We divide

$$(E_t *_x f)(x) = \left(\int_{2|y| \leq |x|} + \int_{2|y| > |x|} \right) E_t(x-y) f(y)\, dy = I + II.$$

We notice that

$$E_t(x-y) \leq C' \exp\left(-\frac{|x-y|^2}{8T} \right) E_{2t}(x-y), \quad t < T,$$

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with some constant C' independent of $x, y \in \mathbf{R}$ and $t < T$. Since $2|y| \leq |x|$ implies $|x-y| \geq |x| - |y| \geq |x|/2$, we obtain that

$$|I| \leq C' \exp\left(-\frac{|x|^2}{32T}\right) |E_{2t} *_x |f|(x)| \leq C' \exp\left(-\frac{|x|^2}{32T}\right) \|f\|_{L^\infty(\mathbf{R})}.$$

For II, we proceed

$$|x||II| \leq \int_{2|y|>|x|} E_t(x-y) 2|y|\,|f(y)|\ dy \leq \sup_{|y|\geq|x|/2} |y|\,|2f(y)| \to 0 \quad \text{as} \quad |x| \to \infty.$$

We thus obtain Eq. (3.1). The proof is now complete. □

We also need to estimate decay as $|x| \to \infty$ for

$$\int_0^t e^{\Delta_a (t-s)} g(s)\ ds$$

when $g(s) = h(s)\partial_x(1_{x>0})$, $h \in L^\infty(0,T)$. This quantity equals

$$\int_0^t E^a_{t-s}(x)h(s)\ ds.$$

Proposition 3.2. *For $h \in L^\infty(0,T)$ and $m > 0$,*

$$\lim_{|x|\to\infty} |x|^m \sup_{0\leq t\leq T} \left| \partial_x \int_0^t E^a_{t-s}(x)h(s)\ ds \right| = 0,$$
$$\lim_{|x|\to\infty} |x|^m \sup_{0\leq t\leq T} \left| \int_0^t E^a_{t-s}(x)h(s)\ ds \right| = 0.$$

Proof. Again, we may assume that $a = 0$. Since

$$(\partial_x E_t) \leq \frac{C}{t^{1/2}} E_{2t}(x) \leq \frac{C''}{t} e^{-M^2/(16t)} e^{-|x|^2/(16T)}$$

for $|x| \geq M$ and $t \leq T$. Since $t^{-1}e^{-M^2/(16t)}$ is integrable near $t = 0$, we see that

$$\left| \partial_x \int_0^t E_{t-s}(x)h(s)\ ds \right| \leq C_0 e^{-|x|^2/(16T)} \|h\|_{L^\infty(0,T)}, \quad |x| \geq M, \quad t \leq T,$$
$$\left| \int_0^t E_{t-s}(x)h(s)\ ds \right| \leq C_0' e^{-|x|^2/(8T)} \|h\|_{L^\infty(0,T)}, \quad |x| \geq M, \quad t \leq T,$$

with some constants C_0, C_0' depending only on M and T. The proof is now complete. □

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We consider a homogeneous version Eq. (1.13) in $\mathbf{R}\times(0,\infty)$. Namely, we consider

$$\begin{cases} \tau_1 w_t = \Delta_a w - 2bw\partial_x(1_{x>0}) \\ w|_{t=0} = w_0. \end{cases} \tag{3.2}$$

PROPOSITION 3.3. *Let w be a bounded solution of Eq. (3.2) with initial data $w_0 \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$. Assume that*

$$\lim_{|x|\to\infty} w_0(x)|x| = 0.$$

Then,

$$\lim_{|x|\to\infty} |x| \sup_{\delta\le t\le T} |\partial_x w(x,t)| = 0, \qquad \lim_{|x|\to\infty} |x| \sup_{0\le t\le T} |w(x,t)| = 0$$

for any δ, T satisfying $T > \delta > 0$.

Proof. We may assume $\tau_1 = 1$. By Duhamel's formula, w is of the form

$$w(x,t) = (e^{\Delta_a t} w_0)(x) - \int_0^t e^{\Delta_a (t-s)} 2b\,(w(s)\partial_x(1_{x>0}))\,ds.$$

Applying propositions 3.1 and 3.2, we obtain desired results. □

Proof. Proof of lemma 1.4. For a function f on $(-L, L)$, we decompose it into its odd and even parts, i.e.,

$$f_{\text{odd}}(x) := \frac{f(x) - f(-x)}{2}, \quad f_{\text{even}}(x) := \frac{f(x) + f(-x)}{2},$$

so that $f = f_{\text{odd}} + f_{\text{even}}$. By the structure of the equation, $V^\varepsilon_{\text{odd}}$ and $V^\varepsilon_{\text{even}}$ solve Eq. (1.13) separately.

At first glance, the locally uniform convergence follows from the maximum or comparison principles for a linear parabolic equation [24]. However, a direct application of the maximum principle is impossible since the domains of functions V^ε and V are different. We first show the convergence where initial data are smooth.

For the odd part, the term $V\partial_y(1_{y>0})$ does not affect since $V=0$ at $y=0$. Thus, Eq. (1.13) is reduced to

$$\begin{cases} \tau_1 V_t &= V_{yy} - a^2(V-1), \quad |y| < L/\varepsilon \\ V|_{t=0} &= V_0^\varepsilon, \qquad\qquad V_y(\pm L/\varepsilon, t) = 0 \quad \text{for} \quad t > 0, \end{cases} \tag{3.3}$$

where $V_0^\varepsilon(y) = V^\varepsilon(y,0)$ for $|y| < L/\varepsilon$. Let V^ε be its solution.

We extend an odd function V_0^ε to $(L/\varepsilon, 3L/\varepsilon)$ for x to be 'even' with respect to L/ε, i.e.,

$$\widetilde{V_0^\varepsilon}(x - L/\varepsilon) = \widetilde{V_0^\varepsilon}(L/\varepsilon - x), \quad x \in (L/\varepsilon, 3L/\varepsilon),$$

where $\widetilde{V_0^\varepsilon}$ is its extension. We extend $\widetilde{V_0^\varepsilon}$ outside $(-L/\varepsilon, 3L/\varepsilon)$ so that the extension $\overline{V_0^\varepsilon}$ is periodic in $\mathbf{R}$ with period $4L/\varepsilon$. Since V_0^ε is even with respect to L/ε,

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$\left(\overline{V_0^\varepsilon}\right)_y(\pm L/\varepsilon) = 0$ if $V_{0y}^\varepsilon(\pm L/\varepsilon) = 0$ and smooth. Solution V^ε is the restriction on $(-L/\varepsilon, L/\varepsilon)$ of a solution W^ε of

$$\begin{cases} \tau_1 W_t = W_{yy} - a^2(W-1), & y \in \mathbf{R} \\ W|_{t=0} = \overline{V_0^\varepsilon}. \end{cases} \tag{3.4}$$

Although the maximum principle implies

$$\|W^\varepsilon - V\|_{L^\infty(\mathbf{R})}(t) \le \left\|\overline{V_0^\varepsilon} - V_0\right\|_{L^\infty(\mathbf{R})},$$

our assumption of the convergence $V_0^\varepsilon \to V_0$ does not guarantee $\left\|\overline{V_0^\varepsilon} - V_0\right\|_{L^\infty(\mathbf{R})} \to 0$. We argue differently.

We approximate V_0 by $V_{0\delta} = V_0 * \rho_\delta$, where ρ_δ is a symmetric mollifier. We also approximate V_0^ε by

$$V_{0\delta}^\varepsilon = \overline{V_0^\varepsilon} * \rho_\delta.$$

We set $W_\delta^\varepsilon = W^\varepsilon * \rho_\delta$, where W^ε is the solution of Eq. (3.4). Since Eq. (3.4) is of constant coefficients, this W_δ^ε solves Eq. $(3.4)_1$ with initial data $V_{0\delta}^\varepsilon$. Let V_δ^ε be the restriction of W_δ^ε on $(-L/\varepsilon, L/\varepsilon)$. It follows from the parity and periodic condition that this V_δ^ε solves Eq. (3.3). We set $V_\delta = V * \rho_\delta$ and observe that V_δ is the bounded solution of Eq. $(3.4)_1$ with initial data $V_{0\delta}$ since Eq. (3.4) is of constant coefficients. For fixed $\delta > 0$, we observe that

$$V_\delta^\varepsilon \to V_\delta \quad \text{in} \quad L^\infty((-M, M) \times (0, T))$$

for $M > 0$. Indeed, by the maximum principle

$$\begin{aligned} \|V_\delta^\varepsilon - 1\|_{\infty,\varepsilon}(t) &\le \|V_{0\delta}^\varepsilon - 1\|_{\infty,\varepsilon}, \\ \|\partial_y V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|\partial_y V_{0\delta}^\varepsilon\|_{\infty,\varepsilon}, \\ \|\partial_t V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|\partial_t V_{0\delta}^\varepsilon\|_{\infty,\varepsilon}, \end{aligned}$$

where $\|\cdot\|_{\infty,\varepsilon}$ is the sup norm on $(-L/\varepsilon, L/\varepsilon)$. Here, the notation $\partial_t V_{0\delta}^\varepsilon$ for a function $V_{0\delta}^\varepsilon$ of y should be interpreted as

$$\partial_t V_{0\delta}^\varepsilon = \left(V_{0\delta yy}^\varepsilon - a^2(V_{0\delta}^\varepsilon - 1)\right)/\tau_1.$$

Because of the mollifier, the right-hand side is bounded by a constant multiple of $\|V_0^\varepsilon\|_{\infty,\varepsilon}$, which is uniformly bounded for $\varepsilon < 1$. By the Arzelà–Ascoli theorem and a diagonal argument, V_δ^ε converges (locally uniformly in $\mathbf{R} \times [0, \infty)$) to a bounded (weak) solution to Eq. (3.4) with initial data $V_{0\delta}$ by taking a subsequence. Since V is bounded, by the uniqueness of the limit problem, the convergence is now full (without taking a subsequence). Note that we only invoke the locally uniform convergence of V_0^ε to V_0 other than the uniform bound on derivatives.

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We note that

$$V - V^\varepsilon = V - V_\delta + V_\delta - V_\delta^\varepsilon + V_\delta^\varepsilon - V^\varepsilon$$

and observe that

$$\|V - V^\varepsilon\|(t) \leq \|V - V_\delta\|(t) + \|V_\delta - V_\delta^\varepsilon\|(t) + \|V_\delta^\varepsilon - V^\varepsilon\|(t) =: I + II + III$$

where the norm is $\|\cdot\|$ taken in $L^\infty(-M, M)$ for $M > 0$. By the maximum principle,

$$\begin{aligned} \|V_\delta^\varepsilon - V^\varepsilon\|_{\infty,\varepsilon}(t) &\leq \|V_{0\delta}^\varepsilon - V_0^\varepsilon\|_{\infty,\varepsilon}, \\ \|V - V_\delta\|_{L^\infty(\mathbf{R})}(t) &\leq \|V_{0\delta} - V_0\|_{L^\infty(\mathbf{R})}. \end{aligned}$$

Since

$$\begin{aligned} V_{0\delta}^\varepsilon &= \rho_\delta * \left(\overline{V_0^\varepsilon} - V_0\right) + \rho_\delta * V_0, \\ V_0^\varepsilon &= \overline{V_0^\varepsilon} - V_0 + V_0 \quad \text{on} \quad (-L/\varepsilon, L/\varepsilon), \end{aligned}$$

and $\|\rho_\delta * f\|_\infty \leq \|f\|_\infty$, we see that

$$\|V_{0\delta}^\varepsilon - V_0^\varepsilon\|_{L^\infty(-L/\varepsilon, L/\varepsilon)} \leq 2\|\overline{V_0^\varepsilon} - V_0\|_{\infty,\varepsilon} + \|\rho_\delta * V_0 - V_0\|_\infty.$$

Thus,

$$\sup_{0<t<T} III \leq 2\|V_0^\varepsilon - V_0\|_{\infty,\varepsilon} + \|\rho_\delta * V_0 - V_0\|_\infty.$$

Fixing $\delta > 0$ and sending $\varepsilon \to 0$, we observe that

$$\overline{\lim_{\varepsilon \downarrow 0}} \sup_{0<t<T} (I + II + III) \leq 2\|\rho_\delta * V_0 - V_0\|_\infty$$

since $V_\delta^\varepsilon \to V_\delta$ in $L^\infty\left((-M, M) \times [0, T]\right)$. Sending $\delta \downarrow 0$, we obtain

$$\overline{\lim_{\varepsilon \downarrow 0}} \sup_{0<t<T} \|V - V^\varepsilon\|(t) = 0$$

since V_0 is uniformly continuous.

We next study the even part. The general strategy is the same but more involved than the odd part. For the even part, we first note that $V_{\text{even}}^\varepsilon$ solves

$$\begin{cases} \tau_1 V_t = V_{yy} - a^2(V - 1), \quad y \in I_\varepsilon := (-L/\varepsilon, 0), \quad t > 0, \\ V_y(0, t) + bV(0, t) = 0, \quad V_y(-L/\varepsilon, t) = 0, \quad t > 0, \\ V|_{t=0}(y) = V_{0\,\text{even}}^\varepsilon, \end{cases} \tag{3.5}$$

where $V_0^\varepsilon(y) = V^\varepsilon(y, 0)$ for $y \in I_\varepsilon$. We suppress the word 'even' from now on. We shall approximate V_0^ε by a smoother function $V_{0\delta}^\varepsilon$ and approximate $V_0 = \lim_{\varepsilon \to 0} V_0^\varepsilon$ by a smoother function uniformly. There are many possible ways, and we rather like

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an abstract way. Let $\mathrm{BUC}(I_\varepsilon)$ denote the space of all bounded uniformly continuous functions in $\overline{I_\varepsilon}$. It is a Banach space equipped with the norm

$$\|f\|_{\infty,\varepsilon} = \sup\left\{|f(x)| \;\middle|\; x \in I_\varepsilon\right\}.$$

If $\varepsilon = 0$, then I_ε should be interpreted as $(-\infty, 0]$. Let A be the operator on $\mathrm{BUC}(I_0)$ defined by

$$Af := (-\partial_y^2 + a^2)f \quad \text{(in the distribution sense)},$$

with

$$D(A) = \left\{f \in \mathrm{BUC}(I_0) \;\middle|\; Af \in \mathrm{BUC}(I_0),\ f_y(0) + bf(0) = 0\right\}.$$

A standard theory [18] implies that $-A$ generates an analytic semigroup e^{-tA} in $\mathrm{BUC}(I_0)$. In particular,

$$\begin{aligned} \left\|Ae^{-tA}f\right\|_{\infty,0} &\le Ct^{-1}\|f\|_{\infty,0}, \\ \left\|e^{-tA}f\right\|_{\infty,0} &\le C\|f\|_{\infty,0}, \end{aligned}$$

with some constant $C > 0$ independent of time $t \in (0,1)$ and $f \in BUC(I_0)$. For a function $h \in BUC(I_\varepsilon)$, we extend it to $\widetilde{h}$ so that $\widetilde{h}(x) = h(-L/\varepsilon)$ for $x < -L/\varepsilon$. For V_0, we set $V_{0\delta} = e^{-\delta A}V_0$. For V_0^ε, we tempt to set $V_{0\delta}^\varepsilon = e^{-\delta A}\widetilde{V_0^\varepsilon}$. However, unfortunately, $V_{0\delta}^\varepsilon$ does not satisfy the boundary condition at $-L/\varepsilon$ although it satisfies $(\partial_y V_{0\delta}^\varepsilon + bV_{0\delta}^\varepsilon)(0) = 0$ and is C^2 (actually smooth). We set $\sigma(x) = \rho_{\delta'} * (1-|x|)_+$ for a fixed $\delta' > 0$ so that $\sigma'(1/2) = -\kappa$ with $\kappa < 0$. We set $\sigma_{\delta''}(y) = \delta''\sigma(y/\delta'')$ for small $\delta'' > 0$. For a given $h \in C^2(-\infty, 0]$, we modify

$$h_{\delta''}(y) = h(y) + c\sigma_{\delta''}(y-\nu),$$

where we take c so that $c\kappa = h_y(-L/\varepsilon)$, $-L/\varepsilon - \nu = \delta''/2$. By this modification,

$$\partial_y h_{\delta''}(-L/\varepsilon) = 0$$

and $h_{\delta''} \to h$ in $L^\infty(-\infty, 0)$ as $\delta'' \to 0$. We set

$$V_{0\delta}^\varepsilon = \left(e^{-\delta A}\widetilde{V_0^\varepsilon}\right)_\delta.$$

Since $V_{0\delta}^\varepsilon$ satisfies the boundary condition on the boundary of I_ε and is smooth, we observe that $\partial_t V_\delta^\varepsilon$ is continuous up to the boundary of $I_\varepsilon \times [0,T)$, where V_δ^ε denotes the solution of Eq. (3.5) with initial data $V_{0\delta}^\varepsilon$. By the maximum principle,

$$\begin{aligned} \|V_\delta^\varepsilon - 1\|_{\infty,\varepsilon}(t) &\le \|V_{0\delta}^\varepsilon - 1\|_{\infty,\varepsilon} \\ \|\partial_y V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|\partial_y V_{0\delta}^\varepsilon\|_{\infty,\varepsilon} + b\|V_\delta^\varepsilon\|_{\infty,\varepsilon} \\ \|\partial_t V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|\partial_t V_{0\delta}^\varepsilon\|_{\infty,\varepsilon}, \end{aligned} \tag{3.6}$$

where $\partial_t V_{0\delta}^\varepsilon$ for initial data $V_{0\delta}^\varepsilon$ should be interpreted as in the proof for the odd part. The term involving b appears because of the Robin type boundary condition.

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As in the case for Eq. (3.4), by the Arzelà–Ascoli theorem and the uniqueness of the limit equation, we can prove that V_δ^ε converges to V_δ locally uniformly in $(-\infty, 0] \times [0, \infty)$. Note that for a fixed $\delta > 0$, the right-hand sides of Eq. (3.6) are uniformly bounded as $\varepsilon \to 0$ since V_0^ε converges to V_0 uniformly. The comparison principle implies that

$$\begin{aligned} \|V_\delta^\varepsilon - V^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|V_{0\delta}^\varepsilon - V_0^\varepsilon\|_{\infty,\varepsilon}, \quad t > 0 \\ \|V - V_\delta\|_{\infty,0}(t) &\le \|V_{0\delta} - V_0\|_{\infty,0}, \quad t > 0. \end{aligned}$$

Thus,

$$\begin{aligned} \|V - V^\varepsilon\|(t) &\le \|V - V_\delta\|(t) + \|V_\delta - V_\delta^\varepsilon\|(t) + \|V_\delta^\varepsilon - V^\varepsilon\|(t) \\ &\le \|V_0 - V_{0\delta}\|_{\infty,0} + \|V_\delta - V_\delta^\varepsilon\|(t) + \|V_{0\delta}^\varepsilon - V_0^\varepsilon\|_{\infty,\varepsilon}, \end{aligned}$$

where the norm $\|\cdot\|$ is taken in $L^\infty(0, M)$ for $M > 0$. Taking the supremum in $t \in (0, T)$ and sending $\varepsilon \to 0$, we obtain that

$$\overline{\lim_{\varepsilon\to 0}} \sup_{0<t<T} \|V - V^\varepsilon\|(t) \le 2\|V_{0\delta} - V_0\|_{\infty,0}$$

since we know $\sup_{0<t<T} \|V_\delta - V_\delta^\varepsilon\|(t) \to 0$ as $\varepsilon \to 0$. Sending $\delta \to 0$, we conclude that V^ε converges to V locally uniformly in $[0, \infty) \times [0, \infty)$.

Since we know that V^ε is continuous up to $y = 0$ and $t = 0$, this gives the local uniform convergence of $V^\varepsilon(0, t)$ in $[0, \infty)$.

So far, we do not invoke the assumption that $(V_0(x) - 1)\,|x| \to 0$ as $|x| \to \infty$. We shall prove

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le T} \|V - V^\varepsilon\|_{\infty,\varepsilon}(t) = 0.$$

We only discuss the case of even initial data since the odd case is easier. Since

$$\begin{aligned} \|V - V^\varepsilon\|_{\infty,\varepsilon}(t) &\le \|V - V_\delta\|_{\infty,\varepsilon}(t) + \|V_\delta - V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) + \|V_\delta^\varepsilon - V^\varepsilon\|_{\infty,\varepsilon}(t) \\ &\le \|V_0 - V_{0\delta}\|_{\infty,0}(t) + \|V_\delta - V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) + \|V_{0\delta}^\varepsilon - V_0^\varepsilon\|_{\infty,\varepsilon}, \end{aligned}$$

it suffices to prove that

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le T} \|V_\delta - V_\delta^\varepsilon\|_{\infty,\varepsilon}(t) = 0. \tag{3.7}$$

By our construction, the function $V_\delta(y, t)$ is

$$V_\delta(y, t) = w(y, t + \delta) + 1, \quad x < 0$$

for w solving Eq. (3.2) with $w_0 = V_0 - 1$. By proposition 3.3,

$$\lim_{y\to\infty} |y| \sup_{0\le t\le T} |\partial_y V_\delta(y, t)| = 0. \tag{3.8}$$

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The difference $V_\delta - V_\delta^\varepsilon = u^\varepsilon$ satisfies

$$\begin{cases} \tau_1 u_t = \Delta_a u, \quad -L/\varepsilon < y < 0, \\ u(0,t) = (V_\delta - V_\delta^\varepsilon)(0,t) =: u_b^\varepsilon(t), \quad t > 0, \\ \partial_y u(-L/\varepsilon, t) = \partial_y V_\delta(-L/\varepsilon, t) - 0, \quad t > 0, \\ u(y,0) = V_{0\delta} - V_{0\delta}^\varepsilon =: u_i^\varepsilon(y), \quad -L/\varepsilon < y < 0. \end{cases}$$

Let $\bar{u}^\varepsilon$ be the solution of

$$\begin{cases} \tau_1 \bar{u}_t = \Delta_a \bar{u}, \quad -L/\varepsilon < y < 0, \\ \bar{u}(0,t) = u_b^\varepsilon(t), \quad t > 0, \\ \partial_y \bar{u}(-L/\varepsilon, t) = 0 \quad t > 0, \\ \bar{u}(y,0) = u_i^\varepsilon(y), \quad -L/\varepsilon < y < 0. \end{cases}$$

By the maximum principle,

$$\|\bar{u}^\varepsilon\|_{\infty,\varepsilon}(t) \le \sup_{0 \le t \le T} |u_b^\varepsilon(t)| + \|u_i^\varepsilon\|_{\infty,\varepsilon}.$$

Since we know that $\sup_{0<t<T} |u_b^\varepsilon(x)| \to 0$ as $\varepsilon \to 0$, sending $\varepsilon \to 0$ yields

$$\overline{\lim_{\varepsilon \downarrow 0}} \|\bar{u}^\varepsilon\|_{\infty,\varepsilon}(t) \le 0.$$

We set $r = u^\varepsilon - \bar{u}^\varepsilon$ and observe that r satisfies

$$\begin{cases} \tau_1 r_t = \Delta_a r, \quad -L/\varepsilon < y < 0, \\ r(0,t) = 0, \quad t > 0, \\ \partial_y r(-L/\varepsilon, t) = \partial_y V_\delta(-L/\varepsilon, t), \\ r(y,0) = 0, \quad -L/\varepsilon < y < 0. \end{cases}$$

We set $\tilde{r} = e^{a^2 t} r$ and observe that $\tilde{r}$ satisfies

$$\begin{cases} \tau_1 \tilde{r}_t = \Delta \tilde{r} \\ \tilde{r}(0,t) = \tilde{r}(y,0) = 0, \quad -L/\varepsilon < y < 0, \quad t > 0, \\ \partial_y \tilde{r} = e^{a^2 t} \partial_y V_{0\delta} \quad \text{for} \quad x = -L/\varepsilon, \quad t > 0. \end{cases}$$

We take $M^\varepsilon = \sup_{0 \le t \le T} e^{a^2 t} |(\partial_y V_\delta)(-L/\varepsilon, 0)|$ and observe that $-yM$ and yM are the super- and subsolution, respectively. By the comparison principle,

$$|\tilde{r}| \le M^\varepsilon |y| \quad \text{or} \quad |r| \le e^{-a^2 t} M^\varepsilon |y| \quad \text{for} \quad y \in (-L/\varepsilon, 0), \quad t \in (0,T).$$

This implies that

$$|r| \le M^\varepsilon L/\varepsilon.$$

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By Eq. (3.8), $M^\varepsilon L/\varepsilon \to 0$ as $\varepsilon \to 0$. We thus conclude that

$$\overline{\lim_{\varepsilon\downarrow 0}} \sup_{0\le t\le T} \|u^\varepsilon - \bar{u}^\varepsilon\|_{\infty,\varepsilon}(t) \le \|V - V_\delta\|_{\infty,\varepsilon}.$$

Sending $\delta \downarrow 0$, we obtain Eq. (3.7). The statement that $\sup_{0\le t\le T} |V-1|(y)|y| \to 0$ as $|y|\to\infty$ follows from proposition 3.3. The proof is now complete. □

If we only assume that the initial data for Eqs. (3.4) and (3.5) are bounded and Lipschitz, we have a similar estimate in Eq. (3.6) up to the first derivative of the solution. However, the estimate for the time derivative should be altered. Since we used such an estimate in lemma 2.2, we state it in the case of $\varepsilon = 0$ for the reader's convenience.

Lemma 3.4. *Let V be the bounded solution of Eq. (1.13) in $\mathbf{R}\times(0,\infty)$ with the bounded and Lipschitz continuous initial data V_0. Then, for each $T>0$, there is a constant C depending only on a, b, and T such that*

$$t^{1/2}\|\partial_t V\|_{L^\infty(\mathbf{R})}(t) \le C\left(\|\partial_y w_0\|_{L^\infty(\mathbf{R})} + \|w_0\|_{L^\infty(\mathbf{R})} + 1\right) \quad \text{for} \quad t\in(0,T).$$

Proof. We give direct proof. We may assume that $\tau_1 = 1$. We set $w = V-1$ and $u = e^{a^2t}w$ to get

$$u_t = u_{xx} - 2b\partial_x\{1_{x>0}\}\left(u + e^{a^2t}\right),$$

where we denote by x instead of y. We consider this equation with initial data $w_0 = V_0 - 1$. It suffices by simple scaling $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ to prove the desired estimate for some T independent of w_0.

Let $E(x,t)$ be the Gauss kernel as before. Then, the solution can be represented as

$$u(x,t) = (E*_x w_0)(x,t) - \int_0^t E(x,t-\tau)h(\tau)\,d\tau, \quad h(t) = 2b\left(u(0,t) + e^{a^2t}\right), \tag{3.9}$$

where

$$(E*_x w_0)(x,t) := \int_{-\infty}^{\infty} E(x-y,t)w_0(y)\,dy.$$

Since we can approximate a smooth w_0, establishing

$$\|\partial_t u\|_{L^\infty(\mathbf{R})}(t) \le Ct^{-1/2}\left(\|\partial_x w_0\|_{L^\infty(\mathbf{R})} + \|w_0\|_{L^\infty(\mathbf{R})} + 1\right) \quad \text{for} \quad t\in(0,T) \tag{3.10}$$

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with some positive constants C and T independent of w_0 suffices, assuming that $\partial_t u$ exists and is bounded in $\mathbf{R} \times (0,T)$ for small T. By the maximum principle Eq. (3.6) and the corresponding estimate for the odd part, we know that

$$\|u\|_{L^\infty(\mathbf{R})}(t) \leq c\left(1 + \|w_0\|_{L^\infty(\mathbf{R})}\right), \tag{3.11}$$

with c independent of w_0 and $t > 0$. We estimate $\partial_t u$ in Eq. (3.9). Since $\|f *_x g\|_{L^\infty(\mathbf{R})} \leq \|f\|_{L^1(\mathbf{R})}\|g\|_{L^\infty(\mathbf{R})}$ and

$$\partial_t(E *_x w_0) = (\partial_x E) *_x \partial_x w_0,$$

we easily see (cf. [5, Chapter 1]) that

$$\|\partial_t(E *_x w_0)\|_{L^\infty(\mathbf{R})}(t) \leq \frac{c'}{t^{1/2}}\|\partial_x w_0\|_{L^\infty(\mathbf{R})} \quad \text{for} \quad t > 0, \tag{3.12}$$

with c' independent of w_0. The second term of the right-hand side of Eq. (3.9) is more involved than the first term because h contains u. We observe that

$$\partial_t \int_0^t E(x, t-\tau)h(\tau)\,d\tau = \int_0^t E(x, t-\tau)\partial_\tau h(\tau)\,d\tau + E(x,t)h(0).$$

Since $|E| \leq (4\pi t)^{-1/2}$, it holds that

$$\left|\int_0^t E(x, t-\tau)\partial_\tau h(\tau)\,d\tau\right| \leq \int_0^t \frac{1}{(4\pi(t-\tau))^{1/2}}\frac{1}{\tau^{1/2}}\,d\tau \cdot \sup_{0<t<T}|t^{1/2}\partial_t h(t)|$$

for $t \in (0,T)$.

Thus,

$$\begin{aligned}\sup_{0<t<T}\left\|\partial_t \int_0^t E(x,t-\tau)h(\tau)\,d\tau\right\|_\infty &\leq C_1 \sup_{0<t<T}\left|t^{1/2}\partial_t h(t)\right| + (4\pi t)^{-1/2}|h(0)| \\ &\leq C_2\left(\sup_{0<t<T}\|t^{1/2}\partial_t u\|_{L^\infty(\mathbf{R})}(t) + 1\right) + C_3 t^{-1/2}\left(\|u\|_{L^\infty(\mathbf{R})}(t) + 1\right)\end{aligned}$$

with some constants C_j $(j = 1,2,3)$. By Eqs. (3.9) and (3.12), we now observe that

$$\begin{aligned}\sup_{0<t<T}\left\|t^{1/2}\partial_t u\right\|_{L^\infty(\mathbf{R})}(t) \leq C_4\|\partial_x w_0\|_{L^\infty(\mathbf{R})} &+ C_2 T^{1/2}\left(\sup_{0<t<T}\|t^{1/2}\partial_t u\|_{L^\infty(\mathbf{R})} + 1\right) \\ &+ C_3 \sup_{0<t<T}\left(\|u\|_{L^\infty(\mathbf{R})}(t) + 1\right),\end{aligned}$$

with C_4 independent of w_0, u, and T. Applying estimate for $\|u\|_\infty$ in Eq. (3.11), we conclude Eq. (3.10) for sufficiently small T by absorbing $C_2 T^{1/2}\sup_{0<t<T}\|t^{1/2}\partial_t u\|_{L^\infty(\mathbf{R})}$ on the right-hand side to the left. □

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4. Dirichlet condition for the total variation flow

In this section, we recall a notion of total variation flow for a given v and prove lemma 1.1. We consider

$$\partial_t u = \operatorname{div}(\beta \nabla u / |\nabla u|) \quad \text{in} \quad I \times (0, T), \tag{4.1}$$

where $\beta \in C(I \times [0, T])$ is a given non-negative function; here, $I = (p_0, p_1)$ is an open interval and $T > 0$. If we impose the Dirichlet boundary condition

$$u = g \quad \text{on} \quad \partial I, \tag{4.2}$$

Eq. (4.1) with Eq. (4.2) should be interpreted as an L^2-gradient flow of a time-dependent total variation type energy

$$\Phi^t(u) = \int_I \beta(x, t)|u_x| + \sum_{i=0}^{1} |\gamma u - g|(p_i)\beta(p_i, t)$$

when $\int \beta |u_x|$ is a weighted total variation of β and γu is a trace of u on ∂I. We consider this energy in $L^2(I)$ by $\Phi^t(u) = \infty$ when $\Phi^t(u)$ is not finite. It is clear that Φ^t is convex in $L^2(I)$. If β is spatially constant, it is well known that Φ^t is also lower semicontinuous; see e.g. [1]. The solution of Eq. (4.1) with Eq. (4.2) should be interpreted as the gradient flow of form

$$u_t \in -\partial \Phi^t(u), \tag{4.3}$$

where $\partial \Phi^t$ denotes the subdifferential of Φ^t in $L^2(I)$, i.e.,

$$\partial \Phi^t(u) = \left\{ f \in L^2(I) \;\middle|\; \Phi^t(u + h) - \Phi^t(u) \geq \int_I hf\, dx \text{ for all } h \in L^2(I) \right\}.$$

It is standard that Eq. (4.3) is uniquely solvable for given initial data $u_0 \in L^2(I)$ if Φ^t does not depend on time and is lower semicontinuous and convex on the Hilbert space $L^2(I)$ (see, for instance, [2, 16]). It applies to the total variation flow case when β is a constant. In one-dimensional case, $u \in BV(I)$ implies $u \in L^\infty(I)$, so the subdifferential becomes

$$\begin{aligned}\partial \Phi(u) = \Bigg\{ v \in L^2(I) \;\Bigg|\; & v = -(\beta z)_x,\ \|z\|_\infty \leq 1,\ \int_I (\beta z, u_x) = \int_I \beta |u_x|, \\ & -(\beta z)(p_i)(-1)^i (g - \gamma u)(p_i) = |\gamma u - g|(p_i)\beta(p_i) \text{ for } i = 1, 2 \Bigg\},\end{aligned}$$

when $\Phi = \Phi^t$; see [1, Proposition 5.10 and Lemma 5.13][1.]. Here, $(\beta z, u_x)$ denotes the Anzellotti pair [1] and

$$\int_I (\beta z, u_x) = -\int_I (\beta z)_x u\, dx - \sum_{i=0}^{1} (\beta z)(p_i)(-1)^i \gamma u(p_i).$$

[1.] In [1], the case $\beta \equiv 1$ is discussed, but its extension to $\beta \in C(\bar{I})$ is straightforward.

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Eq. (4.3) is

$$u_t = (\beta z)_x,$$

with $|z| \le 1$ in I and

$$\begin{aligned}
&-\int_I (\beta z)_x u\, dx \\
&= \int_I \beta |u_x| + \sum_{i=0}^{1} (\beta z)(p_i)(-1)^i \gamma u(p_i) \\
&= \int_I \beta |u_x| + \sum_{i=0}^{1} \left\{ (\beta z)(p_i)(-1)^i \left(\gamma u(p_i) - g(p_i)\right) + (\beta z)(p_i)(-1)^i g(p_i) \right\} \\
&= \Phi(u) + \sum_{i=0}^{1} (\beta z)(p_i)(-1)^i g(p_i),
\end{aligned}$$

with $-(\beta z)(p_i)(-1)^i(g - \gamma u)(p_i) = \beta|\gamma u - g|(p_i)$ for $i = 0, 1$. We mimic this notion of the solution. A function $u \in C\left([0, T), L^2(I)\right)$ is a solution to Eq. (4.1) with Eq. (4.2) if there is $z \in L^\infty\left(I \times (0, T)\right)$ such that

$$u_t = (\beta z)_x \quad \text{in} \quad I \times (0, T) \tag{4.4}$$

$$|z| \le 1 \quad \text{in} \quad I \times (0, T) \tag{4.5}$$

$$\begin{aligned}
&-\int_I (\beta z)_x u = \Phi(u) + \sum_{i=0}^{1} (\beta z)(p_i)(-1)^i g(p_i), \quad \text{with} \\
&-(\beta z)(p_i)(-1)^i (g - \gamma u)(p_i) = |\gamma u - g|(p_i)\beta(p_i) \quad \text{for} \quad i = 1, 2.
\end{aligned} \tag{4.6}$$

Under this preparation, we shall prove lemma 1.1.

Proof. Proof of lemma 1.1. We set $p_0 = -L$, $p_1 = L$ so that $I = (-L, L)$. Since $u^b_t = 0$, Eq. (4.4) says that βz is a constant c. The condition Eq. (4.5) is equivalent to saying that $|c| \le \min \beta = \beta(0)$. Since

$$\int_I \beta |u^b_x| = \beta(0) b \quad \text{and} \quad u^b = g \quad \text{on} \quad \partial I$$

with $g(L) = b$, $g(-L) = 0$, Eq. (4.6) is equivalent to

$$0 = \beta(0) b - (\beta z)(L) b.$$

In other words, c must be $\beta(0)$. Thus, the existence of z satisfying $|z| \le 1$ is guaranteed if and only if

$$c = \beta(0) \le \beta(x) \quad \text{for all} \quad x \in (-L, L).$$

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Eq. (4.6) is fulfilled with $u = u^b$ by taking $g(-L) = 0$ and $g(L) = b$. Thus, u^b is a stationary solution to Eq. (1.4) with Eq. (1.7). □

5. Numerical experiment

In this section, we calculate the solution of Eqs. (1.10)–(1.12) with $v_0(x) = 1 - ce^{-a|x|/\varepsilon}$ and compare its value at $x = 0$ with an explicit solution of Eq. (2.16) whose explicit form is given in Eq. (2.18).

5.1. Numerical scheme

Since the initial function, v_0, is an even function, the original problem Eqs. (1.10)–(1.12) is reduced to

$$\begin{cases} \frac{\tau_1}{\varepsilon} v_t = \varepsilon v_{xx} - \frac{a^2(v-1)}{\varepsilon} & \text{in } (0, L) \times (0, \infty), \\ -v_x(0, t) + \frac{b}{\varepsilon} v(0, t) = 0, & \text{for } t > 0, \\ v_x(L, t) = 0, & \text{for } t > 0, \\ v(x, 0) = v_0(x), & \text{for } x \in [0, L]. \end{cases}$$

The computational region $[0, L]$ is divided into uniform mesh partitions:

$$x_i = i\Delta x, \quad i = -1, 0, \ldots, N, N+1, \quad \Delta x = \frac{L}{N}.$$

The points x_{-1} and x_{N+1} are needed to handle the Neumann boundary conditions.

The approximation of v at $x = x_i$ is written as v_i. The central finite difference approximates the Laplace operator, and the time derivative is approximated by the backward difference, yielding the following linear system:

$$\frac{\tau_1}{\varepsilon} \frac{v_i - \hat{v}_i}{\Delta t} = \varepsilon \frac{v_{i-1} - 2v_i + v_{i+1}}{(\Delta x)^2} - \frac{a^2(v_i - 1)}{\varepsilon}, \quad i = 0, 1, \ldots, N,$$

where $\hat{v}_i$ is the value known at the current time and v_i is the value to be found at the next time. The Neumann boundary conditions at $x = 0$ and $x = L$ are approximated as

$$-\frac{v_1 - v_{-1}}{2\Delta x} + \frac{b}{\varepsilon} v_0 = 0 \quad \text{and} \quad \frac{v_{N+1} - v_{N-1}}{2\Delta x} = 0$$

by the central finite differences.

5.2. Results

Some results are shown for different values of c for parameters

$$\tau_1 = 1, \quad L = 1, \quad a = 1, \quad N = 200, \quad \Delta t = (\Delta x)^2, \quad b = 1.$$

The results of the numerical experiments are summarized in figure 2. In (a), (b),

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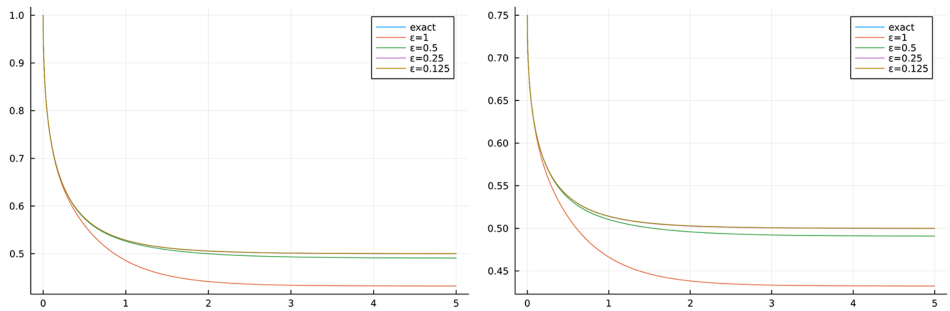
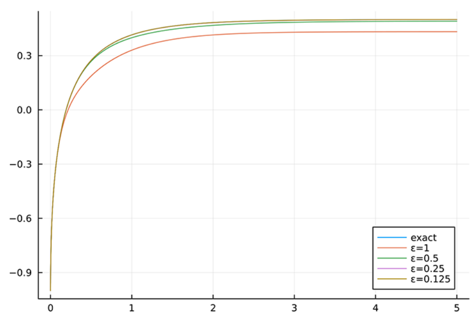


(a) (b)



(c)

ε \ c	0	1/4	2
1	$6.8 \cdot 10^{-2}$	$6.8 \cdot 10^{-2}$	$6.7 \cdot 10^{-2}$
2^{-1}	$9.1 \cdot 10^{-3}$	$9.1 \cdot 10^{-3}$	$9.3 \cdot 10^{-3}$
2^{-2}	$1.4 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$
2^{-3}	$4.6 \cdot 10^{-5}$	$4.6 \cdot 10^{-5}$	$4.6 \cdot 10^{-5}$

(d)

Figure 2. Results of numerical experiments: (a) $c = 0$, (b) $c = 1/4$, (c) $c = 2$, (d) table of L^∞-errors for different ε and c values.

and (c), the horizontal axis represents time, and the vertical axis represents the value at the origin. As ε is decreased, the numerical solution converges to the exact solution to the extent that the exact and numerical solutions overlap. Indeed, the table of L^∞-errors for different values of ε and c is shown in (d). The errors for $\varepsilon = 2^{-3}$ are of order 10^{-5}, indicating that the solution for small ε is an excellent approximation to the solution of the fractional time differential equation obtained as the singular limit.

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