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SYMMETRIC FORMS

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1. Let R_m denote a m dimensional Euclidean space. When $\mathbf{x} \in R_m$ we will write $\mathbf{x}=(x_1, x_2, \ldots, x_m)$. Let $R_m^+=\{\mathbf{x}: \mathbf{x} \in R_m, x_i>0$ for all $i\}$ and $R_m^-=\{\mathbf{x}: \mathbf{x} \in R_m, x_i<0$ for all $i\}$. In this paper we consider a class of functions which consists of mappings, $E_r(\mathbf{K})$ and $H_r(\mathbf{K})$ of R_m into R which are indexed by $\mathbf{K} \in R_m^+$ and $\mathbf{K} \in R_m^-$ respectively, and defined at any point $\boldsymbol{\alpha} \in R_m$ by

$$E_r(\mathbf{K}) = \sum_{i_1+i_2+\ldots+i_m=r} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_m^{i_m} \tag{1.1}$$

where $\lambda_{i_t} = \binom{K_t}{i_t}$ $(\mathbf{K} \in R_m^+)$ and

$$H_r(\mathbf{K}) = \sum_{i_1+i_2+\ldots+i_m=r} \delta_{i_1}\delta_{i_2}\ldots\delta_{i_m}\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_m^{i_m} \tag{1.2}$$

where $\delta_{i_t} = (-1)^{i_t}\binom{K_t}{i_t}$ $(\mathbf{K} \in R_m^-)$.

Let $\mathbf{1} \in R_m$ denote the vector each of whose coordinates is 1. Then $E_r(\mathbf{1})$ and $H_r(-\mathbf{1})$ are, respectively, the elementary and complete symmetric functions of the rth order. On setting $\mathbf{K}=K(\mathbf{1})(K>0)$ in (1.1) and $\mathbf{K}=K\mathbf{1}(K<0)$ in (1.2) we obtain the class of symmetric functions introduced by Whiteley [4]. Clearly $E_r(\mathbf{K})$ and $H_r(\mathbf{K})$ are generalisations of the symmetric functions given by Whiteley [4].

It is shown in [1] that

$$E_{a-\lambda}(\mathbf{1})E_{b+\lambda}(\mathbf{1}) \geq E_{a-\lambda-1}(\mathbf{1})E_{b+\lambda+1}(\mathbf{1}),$$

provided $0 \leq \lambda < a$, and $b \geq A$. In [2] the same inequality with E replaced by H was obtained for the same range of a, b, λ. In this paper we prove that this inequality continues to hold for $E(H)$ on its domain of definition and for the same range of a, b, and λ when $\mathbf{1}$ $(-\mathbf{1})$ is replaced by $\mathbf{K}$. The proofs of these results rely on the classical method of maxima and minima as in [3] and [4] and use the generating series for E and H which are, respectively

$$1+\sum E_r(\mathbf{K})x^r = \prod_{i=1}^{m} (1+\alpha_i x)^{K_i}. \tag{1.3}$$

and

$$1+\sum H_r(\mathbf{K})x^r = \prod_{i=1}^{m} (1-\alpha_i x)^{K_i}. \tag{1.4}$$

2. LEMMA 1. *If $r=1$, then for all m,*

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(2.1) $$[H_r(\mathbf{K})]^2 \geq H_{r-1}(\mathbf{K})H_{r+1}(\mathbf{K}) \quad (K_i \leq -1, \textit{for all } i)$$

and

(2.2) $$[E_r(\mathbf{K})]^2 \geq E_{r-1}(\mathbf{K})E_{r+1}(\mathbf{K}) \quad (K_i > 0 \textit{ for all } i).$$

For (2.2) $r<K$ *when* $K=\min_i K_i$ *is not an integer.*

Proof. We prove (2.1) by induction. If $m = 1$, then $H_1(\mathbf{K}) = \binom{|K_1|}{1}\alpha_1$ and $H_2(\mathbf{K}) = \binom{|K_1|+1}{2}\alpha_1^2$. Hence $[H_1(\mathbf{K})]^2 \geq H_2(\mathbf{K})H_0(\mathbf{K})$ where $H_0(\mathbf{K})=1$. Assume the induction hypothesis holds and consider the $(m+1)$-dimensional case. Observe that

$$1+\sum H_r(\mathbf{K})x^r = (1-\alpha_{m+1}x)^{K_{m+1}}(1+\sum H_r(\mathbf{K}^*)x^r)$$

where $\mathbf{K}^*$ is obtained from $\mathbf{K}$ by deleting K_{m+1}. Thus

$$H_r(\mathbf{K}) = \sum_{j=0}^{r} \binom{|K_{m+1}|+j-1}{j} H_{r-j}(\mathbf{K}^*)\alpha_{m+1}^j$$

and consequently, using the induction hypothesis, we have $[H_1(\mathbf{K})]^2 \geq H_2(\mathbf{K})H_0(\mathbf{K})$. Inequality (2.1) is thereby proved and (2.2) is obtained in a similar fashion.

3. LEMMA 2. *If* $m=1$, *then for all* r,

(3.1) $$[H_r(\mathbf{K})]^2 \geq H_{r-1}(\mathbf{K})H_{r+1}(\mathbf{K}) \quad (K_i \leq -1)$$

and

(3.2) $$[E_r(\mathbf{K})]^2 \geq E_{r-1}(\mathbf{K})E_{r+1}(\mathbf{K}) \quad (K_i > 0).$$

For (3.2) $r<K$, *when* $K=\min_i K_i$ *is not an integer.*

Proof. $H_r=\binom{|K_1|+r-1}{r}\alpha_1^r$. Hence

$$[H_r(\mathbf{K})]^2-H_{r-1}(\mathbf{K})H_{r+1}(\mathbf{K}) = \frac{\binom{|K_1|+r-1}{r}\alpha_1^{2r}(|K_1|-1)}{(r+1)(|K_1|+r-1)}.$$

Therefor we have (3.1). For the proof of (3.2), observe that the restriction on r makes all the terms positive and hence (3.2) can be proved in a similar fashion.

4. LEMMA 3.

(4.1) $$\sum_{i=1}^{m} \frac{\partial}{\partial\alpha_i} H_r(\mathbf{K}) = (-\mathbf{K1}'+r-1)H_{r-1}(\mathbf{K}) \quad (K_i < 0 \textit{ for all } i)$$

and

(4.2) $$\sum_{i=1}^{m} \frac{\partial}{\partial\alpha_i} E_r(\mathbf{K}) = (\mathbf{K1}'-r+1)E_{r-1}(\mathbf{K}) \quad (K_i > 0 \textit{ for all } i)$$

where $\mathbf{1}'$ *denotes the transpose of* $\mathbf{1}$.

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Proof. From (1.4) we have

$$\sum_{i=1}^{m} \frac{\partial}{\partial \alpha_i} H_r(\mathbf{K})x^r = \frac{-K_i x}{(1-\alpha_i x)} \sum_{i=1}^{m} (1-\alpha_i x)^{K_i}.$$

Hence

$$\frac{\partial}{\partial \alpha_i} H_r(\mathbf{K}) - \alpha_i \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) = (-K_i) H_{r-1}(\mathbf{K})$$

or

$$\sum_{i=1}^{m} \frac{\partial}{\partial \alpha_i} H_r(\mathbf{K}) - \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) = (-\mathbf{K}\mathbf{1}') H_{r-1}(\mathbf{K}). \tag{4.3}$$

But by Euler's theorem on homogeneous functions

$$\sum_{i=m}^{m} \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) = (r-1) H_{r-1}(\mathbf{K}). \tag{4.4}$$

From (4.4) and (4.3) we get (4.1). Similary (4.2) can be proved by using (1.3).

5. THEOREM 1. *If* $\alpha_i \geq 0$ *and* $K_i \leq -1$, *for every i, then*

$$H_{a-\lambda}(\mathbf{K}) H_{b+\lambda}(\mathbf{K}) \geq H_{a-\lambda-1}(\mathbf{K}) H_{b+\lambda+1}(\mathbf{K}), \tag{5.1}$$

where $(0 \leq \lambda < a)$, $(b \geq a)$. *The inequality is strict unless all but one of the variables are zeros and* $K_1 = K_2 = \cdots = K_m = -1$. *Also strict inequality fails to hold if all the* α_i *are zero, whatever be* K_i, $i = 1, 2, \ldots, m$.

Proof. The proof is by induction on m and r. We shall prove that the theorem is true for all pairs m, r $(m > 2, r > 2)$ provided it is true for all pairs m, r with $m_1 < m$, and all pairs n, r, with $r_1 < r$. Also Lemma 1 shows that the theorem is true for all m, if $r = 1$, and Lemma 2 shows that the theorem is true for all r if $m = 1$. Let

$$C = \{\boldsymbol{\alpha} : \boldsymbol{\alpha} \in R_m,\ H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}) = 1,\ \alpha_i \geq 0 \text{ for all } i\}. \tag{5.2}$$

Then clearly C is a compact subset of R_m. Let us denote by M, the minimum value of $[H_r(\mathbf{K})]^2$ subject to the conditions given in (5.2). If we can prove that $M \geq 1$, then we have

$$[H_r(\mathbf{K})]^2 \geq H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}). \tag{5.3}$$

From (5.3) we have

$$\frac{H_r(\mathbf{K})}{H_{r+1}(\mathbf{K})} \geq \frac{H_{r-1}(\mathbf{K})}{H_r(\mathbf{K})} \geq \cdots \geq \frac{H_0(\mathbf{K})}{H_1(\mathbf{K})}.$$

Hence our theorem is proved if we can prove (5.3).

Suppose that the minimum value M is attained at a point $\boldsymbol{\alpha} \in R_m$ such that

$\alpha_i > 0$ (for all i). This point cannot be a singular point since by Euler's theorem on homogeneous functions

$$\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}) = 2r H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}) = 2r.$$

Hence the first partial derivatives cannot vanish simultaneously. Applying Lagrange's conditions we have

$$\text{(5.4)} \qquad \frac{\partial}{\partial \alpha_i} [H_r(\mathbf{K})]^2 - \lambda^* \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}) = 0 \quad \text{for all } i$$

or

$$\text{(5.5)} \qquad 2H_r(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_r(\mathbf{K}) - \lambda^* \{ H_{r+1}(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) + H_{r-1}(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_{r+1}(\mathbf{K}) \} = 0.$$

Multiplying (5.5) successively by $\alpha_1, \alpha_2, \ldots, \alpha_m$ and adding the results we have

$$\text{(5.6)} \qquad 2 \sum_{i=m}^{m} \alpha_i H_r(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_r(\mathbf{K}) - \lambda^* \left\{ \sum_{i=1}^{m} \alpha_i H_{r+1}(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_{r-1}(\mathbf{K}) + \sum_{i=1}^{m} \alpha_i H_{r-1}(\mathbf{K}) \frac{\partial}{\partial \alpha_i} H_{r+1}(\mathbf{K}) \right\} = 0.$$

Using Euler's theorem on homogeneous functions we have from (5.4)

$$\text{(5.7)} \qquad 2r[H_r(\mathbf{K})]^2 = \lambda^* 2r H_{r-1}(\mathbf{K}) H_{r+1}(\mathbf{K}).$$

From (5.7) and (5.2) we have $\lambda^* = M$. Hence our theorem is proved if we can show that $\lambda^* \geq 1$. From (5.6) and (4.1) we have

$$2(-\mathbf{K1}' + r - 1) H_r(\mathbf{K}) H_{r-1}(\mathbf{K}) = \lambda^* \{ (-\mathbf{K1}' + r - 2) H_{r+1}(\mathbf{K}) H_{r-2}(\mathbf{K}) + (-\mathbf{K1}' + r) H_{r-1}(\mathbf{K}) H_r(\mathbf{K}) \}$$

or

$$\text{(5.8)} \qquad 2(-\mathbf{K1}' + r - 1) - \lambda^*(-\mathbf{K1}' + r) = \frac{\lambda^*(-\mathbf{K1}' + r - 2) H_{r+1}(\mathbf{K}) H_{r-2}(\mathbf{K})}{H_r(\mathbf{K}) H_{r-1}(\mathbf{K})}.$$

Now from (5.7) and (5.8) we get

$$\text{(5.9)} \qquad 2(-\mathbf{K1}' + r - 1) - \lambda^*(-\mathbf{K1}' + r) = \frac{(-\mathbf{K1}' + r - 2) H_{r-2}(\mathbf{K}) H_r(\mathbf{K})}{[H_{r-1}(\mathbf{K})]^2}.$$

But by induction hypothesis

$$\text{(5.10)} \qquad [H_{r-1}(\mathbf{K})]^2 \geq H_r(\mathbf{K}) H_{r-2}(\mathbf{K}).$$

Hence from (5.9) and (5.10)

$$\text{(5.11)} \qquad 2(-\mathbf{K1}' + r - 1) - \lambda^*(-\mathbf{K1}' + r) \leq (-\mathbf{K1}' + r - 2).$$

Hence $\lambda^* \geq 1$.

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In the next place we suppose that the minimum is attained at a point at which one or more of $\alpha_1, \alpha_2, \ldots, \alpha_m$ are zeros. Suppose that $\alpha_1 \neq 0, \alpha_2 \neq 0, \ldots, \alpha_S \neq 0 \quad (S < m)$, and from induction on m we have from (5.7)

$$\lambda^* = \frac{[H_r(\mathbf{K})]^2}{H_{r-1}(\mathbf{K})H_{r+1}(\mathbf{K})}$$

and $\lambda^* \geq 1$. Hence the theorem follows from (5.11).

6. THEOREM 2. *If* $\alpha_i \geq 0$ *and* $K_i > 0$ *for all i, then*

$$E_{a-\lambda}(\mathbf{K})E_{b+\lambda}(\mathbf{K}) \geq E_{a-\lambda-1}(\mathbf{K})E_{b+\lambda+1}(\mathbf{K})$$

provided $(0 \leq \lambda < a)$, $(b \geq a)$ *and* $b+\lambda < K$ when $K = \min_i K_i$ *is not an integer.*

The inequality is strict unless all but one of the variables are zeros and $K_1 = K_2 = \cdots = K_m = 1$. *Also the strict inequality fails to hold if all the* α_i *are zero whatever be* K_i, $i = 1, 2, \ldots, m$.

Proof. The restriction on $b+\lambda$ makes all the terms positive in our considerations. Hence we can apply the method of Theorem 1.

7. THEOREM 3.

$$[H_r(\mathbf{K})]^{1/r} \geq [H_{r+1}(\mathbf{K})]^{1/(r+1)} \tag{7.1}$$

$$[E_r(\mathbf{K})]^{1/r} \geq [E_{r+1}(\mathbf{K})]^{1/(r+1)} \tag{7.2}$$

The inequality is strict unless all but one of the variables are zeros and

$$K_1 = K_2 = \cdots = K_m = -1 \quad \textit{for } (7.1)$$

and

$$K_1 = K_2 = \cdots = K_m = 1 \quad \textit{for } (7.2)$$

Also the strict inequality fails to hold if all the α_i *are zero whatever be* K_i, $i = 1, 2, \ldots, m$. *For* (7.2), $r < K$ *when* $K = \min_i K_i$ *is not an integer.*

Proof. Same as in [1].

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