SYSTEMS OF IDEALS

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Introduction. Some well-known theorems on the representation of an ideal in a commutative ring as an intersection of ideals of a specified type remain valid when attention is restricted to certain subclasses of the ideals of the ring. For example, a homogeneous ideal in a graded Noetherian ring is the intersection of homogeneous primary ideals, and Seidenberg (**16**) has recently proved a similar result for differential rings. Other examples are provided by theorems on the representation of certain differential and difference ideals as intersections of prime differential or difference ideals.

Kolchin (**6**) began the development of a general theory applicable to such phenomena. In this paper, I continue the development of that theory.

In §§ 1 and 2, I have followed Kolchin's definition and discussion of divisible systems with minor modifications and additions. (The terminology is not that of (**6**) but of his more recent and still unpublished notes on differential algebra.) The strengthening of Theorem I to give information on minimal prime divisors generalizes a similar result of Andreian (**1**) for differential algebra. Section 4 also draws on ideas due to Kolchin. In other sections, representation theorems not considered by him are discussed. My own method (**2**) of defining divisible conservative systems by means of "links" is also included since it is often convenient, and since it suggests a classification of divisible conservative systems. (See the conclusion of § 5.) The equivalence of the two methods is proved using a result (Lemma XV below) due to Kolchin. Some of the ideas go back to a paper by Seidenberg (**15**, p. 181).

In a later paper, I plan to give the applications of the theory to basis theorems of difference-differential algebra which motivated my investigation. Kolchin (in the notes referred to above) has applied the theory to give an elegant account of the behaviour of polynomial and differential polynomial ideals under ground field extension, as well as, (**6**), to basis theorems for differential ideals.

Notation. Inclusion is denoted by $\subseteq$, proper inclusion by $\subset$. Throughout, $\mathfrak{A}$ will denote a commutative ring and $\mathscr{S}$ the set of ideals of $\mathfrak{A}$. Let $\mathscr{X} \subseteq \mathscr{S}$ be closed under intersection, and let $\mathfrak{A} \in \mathscr{X}$. Then for $\mathfrak{T} \subseteq \mathfrak{A}$, $(\mathfrak{T}; \mathscr{X})$ denotes the intersection of the members of $\mathscr{X}$ which contain $\mathfrak{T}$. Hence,

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$(\mathfrak{T};\mathscr{X}) \in \mathscr{X}$; and, indeed $(\mathfrak{T};\mathscr{X})$ is the minimal ideal of $\mathscr{X}$ containing $\mathfrak{T}$. $\mathfrak{T}$ is called a set of *$\mathscr{X}$-generators* of $(\mathfrak{T};\mathscr{X})$. A finite set of $\mathscr{X}$-generators of an ideal $\mathfrak{a} \in \mathscr{X}$ is called an *$\mathscr{X}$-basis* of $\mathfrak{a}$. $\mathscr{X}$ is called *Noetherian* if the ascending chain condition holds in $\mathscr{X}$. $\mathscr{X}$ is *additive* if $\mathfrak{a} \in \mathscr{X}$, $\mathfrak{b} \in \mathscr{X}$ implies $(\mathfrak{a}, \mathfrak{b}) \in \mathscr{X}$.

Let $\mathfrak{a} \in \mathscr{S}$, and let $\mathfrak{T}$ be a non-empty, multiplicatively closed subset of $\mathfrak{A}$. Then $\mathfrak{a}_{\mathfrak{T}}$ is the ideal consisting of all $b \in \mathfrak{A}$ such that there exists $t \in \mathfrak{T}$ with $bt \in \mathfrak{a}$. If $x \in \mathfrak{A}$, then $\mathfrak{a}_x$ denotes $\mathfrak{a}_{\mathfrak{T}}$ with $\mathfrak{T}$ the set of positive integral powers of x.

1. Conservative systems. A subset $\mathscr{X}$ of $\mathscr{S}$ will be called *conservative* if it satisfies:

C-1: If $\mathscr{Y} \subseteq \mathscr{X}$, the intersection of the members of $\mathscr{Y}$ is in $\mathscr{X}$;

C-2: If $\mathscr{Y} \subseteq \mathscr{X}$, and $\mathscr{Y}$ is totally ordered by inclusion, then the union of the ideals of $\mathscr{Y}$ is in $\mathscr{X}$;

C-3: $\mathfrak{A} \in \mathscr{X}$. (This follows from C-1 if a common convention is adopted when $\mathscr{Y}$ is empty.)

Conservative systems are frequent, but the desired representation theorems apply only to appropriate subsets of conservative systems. If $\mathscr{X}$ is conservative and $\mathfrak{a} \in \mathscr{S}$, then $\mathfrak{a}$ is called *divisible* with respect to $\mathscr{X}$ if $\mathfrak{a} \in \mathscr{X}$ and, for each $x \in \mathfrak{A}$, $\mathfrak{a}:x \in \mathscr{X}$. We denote by $\mathscr{D}(\mathscr{X})$ the set of ideals divisible with respect to $\mathscr{X}$. If $\mathscr{X} = \mathscr{D}(\mathscr{X})$, then $\mathscr{X}$ is called a *divisible conservative* system. Evidently, $\mathscr{D}(\mathscr{X})$ is conservative and $\mathscr{D}(\mathscr{D}(\mathscr{X})) = \mathscr{D}(\mathscr{X})$. Hence, $\mathscr{D}(\mathscr{X})$ is a divisible conservative system, and, indeed, the maximal such system contained in $\mathscr{X}$. Observe also that if $\mathfrak{T} \subseteq \mathfrak{A}$, $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then $\mathfrak{a}:\mathfrak{T} \in \mathscr{D}(\mathscr{X})$.

If $\mathscr{X}$ is conservative, then $\mathfrak{a}$ is called *perfect* with respect to $\mathscr{X}$ if $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ and $\mathfrak{a} = \sqrt{\mathfrak{a}}$. We denote by $\mathscr{P}(\mathscr{X})$ the set of perfect ideals of $\mathscr{X}$. If $\mathscr{X} = \mathscr{P}(\mathscr{X})$, then $\mathscr{X}$ is called a *perfect conservative* system. It is evident that if $\mathscr{X}$ is conservative, then $\mathscr{P}(\mathscr{X})$ is a perfect conservative system and is the maximal such system contained in $\mathscr{X}$.

If $\mathscr{X}$ is conservative, then $\mathfrak{a}$ is called *partially divisible* with respect to $\mathscr{X}$ if $\mathfrak{a} \in \mathscr{X}$ and for each $x \in \mathfrak{A}$, $\mathfrak{a}_x \in \mathscr{X}$. We denote by $\mathscr{P}\mathscr{D}(\mathscr{X})$ the set of ideals partially divisible with respect to $\mathscr{X}$. If $\mathscr{X} = \mathscr{P}\mathscr{D}(\mathscr{X})$, then $\mathscr{X}$ is called a *partially divisible conservative* system. $\mathscr{P}\mathscr{D}(\mathscr{X})$ is closed under finite intersection and satisfies C-2. If $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$, $x \in \mathfrak{A}$, then $\mathfrak{a}_x \in \mathscr{P}\mathscr{D}(\mathscr{X})$; since $(\mathfrak{a}_x)_y = \mathfrak{a}_{xy}$, $y \in \mathfrak{A}$. If $\mathfrak{a}$ is radical, $\mathfrak{a}_x = \mathfrak{a}:x$. Hence, every radical ideal of $\mathscr{P}\mathscr{D}(\mathscr{X})$ is in $\mathscr{P}(\mathscr{X})$.

If $\mathscr{X}$ is conservative, then $\mathfrak{a}$ is called *complete* with respect to $\mathscr{X}$ if $\mathfrak{a} \in \mathscr{X}$ and $\sqrt{\mathfrak{a}} \in \mathscr{P}(\mathscr{X})$. We denote by $\mathscr{C}(\mathscr{X})$ the set of complete ideals of $\mathscr{X}$. $\mathscr{C}(\mathscr{X})$ is closed under finite intersection.

Remark. In all the preceding definitions except the last, the requirement $\mathfrak{a} \in \mathscr{X}$ is superfluous if $\mathfrak{A}$ contains an identity.

Some of the facts already stated and other obvious relations are summarized in the following lemma.

LEMMA I. *If $\mathscr{X}$ is a conservative system of ideals of $\mathfrak{A}$, then $\mathscr{D}(\mathscr{X})$ and $\mathscr{P}(\mathscr{X})$ are conservative, $\mathscr{D}(\mathscr{X}) = \mathscr{D}(\mathscr{D}(\mathscr{X}))$, $\mathscr{P}(\mathscr{X}) = \mathscr{P}(\mathscr{P}(\mathscr{X}))$; $\mathscr{P}\mathscr{D}(\mathscr{X})$ and $\mathscr{C}(\mathscr{X})$ are closed under finite intersection;*

$$\mathscr{P}(\mathscr{X}) \subseteq \mathscr{D}(\mathscr{X}) \subseteq \mathscr{P}\mathscr{D}(\mathscr{X}) \subseteq \mathscr{X};$$

and $\mathscr{P}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X}) \subseteq \mathscr{X}$.

Remark. It will be shown later that $\mathscr{D}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X})$. However, neither inclusion holds in general between $\mathscr{C}(\mathscr{X})$ and $\mathscr{P}\mathscr{D}(\mathscr{X})$; see Theorem I and § 6, examples (5) and (7).

LEMMA II. *If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ and $\mathfrak{T} \subseteq \mathfrak{A}$ is not empty and is multiplicatively closed, then $\mathfrak{a}_{\mathfrak{T}} \in \mathscr{D}(\mathscr{X})$.*

Proof. Let $x \in \mathfrak{T}$. Then by Lemma I, $\mathfrak{a}_x \in \mathscr{D}(\mathscr{X})$, since it is the union of the ascending chain $\mathfrak{a}:x, \mathfrak{a}:x^2, \ldots$, whose members are in $\mathscr{D}(\mathscr{X})$. Consider the non-empty multiplicatively closed subsets $\mathfrak{U}$ of $\mathfrak{T}$ such that $\mathfrak{a}_{\mathfrak{U}} \in \mathscr{D}(\mathscr{X})$. The preceding remark shows the existence of such subsets. It follows easily from Zorn's lemma that there is a maximal such subset $\mathfrak{T}'$. Let $x \in \mathfrak{T}$, and let $\mathfrak{T}^*$ denote the multiplicatively closed set generated by x and the members of $\mathfrak{T}'$. Then $\mathfrak{a}_{\mathfrak{T}^*} = (\mathfrak{a}_{\mathfrak{T}'})_x \in \mathscr{D}(\mathscr{X})$. Hence, $x \in \mathfrak{T}'$; $\mathfrak{T}' = \mathfrak{T}$.

LEMMA III. *If the radical of every ideal of $\mathscr{P}\mathscr{D}(\mathscr{X})$ is in $\mathscr{X}$, then*

$$\mathscr{P}\mathscr{D}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X}).$$

Proof. Let $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$. By hypothesis, $\mathfrak{a} \in \mathscr{X}$. If $x \in \mathfrak{A}$, then

$$(\sqrt{\mathfrak{a}}):x = \sqrt{\mathfrak{a}_x}.$$

Since $\mathfrak{a}_x \in \mathscr{P}\mathscr{D}(\mathscr{X})$, it follows from the hypothesis that $(\sqrt{\mathfrak{a}}):x \in \mathscr{X}$. Hence, $\sqrt{\mathfrak{a}} \in \mathscr{D}(\mathscr{X})$. Then by the definitions, $\sqrt{\mathfrak{a}} \in \mathscr{P}(\mathscr{X})$; $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$.

LEMMA IV. *If $\mathscr{Y} \subseteq \mathscr{X}$ is a conservative system, then $\mathscr{F}(\mathscr{Y}) \subseteq \mathscr{F}(\mathscr{X})$, where $\mathscr{F}$ is $\mathscr{D}$, $\mathscr{P}\mathscr{D}$, $\mathscr{P}$, or $\mathscr{C}$.*

The proof is obvious.

LEMMA V. *Let $\mathscr{I}$ be an index set and let $\mathscr{Y}_i$, $i \in \mathscr{I}$, be conservative systems of ideals in $\mathfrak{A}$. Let $\mathscr{Y} = \bigcap_{i \in \mathscr{I}} \mathscr{Y}_i$. Then $\mathscr{Y}$ is conservative, and*

$$\mathscr{D}(\mathscr{Y}) = \bigcap_{i \in \mathscr{I}} \mathscr{D}(\mathscr{Y}_i), \qquad \mathscr{P}(\mathscr{Y}) = \bigcap_{i \in \mathscr{I}} \mathscr{P}(\mathscr{Y}_i),$$
$$\mathscr{P}\mathscr{D}(\mathscr{Y}) = \bigcap_{i \in \mathscr{I}} \mathscr{P}\mathscr{D}(\mathscr{Y}_i), \quad \text{and} \quad \mathscr{C}(\mathscr{Y}) = \bigcap_{i \in \mathscr{I}} \mathscr{C}(\mathscr{Y}_i).$$

The proof is obvious.

2. Representation theorems. Let $\mathscr{X} \subseteq \mathscr{S}$ be a conservative system. We can verify at once that every prime ideal, and hence every intersection of prime ideals, of $\mathscr{X}$ is in $\mathscr{P}(\mathscr{X})$, and that every primary ideal, and hence every finite intersection of primary ideals, of $\mathscr{X}$ is in $\mathscr{P}\mathscr{D}(\mathscr{X})$.

Theorem I, which is essentially due to Kolchin (see **6**), shows conversely that every ideal of $\mathscr{P}(\mathscr{X})$ is an intersection of prime ideals of $\mathscr{P}(\mathscr{X})$, thus generalizing a well-known observation of Krull (**8**, p. 9) that every radical ideal is an intersection of prime ideals. Cases in which every ideal of $\mathscr{P}\mathscr{D}(\mathscr{X})$ is an intersection of primary ideals of $\mathscr{P}\mathscr{D}(\mathscr{X})$ will be investigated below.

THEOREM I. *Let $\mathscr{X}$ be a conservative system of ideals of a commutative ring $\mathfrak{A}$. If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then $\sqrt{\mathfrak{a}}$ is an intersection of prime ideals of $\mathscr{X}$. Indeed, every prime ideal minimal among those containing $\mathfrak{a}$ is in $\mathscr{X}$, and $\sqrt{\mathfrak{a}}$ is the intersection of these prime ideals. It follows that $\mathscr{D}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X})$, and that an ideal $\mathfrak{b}$ is an intersection of prime ideals of $\mathscr{X}$ if and only if $\mathfrak{b} \in \mathscr{P}(\mathscr{X})$,*

Proof. Let $\mathfrak{T}$ be a multiplicatively closed subset of $\mathfrak{A}$ not intersecting $\mathfrak{a}$. The set of ideals of $\mathscr{D}(\mathscr{X})$ which contain $\mathfrak{a}$ and do not intersect $\mathfrak{T}$ contains maximal elements by Lemma I. Let $\mathfrak{p}$ be one of these. It will be shown that $\mathfrak{p}$ is prime.

Let b, $c \in \mathfrak{A}$, $bc \in \mathfrak{p}$, $b \notin \mathfrak{p}$. It will be shown first that there exists $t \in \mathfrak{T}$ such that $ct \in \mathfrak{p}$. Since $b \in \mathfrak{p}{:}c$, $\mathfrak{p} \subset \mathfrak{p}{:}c$. Also, $\mathfrak{p}{:}c \in \mathscr{D}(\mathscr{X})$. Hence, by the maximality of $\mathfrak{p}$, there exists $t \in \mathfrak{p}{:}c \cap \mathfrak{T}$. Then $ct \in \mathfrak{p}$. Now let g, $h \in \mathfrak{A}$, $gh \in \mathfrak{p}$, $g \notin \mathfrak{p}$. A first application of the result just obtained yields $t_1h \in \mathfrak{p}$, $t_1 \in \mathfrak{T}$. If $h \notin \mathfrak{p}$, a second application yields $t_1t_2 \in \mathfrak{p}$, a contradiction. Hence $\mathfrak{p}$ is prime.

Now let $\mathfrak{q}$ be a prime ideal minimal among those containing $\mathfrak{a}$, and let $\mathfrak{T} = \mathfrak{A} - \mathfrak{q}$. The result of the preceding paragraph yields a prime ideal of $\mathscr{D}(\mathscr{X})$ not intersecting $\mathfrak{T}$. This ideal can only be $\mathfrak{q}$. Hence, $\mathfrak{q} \in \mathscr{D}(\mathscr{X})$. This proves the statement concerning minimal prime ideals. To complete the proof one may appeal directly to the result of Krull of which Theorem I is a generalization; namely, that every radical ideal is the intersection of the minimal members of the set of prime ideals containing it. Alternatively, one may proceed as in the proof of Krull's result (**5**, p. 13) applying the result of the preceding paragraph to each set $\{x, x^2, \ldots\}$, $x \in \mathfrak{A} - \sqrt{\mathfrak{a}}$.

LEMMA VI. *Let $\mathfrak{s}$ and $\mathfrak{t}$ be subsets of $\mathfrak{A}$. Then $(\mathfrak{s};\mathscr{P}(\mathscr{X})) \cap (\mathfrak{t};\mathscr{P}(\mathscr{X})) = (\mathfrak{st};\mathscr{P}(\mathscr{X}))$. Here $\mathfrak{st}$ denotes the set of all products ab, $a \in \mathfrak{s}$, $b \in \mathfrak{t}$.*

Proof. Let $\mathfrak{q} = (\mathfrak{s};\mathscr{P}(\mathscr{X})) \cap (\mathfrak{t};\mathscr{P}(\mathscr{X}))$. Of course, $(\mathfrak{st};\mathscr{P}(\mathscr{X})) \subseteq \mathfrak{q}$. By Theorem I, $(\mathfrak{st};\mathscr{P}(\mathscr{X})) = \bigcap_{i\in\mathscr{I}}\mathfrak{p}_i$, where $\mathscr{I}$ is a suitable index set and the $\mathfrak{p}_i$ are prime ideals of $\mathscr{X}$. Let $i \in \mathscr{I}$. If $\mathfrak{s} \subseteq \mathfrak{p}_i$, then $(\mathfrak{s};P(\mathscr{X})) \subseteq \mathfrak{p}_i$, and hence $\mathfrak{q} \subseteq \mathfrak{p}_i$. If $\mathfrak{s}$ is not contained in $\mathfrak{p}_i$, then $\mathfrak{t} \subseteq \mathfrak{p}_i$, and therefore $\mathfrak{q} \subseteq \mathfrak{p}_i$ by a similar argument. Hence, $\mathfrak{q} \subseteq \bigcap_{i\in\mathscr{I}}\mathfrak{p}_i = (\mathfrak{st};\mathscr{P}(\mathscr{X}))$.

THEOREM II. *Let $\mathscr{X}$ be a conservative system of ideals of the commutative ring $\mathfrak{A}$. Let $\mathscr{P}(\mathscr{X})$ be Noetherian. Then every ideal of $\mathscr{P}(\mathscr{X})$ can be represented uniquely as the irredundant intersection of finitely many prime ideals of $\mathscr{X}$.*

Proof. It is only necessary to prove that every ideal of $\mathscr{P}(\mathscr{X})$ is the intersection of finitely many prime ideals of $\mathscr{X}$. Then well-known elementary considerations independent of the fact that the prime ideals involved are in $\mathscr{X}$ permit the determination of a subset of these ideals furnishing the unique irredundant representation.

Suppose, to the contrary, that there are ideals in $\mathscr{P}(\mathscr{X})$ which are not intersections of finitely many prime ideals of $\mathscr{X}$. Then the ascending chain condition shows the existence of an ideal $\mathfrak{a}$ maximal among those with this property. Of course, $\mathfrak{a}$ is not prime. Let $fg \in \mathfrak{a}, f \notin \mathfrak{a}, g \notin \mathfrak{a}$. Lemma VI shows that $\mathfrak{a} \supseteq (\mathfrak{a}, f; \mathscr{P}(\mathscr{X})) \cap (\mathfrak{a}, g; \mathscr{P}(\mathscr{X}))$. Since the opposite inclusion is obvious, this yields $\mathfrak{a} = (\mathfrak{a}, f; \mathscr{P}(\mathscr{X})) \cap (\mathfrak{a}, g; \mathscr{P}(\mathscr{X}))$. However, by the maximality of $\mathfrak{a}$, the ideals on the right-hand side of the last equation are each intersections of finitely many prime ideals of $\mathscr{X}$. Hence, we obtain the contradiction that $\mathfrak{a}$ itself is such an intersection.

THEOREM III. *Let $\mathscr{X}$ be a conservative system in a commutative ring $\mathfrak{A}$ and let $\mathfrak{a} \in \mathscr{X}$ be the intersection of finitely many primary ideals of $\mathfrak{A}$.*

(1) *If $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$ and $\mathfrak{b}$ is an isolated ideal component of $\mathfrak{a}$, then there exist finitely many $x_i \in \mathfrak{A}$ such that $\mathfrak{b} = \bigcap \mathfrak{a}_{x_i}$. Hence, all the isolated ideal components of $\mathfrak{a}$ are in $\mathscr{P}\mathscr{D}(\mathscr{X})$.*

(2) *If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then the isolated ideal components of $\mathfrak{a}$ are in $\mathscr{D}(\mathscr{X})$.*

(3) *If $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$, then the minimal associated prime ideals of $\mathfrak{a}$ are in $\mathscr{P}(\mathscr{X})$.*

(4) *If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ and $\mathfrak{p}$ is an associated prime ideal of $\mathfrak{a}$, then there exists $y \in \mathfrak{A}$ such that $\mathfrak{p} = \sqrt{(\mathfrak{a}:y)}$. Hence, all the associated prime ideals of $\mathfrak{a}$ are in $\mathscr{P}(\mathscr{X})$.*

Proof. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be an irredundant representation of $\mathfrak{a}$ as an intersection of primary ideals. Let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, $i = 1, \ldots, r$. Let $\mathfrak{a}_j$, $j = 1, \ldots, r$, denote the intersection of those $\mathfrak{q}_i$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$. Then the isolated ideal components of $\mathfrak{a}$ are intersections of the $\mathfrak{a}_j$.

(1) Let $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$. For $x \in \mathfrak{A}$, $\mathfrak{a}_x = (\mathfrak{q}_1)_x \cap \ldots \cap (\mathfrak{q}_r)_x$. Since $(\mathfrak{q}_i)_x$ is $\mathfrak{q}_i$ if $x \notin \mathfrak{p}_i$, and $(\mathfrak{q}_i)_x$ is $\mathfrak{A}$ if $x \in \mathfrak{p}_i$, we see that $\mathfrak{a}_x$ is the intersection of those $\mathfrak{q}_i$ such that $x \notin \mathfrak{p}_i$. For each j, $1 \leqq j \leqq r$, there exists x_j such that $x_j \notin \mathfrak{p}_j$, however $x_j \in \mathfrak{p}_i$ for each $\mathfrak{p}_i$ not contained in $\mathfrak{p}_j$. Then $\mathfrak{a}_j = \mathfrak{a}_{x_j} \in \mathscr{P}\mathscr{D}(\mathscr{X})$.

(2) This follows from (1) and Lemma I.

(3) Let $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$. Then $\sqrt{\mathfrak{a}} \in \mathscr{P}(\mathscr{X})$, therefore $\sqrt{\mathfrak{a}} \in \mathscr{P}\mathscr{D}(\mathscr{X})$. However, the isolated ideal components of $\sqrt{\mathfrak{a}}$ include the minimal associated prime ideals of $\mathfrak{a}$. Hence, these are in $\mathscr{P}\mathscr{D}(\mathscr{X})$. It has already been noted that every prime ideal in $\mathscr{X}$ is in $\mathscr{P}(\mathscr{X})$. Hence, the minimal associated prime ideals of $\mathfrak{a}$ are in $\mathscr{P}(\mathscr{X})$.

(4) Let $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$. For each $j, 1 \leqq j \leqq r$, there exists y_j such that $y_j \notin \mathfrak{q}_j, y_j \in \mathfrak{q}_i, 1 \leqq i \neq j \leqq r$. Then $\mathfrak{a}:y_j = \mathfrak{q}_j:y_j \subseteq \mathfrak{p}_j$; thus,

$$\mathfrak{p}_j \supseteq \sqrt{(\mathfrak{a}:y_j)}.$$

However, $\mathfrak{q}_j \subseteq \mathfrak{q}_j:y_j = \mathfrak{a}:y_j$ also. Therefore, $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j} \subseteq \sqrt{(\mathfrak{a}:y_j)}$. Thus, $\mathfrak{p}_j = \sqrt{(\mathfrak{a}:y_j)}$. Since $\mathfrak{a}:y_j \in \mathscr{D}(\mathscr{X})$, it follows from Theorem I that

$$\mathfrak{p}_j \in \mathscr{P}(\mathscr{X}).$$

COROLLARY I. *If $\mathfrak{a} \in \mathscr{P}(\mathscr{X})$, and $\mathfrak{a}$ is the irredundant intersection of finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, then the $\mathfrak{p}_i$ are in $\mathscr{P}(\mathscr{X})$.*

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Proof. $\mathscr{P}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X})$. Apply (3) of Theorem III.

COROLLARY II. *If $\mathfrak{A}$ is Noetherian, then the isolated ideal components of every ideal of $\mathscr{P}\mathscr{D}(\mathscr{X})$ are in $\mathscr{P}\mathscr{D}(\mathscr{X})$.*

The proof is obvious from (1).

Although the case of homogeneous ideals might lead one to expect that not only are the isolated ideal components partially divisible in the situation of (1) of Theorem III, but that also the primary ideals themselves may be selected to be partially divisible, this is not so in general. (Some circumstances under which such a selection is possible will be examined later.)

Counterexample. Let $\mathfrak{A}$ be a Noetherian ring containing primary ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ such that $\mathfrak{q}_1$ does not contain $\mathfrak{q}_2$ but $\mathfrak{p}_1 = \sqrt{\mathfrak{q}_1} \supset \mathfrak{p}_2 = \sqrt{\mathfrak{q}_2}$. Let $\mathscr{X}$ consist of all ideals of $\mathfrak{A}$ contained in $\mathfrak{p}_2$, and of $\mathfrak{A}$ itself. Then $\mathscr{X}$ is a conservative system. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2$. Then for $x \in \mathfrak{A}$, $\mathfrak{a}_x$ is either $\mathfrak{a}$, $\mathfrak{q}_2$ or $\mathfrak{A}$. Hence $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$. However, in every representation of $\mathfrak{a}$ as an irredundant intersection of primary ideals, one of the components will have the associated prime ideal $\mathfrak{p}_1$, and therefore not be in $\mathscr{X}$.

3. Comaximal representations. Throughout this section, $\mathfrak{A}$ is assumed to have an identity element 1. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{A}$, then they are said to be *comaximal* if $1 \in (\mathfrak{a}, \mathfrak{b})$. If, in particular, $\mathfrak{A}$ is Noetherian, then every ideal of $\mathfrak{A}$ is the intersection of a uniquely determined finite set of pairwise comaximal ideals, no one of which is itself the intersection of two pairwise comaximal ideals different from (1). Ritt (**11**, p. 14; **10**, p. 687) has obtained comparable theorems for differential and difference ideals. These results will be generalized for the ideals of $\mathscr{D}(\mathscr{X})$ and of $\mathscr{C}(\mathscr{X}) \cap \mathscr{P}\mathscr{D}(\mathscr{X})$, where $\mathscr{X}$ is any conservative system of ideals in $\mathfrak{A}$. A more direct generalization of Ritt's results will be given later for certain conservative systems.

The following results are well known (**17**, p. 177).

LEMMA VII. *If $\mathfrak{a}, \mathfrak{b}_1, \ldots, \mathfrak{b}_k$ are ideals of $\mathfrak{A}$ and $\mathfrak{a}$ is comaximal with each $\mathfrak{b}_i$, then $\mathfrak{a}$ is comaximal with $\bigcap_{i=1}^k \mathfrak{b}_i$.*

LEMMA VIII. *If $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are pairwise comaximal, then*

$$\bigcap_{i=1}^{k} \mathfrak{a}_i = \prod_{i=1}^{k} \mathfrak{a}_i.$$

Lemma VIII permits the replacement of intersections by products in the theorems which follow.

The proof of the next result is a simplified version of a proof by Ritt (**10**, p. 687) of a theorem about difference ideals.

LEMMA IX. *Let $\mathfrak{a}$ be an ideal of $\mathfrak{A}$, $\mathfrak{b} = \sqrt{\mathfrak{a}}$. If $\mathfrak{b} = \bigcap_{i=1}^k \mathfrak{b}_i$, where the $\mathfrak{b}_i$ are pairwise comaximal, then the $\mathfrak{b}_i$ are radical ideals, and there exist uniquely determined ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ in $\mathfrak{A}$ such that $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{a}_i$, $\mathfrak{b}_i = \sqrt{\mathfrak{a}_i}$, $i = 1, \ldots, k$. The $\mathfrak{a}_i$ are pairwise comaximal.*

Proof. We suppose that $k = 2$. The general case then follows easily by induction using Lemma VII. By comaximality, there exist $x \in \mathfrak{b}_1$, $y \in \mathfrak{b}_2$ such that $x + y = 1$. Then $xy \in \mathfrak{b}$, and there is a positive integer t such that $(xy)^t \in \mathfrak{a}$. In the binomial expansion of $(x + y)^{2t}$, let c be the sum of those terms of degree in x not less than t, and let d be the sum of the remaining terms. Then $c \in \mathfrak{b}_1$, $d \in \mathfrak{b}_2$, $c + d = 1$, $cd \in \mathfrak{a}$. Let $\mathfrak{a}_1 = (\mathfrak{a}, c)$, $\mathfrak{a}_2 = (\mathfrak{a}, d)$. Clearly, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are comaximal.

It will be shown first that $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. It is sufficient to show that $\mathfrak{a} \supseteq \mathfrak{a}_1 \cap \mathfrak{a}_2$. Let $g \in \mathfrak{a}_1 \cap \mathfrak{a}_2$. Then $g = u + vc$, $u \in \mathfrak{a}$, $v \in \mathfrak{A}$. Hence, $gd = ud + vcd \in \mathfrak{a}$. Furthermore, $g = u' + v'd$, $u' \in \mathfrak{a}$, $v' \in \mathfrak{A}$, and therefore $gc \in \mathfrak{a}$. However, $g = gd + gc$; thus $g \in \mathfrak{a}$.

Let $m^r \in \mathfrak{b}_1$. Then $m^r y \in \mathfrak{b}$. Hence $my \in \mathfrak{b} \subseteq \mathfrak{b}_1$. Since $my = m - mx$, and $x \in \mathfrak{b}_1$, it follows that $m \in \mathfrak{b}_1$. Thus, $\mathfrak{b}_1$ is a radical ideal. To show that $\mathfrak{b}_1 = \sqrt{\mathfrak{a}_1}$ it suffices, since $\mathfrak{a}_1 \subseteq \mathfrak{b}_1$ by construction, to show, given $n \in \mathfrak{b}_1$, that $n \in \sqrt{\mathfrak{a}_1}$. Now $nd = n - nc \in \mathfrak{b} \subseteq \sqrt{\mathfrak{a}_1}$. Since $c \in \mathfrak{a}_1$, this implies that $n \in \sqrt{\mathfrak{a}_1}$.

To prove uniqueness, let $\mathfrak{a} = \mathfrak{a}_1' \cap \mathfrak{a}_2'$, $\mathfrak{b}_1 = \sqrt{\mathfrak{a}_1'}$, $\mathfrak{b}_2 = \sqrt{\mathfrak{a}_2'}$. Let $g \in \mathfrak{a}_1'$. With c and d as above, we have $d^p \in \mathfrak{a}_2$, for some positive integer p. Then $gd^p = g(1 - c)^p \in \mathfrak{a} \subseteq \mathfrak{a}_1$. Expanding the second factor and using $c \in \mathfrak{a}_1$, we find that $g \in \mathfrak{a}_1$. Hence $\mathfrak{a}_1' \subseteq \mathfrak{a}_1$. Now let $h \in \mathfrak{a}_1$. Then $hd = h(1 - c) \in \mathfrak{a} \subseteq \mathfrak{a}_1'$. Then $h(1 - c^n) \in \mathfrak{a}_1'$, $n = 1, 2, \ldots$. For n large, $c^n \in \mathfrak{a}_1'$. Hence $h \in \mathfrak{a}_1'$. It follows that $\mathfrak{a}_1 = \mathfrak{a}_1'$.

LEMMA X. *Let $\mathscr{X}$ be a conservative system in the commutative ring $\mathfrak{A}$ with identity. Let $\mathfrak{a}$ be an ideal of $\mathfrak{A}$ and suppose that $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{a}_i$, where the $\mathfrak{a}_i$ are pairwise comaximal ideals of $\mathfrak{A}$. Then if $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$, each $\mathfrak{a}_i \in \mathscr{P}\mathscr{D}(\mathscr{X})$. If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$.*

Proof. By Lemma VII it suffices to consider $k = 2$. Let $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$. Let $x \in \mathfrak{a}_1$, $y \in \mathfrak{a}_2$ be such that $x + y = 1$. Since $\mathfrak{a}_x = (\mathfrak{a}_1)_x \cap (\mathfrak{a}_2)_x = (\mathfrak{a}_2)_x$, we see that

$$(\mathfrak{a}_2)_x \in \mathscr{P}\mathscr{D}(\mathscr{X}).$$

If $g \in (\mathfrak{a}_2)_x$, then for some positive integer r, $gx^r \in \mathfrak{a}_2$. Putting $x = 1 - y$, expanding, and using $y \in \mathfrak{a}_2$, we find that $g \in \mathfrak{a}_2$. Hence $\mathfrak{a}_2 = (\mathfrak{a}_2)_x \in \mathscr{P}\mathscr{D}(\mathscr{X})$. Similarly, $\mathfrak{a}_1 \in \mathscr{P}\mathscr{D}(\mathscr{X})$.

The proof for $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ may be given similarly or by recalling that $\mathscr{D}(\mathscr{X})$ is conservative and

$$\mathscr{D}(\mathscr{X}) \supseteq \mathscr{P}\mathscr{D}(\mathscr{D}(\mathscr{X})) \supseteq \mathscr{D}(\mathscr{D}(\mathscr{X})) = \mathscr{D}(\mathscr{X}).$$

THEOREM IV. *Let $\mathscr{X}$ be a conservative system in the commutative ring $\mathfrak{A}$ with identity. Let $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$, and let $\mathfrak{b}$ denote $\sqrt{\mathfrak{a}}$. If $\mathfrak{b} = \bigcap_{i=1}^k \mathfrak{b}_i$, where the $\mathfrak{b}_i$ are radical, pairwise comaximal ideals, then there exist uniquely determined ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ such that $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{a}_i$, $\mathfrak{b}_i = \sqrt{\mathfrak{a}_i}$, $i = 1, \ldots, k$. The $\mathfrak{a}_i$ are pairwise comaximal. Each $\mathfrak{a}_i \in \mathscr{P}\mathscr{D}(\mathscr{X})$. If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$. If $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$, then each $\mathfrak{a}_i \in \mathscr{C}(\mathscr{X})$.*

Proof. The existence, uniqueness, and comaximality of the $\mathfrak{a}_i$ follow from Lemma IX. Each $\mathfrak{a}_i \in \mathscr{P}\mathscr{D}(\mathscr{X})$ and, if $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$ by Lemma X. If $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$, then $\mathfrak{b} \in \mathscr{P}(\mathscr{X})$, so that Lemma X shows that each $\mathfrak{b}_i \in \mathscr{D}(\mathscr{X})$ and, hence, since the $\mathfrak{b}_i$ are radical, each $\mathfrak{b}_i \in \mathscr{P}(\mathscr{X})$. Then each $\mathfrak{a}_i \in \mathscr{C}(\mathscr{X})$.

Suppose that $\mathscr{P}(\mathscr{X})$ is Noetherian. Then if $\mathfrak{b} \in \mathscr{P}(\mathscr{X})$, $\mathfrak{b} \neq (1)$, there is a uniquely determined set $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$ of ideals of $\mathscr{P}(\mathscr{X})$ distinct from (1) such that:

(α) The $\mathfrak{b}_i$ are pairwise comaximal;

(β) None of the $\mathfrak{b}_i$ is the intersection of two pairwise comaximal ideals distinct from (1);

(γ) $\mathfrak{b} = \bigcap_{i=1}^{k} \mathfrak{b}_i$.

The $\mathfrak{b}_i$ are uniquely determined except for order.

To find the $\mathfrak{b}_i$, apply Theorem II to express $\mathfrak{b}$ as the irredundant intersection of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathscr{P}(\mathscr{X})$. Two of these prime ideals, $\mathfrak{p}_i$ and $\mathfrak{p}_j$, will be called equivalent if there exists a chain $\mathfrak{p}_{i1} = \mathfrak{p}_i, \ldots, \mathfrak{p}_{ih} = \mathfrak{p}_j$ of these ideals such that no two adjacent members of the chain are comaximal. Each $\mathfrak{b}_i$ is the intersection of the members of an equivalence class. Then (α) follows easily from Lemma VII and (γ) is obvious. To prove (β) suppose, say, that $\mathfrak{b}_1 = \mathfrak{b}' \cap \mathfrak{b}''$, where $\mathfrak{b}'$ and $\mathfrak{b}''$ are comaximal ideals distinct from (1). Then $\sqrt{\mathfrak{b}'}$ and $\sqrt{\mathfrak{b}''}$ are comaximal ideals distinct from (1) and $\mathfrak{b}_1 = \sqrt{\mathfrak{b}'} \cap \sqrt{\mathfrak{b}''}$. By Lemma X, $\sqrt{\mathfrak{b}'}$ and $\sqrt{\mathfrak{b}''}$ are in $\mathscr{P}(\mathscr{X})$. Then $\sqrt{\mathfrak{b}'}$ and $\sqrt{\mathfrak{b}''}$ are irredundant intersections of prime ideals of $\mathscr{P}(\mathscr{X})$, say of $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ and $\mathfrak{r}_1, \ldots, \mathfrak{r}_t$, respectively. Since the $\mathfrak{q}_i$ and $\mathfrak{r}_j$ satisfy no inclusion relations and their intersection is $\mathfrak{b}_1$, they constitute one of the equivalence classes into which $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ have been divided. But then there must be a chain in which some $\mathfrak{q}_i$ is adjacent to some $\mathfrak{r}_j$, which is impossible since each $\mathfrak{q}_i$ is comaximal with each $\mathfrak{r}_j$.

To show the uniqueness of k and the $\mathfrak{b}_i$, suppose that a set $\mathfrak{b}_1^*, \ldots, \mathfrak{b}_m^*$ of ideals of $\mathscr{P}(\mathscr{X})$ distinct from (1) with the properties corresponding to (α), (β), and (γ) is given. Representing the $\mathfrak{b}_i^*$ as irredundant intersections of prime ideals, one finds that these prime ideals have intersection $\mathfrak{b}$, and that no prime ideal obtained from a $\mathfrak{b}_i^*$ contains a prime ideal obtained from a $\mathfrak{b}_j^*$, $i \neq j$, since that would contradict comaximality. Hence, these prime ideals are the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of the preceding paragraph. It remains only to show that each $\mathfrak{b}_i^*$ is an intersection of all the ideals of one of the equivalence classes previously defined. Let $\mathfrak{b}_1^*$, say, be the irredundant intersection of the prime ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_h$. Then the $\mathfrak{c}_i$ all belong to the same equivalence class $\mathscr{C}$ of the $\mathfrak{p}_j$. For, if not, then on defining equivalence classes among the $\mathfrak{c}_i$ themselves, there would be at least two such classes. It would follow that $\mathfrak{b}_1^*$ is an intersection, in contradiction to (β). $\mathscr{C}$ can contain no ideals other than the $\mathfrak{c}_i$. Otherwise, there would be a chain leading from one of the $\mathfrak{c}_i$ to a $\mathfrak{p}_j$ which is not one of the $\mathfrak{c}_i$. Let $\mathfrak{c}_s$ be the last of the $\mathfrak{c}_i$ in this chain and $\mathfrak{p}_t$ the next ideal of the chain. Then $\mathfrak{p}_t$ contains some $\mathfrak{b}_n^*$, $n \neq 1$. Since $\mathfrak{c}_s$ and $\mathfrak{p}_t$ are not comaximal, $\mathfrak{b}_1^* \subseteq \mathfrak{c}_s$ and $\mathfrak{b}_n^* \subseteq \mathfrak{p}_t$ are not comaximal, contradicting (α).

Combining these remarks with Theorem IV we obtain a decomposition of a class of ideals into pairwise comaximal ideals.

THEOREM V. *Let X be a conservative system in the commutative ring $\mathfrak{A}$ with identity, and let $\mathscr{P}(\mathscr{X})$ satisfy the ascending chain condition. If*

$$\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X}) \cap \mathscr{C}(\mathscr{X}), \qquad \mathfrak{a} \neq (1),$$

then there is a uniquely determined set $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ of ideals of $\mathscr{P}\mathscr{D}(\mathscr{X}) \cap \mathscr{C}(\mathscr{X})$ distinct from (1) *and such that:*

(a) *The $\mathfrak{a}_i$ are pairwise comaximal;*

(b) *None of the $\mathfrak{a}_i$ is the intersection of two pairwise comaximal ideals distinct from* (1)*;*

(c) $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{a}_i$.

If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$.

Proof. Let $\mathfrak{b} = \sqrt{\mathfrak{a}}$. Then $\mathfrak{b} \in \mathscr{P}(\mathscr{X})$ so that there exist ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$ in $\mathscr{P}(\mathscr{X})$ satisfying (α), (β), and (γ), and distinct from (1). Applying Theorem IV, one obtains ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ of $\mathscr{P}\mathscr{D}(\mathscr{X}) \cap \mathscr{C}(\mathscr{X})$ satisfying (a) and (c). If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$.

Suppose that $\mathfrak{a}_1 = \mathfrak{a}_1' \cap \mathfrak{a}_1''$, where $\mathfrak{a}_1'$ and $\mathfrak{a}_1''$ are comaximal and distinct from (1). Let $\mathfrak{b}_1' = \sqrt{\mathfrak{a}_1'}$, $\mathfrak{b}_1'' = \sqrt{\mathfrak{a}_1''}$. Then $\mathfrak{b}_1 = \mathfrak{b}_1' \cap \mathfrak{b}_1''$, and this contradicts ($\beta$). Hence, (b) is satisfied.

To prove uniqueness, suppose that a set $\mathfrak{a}_1^*, \ldots, \mathfrak{a}_m^*$ of ideals of $\mathscr{P}\mathscr{D}(\mathscr{X}) \cap \mathscr{C}(\mathscr{X})$ and distinct from (1) with properties corresponding to (a), (b), and (c) is given. Let $\mathfrak{b}_i^* = \sqrt{\mathfrak{a}_i^*}$, $i = 1, \ldots, m$. Then $\mathfrak{b} = \bigcap_{i=1}^m \mathfrak{b}_i^*$. The $\mathfrak{b}_i^*$ are in $\mathscr{P}(\mathscr{X})$ and pairwise comaximal. No $\mathfrak{b}_i^*$ is the intersection of two comaximal ideals distinct from (1). If it were, Theorem IV would show that $\mathfrak{a}_i^*$ is also such an intersection, contradicting the hypothesis. By the uniqueness of sets of ideals satisfying (α), (β), and (γ), the $\mathfrak{b}_i^*$ coincide with the $\mathfrak{b}_i$ except for order. By Theorem IV, the $\mathfrak{a}_i^*$ coincide with the $\mathfrak{a}_i$ except for order.

Definition. Let $\mathfrak{A}$ be a commutative ring, $\mathscr{Y}$ a set of ideals of $\mathfrak{A}$ closed under intersection and with $\mathfrak{A} \in \mathscr{Y}$. Ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{A}$ are said to be *comaximal* in $\mathscr{Y}$ if $\mathfrak{A} = (\mathfrak{a}, \mathfrak{b}; \mathscr{Y})$.

In the terminology originally introduced by Ritt for the conservative system $\mathscr{X}$ of difference ideals, comaximal ideals are said to be strongly separated, while ideals comaximal in $\mathscr{P}(\mathscr{X})$ are said to be separated. From the standpoint of studying manifolds of solutions related to a conservative system $\mathscr{X}$, the concept of comaximality in $\mathscr{P}(\mathscr{X})$ is more relevant than that of comaximality (or comaximality in $\mathscr{X}$); cf. **3**, p. 116.

LEMMA XI. *Let $\mathfrak{A}$ be a commutative ring $\mathfrak{a}, \mathfrak{b}_1, \ldots, \mathfrak{b}_k$ ideals of $\mathfrak{A}$, and $\mathscr{X}$ a* *conservative system of ideals of* $\mathfrak{A}$. *If* $\mathfrak{a}$ *and* $\mathfrak{b}_i$ *are comaximal in* $\mathscr{P}(\mathscr{X})$ $i = 1, \ldots, k$, *then* $\mathfrak{a}$ *and* $\mathfrak{b} = \bigcap_{i=1}^{k} \mathfrak{b}_i$ *are comaximal in* $\mathscr{P}(\mathscr{X})$.

Proof. It is sufficient to consider $k = 2$. Using Lemma VI, one finds that $\mathfrak{A} = (\mathfrak{a}, \mathfrak{b}_1; \mathscr{P}(\mathscr{X})) \cap (\mathfrak{a}, \mathfrak{b}_2; \mathscr{P}(\mathscr{X})) \subseteq (\mathfrak{a}, \mathfrak{b}_1 \cap \mathfrak{b}_2; \mathscr{P}(\mathscr{X}))$.

THEOREM VI. *Let* $\mathscr{X}$ *be a conservative system in the commutative ring* $\mathfrak{A}$, *and let* $\mathscr{P}(\mathscr{X})$ *be Noetherian. If* $\mathfrak{b} \in \mathscr{P}(\mathscr{X})$, $\mathfrak{b} \neq \mathfrak{A}$, *then there is a uniquely determined set* $\mathfrak{b}_1, \ldots, \mathfrak{b}_k$ *of ideals of* $\mathscr{P}(\mathscr{X})$ *distinct from* $\mathfrak{A}$ *such that:*

(a) *The* $\mathfrak{b}_i$ *are pairwise comaximal in* $\mathscr{P}(\mathscr{X})$;

(b) *None of the* $\mathfrak{b}_i$ *is the intersection of two ideals of* $\mathscr{C}(\mathscr{X})$ *pairwise comaximal in* $\mathscr{P}(\mathscr{X})$ *and distinct from* $\mathfrak{A}$;

(c) $\mathfrak{b} = \bigcap_{i=1}^{k} \mathfrak{b}_i$.

Proof. The proof is like that of the statements preceding Theorem V with comaximality in $\mathscr{P}(\mathscr{X})$ replacing comaximality in the definition of the equivalence classes. Note that (b) is weaker than the precise analogue of (β) due to the absence of an analogue of Lemma X.

4. Extension-contraction maps. Throughout this section we consider, in addition to the ring $\mathfrak{A}$, a second ring $\mathfrak{B}$ and the set $\mathscr{T}$ of ideals of $\mathfrak{B}$.

Maps $e: \mathscr{S} \to \mathscr{T}$, and $c: \mathscr{T} \to \mathscr{S}$ will be called a pair of *extension-contraction* maps from $\mathfrak{A}$ to $\mathfrak{B}$ if they satisfy:

EC-1: $c(\bigcap_{i \in \mathscr{I}} \mathfrak{t}_i) = \bigcap_{i \in \mathscr{I}} c\mathfrak{t}_i$, $\mathfrak{t}_i \in \mathscr{T}$;

EC-2: $c(\bigcup_{i \in \mathscr{I}} \mathfrak{t}_i) = \bigcup_{i \in \mathscr{I}} c\mathfrak{t}_i$, $\mathfrak{t}_i \in \mathscr{T}$, if the $\mathfrak{t}_i$ are totally ordered by inclusion;

EC-3: $e(\bigcup_{i \in \mathscr{I}} \mathfrak{s}_i) = \bigcup_{i \in \mathscr{I}} e\mathfrak{s}_i$, $\mathfrak{s}_i \in \mathscr{S}$, if the $\mathfrak{s}_i$ are totally ordered by inclusion.

EC-4: $cec = c$; $ece = e$.

EC-5: $c\mathfrak{B} = \mathfrak{A}$.

Remark. It follows from EC-2 and EC-3 that c and e are inclusion-preserving.

If e, c is a pair of extension-contraction maps from $\mathfrak{A}$ to $\mathfrak{B}$, then $\phi: \mathfrak{A} \to \mathfrak{B}$ will be said to *agree* with e and c if it satisfies:

(A): $c\mathfrak{t}{:}a = c(\mathfrak{t}{:}\phi a)$, $\mathfrak{t} \in \mathscr{T}$, $a \in \mathfrak{A}$.

Remark. One can generalize the preceding definitions to the case of maps $e': \mathscr{S}' \to \mathscr{T}'$ and $c': \mathscr{T}' \to \mathscr{S}'$, where $\mathscr{S}' \subseteq \mathscr{S}$ and $\mathscr{T}' \subseteq \mathscr{T}$ are conservative (or, where needed, divisible conservative) systems.

The following lemmas remain valid for sub-systems of $\mathscr{S}'$ and $\mathscr{T}'$.

Remark. Wherever the lemmas below call for the existence of ϕ, one could replace this requirement by the following condition: For each $a \in \mathfrak{A}$, $t \in \mathfrak{T}$, there exists $b \in \mathfrak{B}$ such that $c\mathfrak{t}{:}a = c(\mathfrak{t}{:}b)$. The statement (c) of Lemma XIII must of course be modified accordingly.

Remark. Given c satisfying EC-1, EC-2, and EC-5, there exists at least

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one map $e: \mathscr{S} \to \mathscr{T}$, such that EC-3 and EC-4 hold. One may define e by $e\mathfrak{b} = \bigcap \{\mathfrak{a} \in \mathscr{T}; c\mathfrak{a} \supseteq \mathfrak{b}\}$. EC-3 and EC-4 do not in general determine e uniquely.

If $\mathscr{X} \subseteq \mathscr{S}$, then $e\mathscr{X}$ will denote $\{e\mathfrak{s}: \mathfrak{s} \in \mathscr{X}\}$; $c\mathscr{Y}, \mathscr{Y} \subseteq \mathscr{T}$, is defined similarly.

LEMMA XII. *Let e, c be a pair of extension-contraction maps from $\mathfrak{A}$ to $\mathfrak{B}$.*

(a) *If $\mathscr{Y} \subseteq \mathscr{T}$ is conservative and $ec\mathscr{Y} \subseteq \mathscr{Y}$, then $c\mathscr{Y}$ is conservative.*

(b) *If $\mathscr{Y}$ is as in* (a), *and there exists $\phi: \mathfrak{A} \to \mathfrak{B}$, agreeing with e, c, then $\mathscr{D}(c\mathscr{Y}) \supseteq c(\mathscr{D}(\mathscr{Y}))$; and if $\mathscr{Y}$ is divisible, $c\mathscr{Y}$ is divisible.*

Proof. (a) Let $\mathfrak{s}_1, \mathfrak{s}_2 \in c\mathscr{Y}$; $\mathfrak{s}_1 \subseteq \mathfrak{s}_2$. There exist $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathscr{Y}$ such that $\mathfrak{s}_i = c\mathfrak{t}_i$, $i = 1, 2$. Let $\mathfrak{t}_i' = e\mathfrak{s}_i = ec\mathfrak{t}_i$, $i = 1, 2$. Since e is inclusion-preserving, $\mathfrak{t}_1' \subseteq \mathfrak{t}_2'$. By EC-4, $\mathfrak{s}_i = c\mathfrak{t}_i'$, $i = 1, 2$; and the $\mathfrak{t}_i' \in \mathscr{Y}$ by hypothesis. Hence, an ascending chain in $c\mathscr{Y}$ is found by applying c to an ascending chain in $\mathscr{Y}$. It now follows by EC-2 that $c\mathscr{Y}$ satisfies C-2; and $c\mathscr{Y}$ satisfies C-1 by EC-1, C-3 by EC-5.

(b) The existence of ϕ yields at once $\mathscr{D}(c\mathscr{Y}) \supseteq c(\mathscr{D}(\mathscr{Y}))$. If $\mathscr{Y} = \mathscr{D}(\mathscr{Y})$, this yields $c\mathscr{Y} = \mathscr{D}(c\mathscr{Y})$.

LEMMA XIII. *Let e, c be a pair of extension-contraction maps from $\mathfrak{A}$ to $\mathfrak{B}$, and let ec be the identity map.*

(a) *If $\mathscr{X} \subseteq \mathscr{S}$ is conservative and $ce\mathscr{X} \subseteq \mathscr{X}$, then $e\mathscr{X}$ is conservative.*

(b) *If $\mathscr{X}$ is as in* (a), *and there exists $\phi: \mathfrak{A} \to \mathfrak{B}$, agreeing with e, c, then $\mathscr{D}(e\mathscr{X}) \subseteq e(\mathscr{D}(\mathscr{X}))$.*

(c) *Let $\mathscr{X}$, ϕ be as above, and suppose that $\mathscr{X}$ is divisible. If for each $\mathfrak{t} \in \mathscr{T}$, $b \in \mathfrak{B}$ there exists $a \in \mathfrak{A}$ such that $\mathfrak{t}:b = \mathfrak{t}:\phi a$, then $e\mathscr{X}$ is divisible.*

Proof. (a) The proof that $e\mathscr{X}$ satisfies C-2 is "dual" to the proof of C-2 in Lemma XII. Let $\mathfrak{t}_i$, $i \in \mathscr{I}$, be ideals of $e\mathscr{X}$, and let $\mathfrak{t} = \bigcap_{i \in \mathscr{I}} \mathfrak{t}_i$. Defining $\mathfrak{s}_i = c\mathfrak{t}_i$, $i \in \mathscr{I}$, we find that $\mathfrak{s}_i \in \mathscr{X}$, $\mathfrak{t}_1 = e\mathfrak{s}_i$, $i \in \mathscr{I}$. Let $\mathfrak{s} = \bigcap_{i \in \mathscr{I}} \mathfrak{s}_i$. Then $\mathfrak{s} \in \mathscr{X}$, and from the hypothesis and EC-1 we find that

$$\mathfrak{t} = ec\mathfrak{t} = e\Big(\bigcap_{i \in \mathscr{I}} c\mathfrak{t}_i\Big) = e\mathfrak{s}.$$

Hence, $e\mathscr{X}$ satisfies C-1, and C-3 results from $ec\mathfrak{B} = \mathfrak{B}$.

(b) Let $\mathfrak{t} \in \mathscr{D}(e\mathscr{X})$. Define $\mathfrak{s} = c\mathfrak{t}$, so that $\mathfrak{s} \in \mathscr{X}$, $\mathfrak{t} = e\mathfrak{s}$. We must prove that $\mathfrak{s} \in \mathscr{D}(\mathscr{X})$. Let $a \in \mathfrak{A}$. Then $\mathfrak{s}:a = c\mathfrak{t}:a = c(\mathfrak{t}:\phi a)$. Since $\mathfrak{t}:\phi a \in e\mathscr{X}$, there exists $\mathfrak{v} \in \mathscr{X}$ with $\mathfrak{t}:\phi a = e\mathfrak{v}$. Hence $\mathfrak{s}:a = ce\mathfrak{v} \in \mathscr{X}$.

(c) Let $\mathfrak{t} \in e\mathscr{X}$, $b \in \mathfrak{B}$. Define $\mathfrak{s} = c\mathfrak{t}$, so that $\mathfrak{s} \in \mathscr{X} = \mathscr{D}(\mathscr{X})$. Choose $a \in \mathfrak{A}$ such that $\mathfrak{t}:b = \mathfrak{t}:\phi a$. Then $c(\mathfrak{t}:b) = c(\mathfrak{t}:\phi a) = \mathfrak{s}:a \in \mathscr{X}$. Then $\mathfrak{t}:b = ec(\mathfrak{t}:b) \in e\mathscr{X}$.

As an example, let $\phi: \mathfrak{A} \to \mathfrak{B}$ be a homomorphism; and define e, c in the expected way, that is, $e\mathfrak{s}$, $\mathfrak{s} \in \mathscr{S}$, is the ideal generated by $\phi\mathfrak{s}$ in $\mathfrak{B}$ and $c\mathfrak{t}$, $\mathfrak{t} \in \mathscr{T}$, is the complete pre-image of $\mathfrak{t}$. Then e, c is a pair of extension-contraction maps from $\mathfrak{A}$ to $\mathfrak{B}$, and ϕ agrees with e and c. (These are the maps studied in (**17**, p. 218).) It follows from Lemma XII that if $\mathscr{Y} \subseteq \mathscr{T}$ is such that $ec\mathscr{Y} \subseteq \mathscr{Y}$,

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then $c\mathscr{Y}$ is conservative if $\mathscr{Y}$ is conservative and divisible if $\mathscr{Y}$ is divisible. In two important cases, ec is the identity so that $ec\mathscr{Y} \subseteq \mathscr{Y}$ must be satisfied: namely, if ϕ is an epimorphism, or if $\mathfrak{B}$ is the quotient ring of $\mathfrak{A}$ with respect to a multiplicatively closed system and ϕ is the canonical map. Suppose, now, that ϕ is an epimorphism and $\mathscr{X} \subseteq \mathscr{S}$ an additive, conservative system with $\ker \phi \in \mathscr{X}$. Then $ce\mathscr{X} \subseteq \mathscr{X}$, and for $\mathfrak{t} \in \mathscr{T}$, $b \in \mathfrak{B}$, there exists $a \in \mathfrak{A}$ such that $\phi a = b$, and hence $\mathfrak{t}:b = \mathfrak{t}:\phi a$. It follows from Lemma XIII that $e\mathscr{X}$ is an (additive) conservative system satisfying $\mathscr{D}(e\mathscr{X}) \subseteq e(\mathscr{D}(\mathscr{X}))$, and that $e\mathscr{X}$ is divisible if $\mathscr{X}$ is divisible.

5. Links. Several important conservative systems are defined by means of closure under certain operators. The following definitions generalize this notion.

Definition. Let A be a commutative ring. A *link* Λ in $\mathfrak{A}$ consists of an index set $\mathscr{I}$ and mappings λ_i, $i \in \mathscr{I}$, μ of $\mathfrak{A}$ into $\mathfrak{A}$. The λ_i are called the *antecedents*, μ the *consequent* of Λ. We write $\Lambda = \{\lambda_i, i \in \mathscr{I}; \mu\}$.

Definition. An ideal $\mathfrak{a}$ in the commutative ring $\mathfrak{A}$ is said to *admit* the link $\{\lambda_i, i \in \mathscr{I}; \mu\}$ if for $x \in \mathfrak{A}$ the relations $\lambda_i x \in \mathfrak{a}$, $i \in \mathscr{I}$, imply $\mu x \in \mathfrak{a}$.

The set of ideals of $\mathfrak{A}$ admitting links Λ_j, $j \in \mathscr{J}$, satisfy C-1 and C-3. To assure C-2, a further restriction is necessary.

Definition. A link $\{\lambda_i, i \in \mathscr{I}; \mu\}$ in the ring $\mathfrak{A}$ is called *finitary* if for each $x \in \mathfrak{A}$ the set of disctinct $\lambda_i x$, $i \in \mathscr{I}$, is finite. If $\mathscr{I}$ is finite, then Λ is called *finite.*

Remark. A (finitary) link Λ may also be described by a collection of (finite) subsets $\mathfrak{S}_i \subseteq \mathfrak{A}$ and elements $x_i \in \mathfrak{A}$. Then an ideal $\mathfrak{a}$ admits Λ if and only if for each i, $\mathfrak{S}_i \subseteq \mathfrak{a}$ implies $x_i \in \mathfrak{a}$.

The proof of the following result is straightforward.

LEMMA XIV. *Let $\mathfrak{A}$ be a commutative ring, $\mathscr{J}$ an index set,*

$$\Lambda_j = \{\lambda_{ij}, i \in \mathscr{I}_j; \mu_j\}, \qquad j \in \mathscr{J},$$

a set of finitary links in $\mathfrak{A}$, and $\mathscr{X}$ the set of ideals of $\mathfrak{A}$ admitting the Λ_j. Then $\mathscr{X}$ is conservative. If $\mathfrak{a} \in \mathscr{X}$, then $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ if and only if for each $j \in \mathscr{J}$, $x \in \mathfrak{A}, y \in \mathfrak{A}$, the relations $x\lambda_{ij}y \in \mathfrak{a}$, $i \in \mathscr{I}_j$, imply $x\mu_j y \in \mathfrak{a}$; and $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$ if and only if for each $j \in \mathscr{J}$, $x \in \mathfrak{A}$, $y \in \mathfrak{A}$, the relations $x\lambda_{ij}y \in \mathfrak{a}$, $i \in \mathscr{I}_j$, imply that there is a positive integer r such that $x^r\mu_j y \in \mathfrak{a}$.

Remark. The conditions given in the lemma for $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$ may be regarded as the requirement that $\mathfrak{a}$ admit a certain set of links; but the conditions for $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$ are not of this form.

Let $\mathscr{X}$ be the system of ideals admitting the links Λ_j. Let $\mathfrak{M} \subseteq \mathfrak{A}$. It is possible to describe in a more or less constructive fashion how $(\mathfrak{M}; \mathscr{X})$ is generated from $\mathfrak{M}$. The notation will be as in Lemma XIV. For $\mathfrak{N} \subseteq \mathfrak{A}$, define $\mathfrak{N}^*$ to consist precisely of those elements whose presence in any ideal containing $\mathfrak{N}$ is required if the ideal admits the Λ_j. That is, if for some $j \in \mathscr{J}$, $x \in \mathfrak{A}$, the $\lambda_{ij}x \in \mathfrak{N}$, $i \in \mathscr{I}_j$, then $\mu_j x \in \mathfrak{N}^*$. Now let $\mathfrak{M}_1 = (\mathfrak{M})$, $\mathfrak{M}_{i+1} = (\mathfrak{M}_i, \mathfrak{M}_i^*)$, $i = 1, 2, \ldots$. Then it is easily verified that

$$(\mathfrak{M}; \mathscr{X}) = \bigcup_{i=1}^{\infty} \mathfrak{M}_i.$$

Since $\mathscr{D}(\mathscr{X})$ is also determined by a system of links, as shown in the remark above, there is a description similar to the foregoing for $(\mathfrak{M}; \mathscr{D}(\mathscr{X}))$. Since $\mathscr{P}(\mathscr{X})$ consists of the radical ideals of $\mathscr{D}(\mathscr{X})$ one may also find such a description for $(\mathfrak{M}; \mathscr{P}(\mathscr{X}))$ using the links for $\mathscr{D}(\mathscr{X})$ but redefining $\mathfrak{M}_1 = \sqrt{(\mathfrak{M})}$, $\mathfrak{M}_{i+1} = \sqrt{(\mathfrak{M}_i, \mathfrak{M}_i^*)}$, $i = 1, 2, \ldots$. It will be seen in the next section that the links determining $\mathscr{D}(\mathscr{X})$ or $\mathscr{P}(\mathscr{X})$ may sometimes be replaced by simpler ones than those resulting from Lemma XIV. With these replacements one obtains, for example, the standard descriptions of the generation of perfect differential and of perfect difference ideals from generators.

LEMMA XV (Kolchin). *Let $\mathscr{X}$ be a conservative system of ideals in the ring $\mathfrak{A}$, and let $\mathfrak{M} \subseteq \mathfrak{A}$. If $x \in (\mathfrak{M}; \mathscr{X})$, then there exists a finite subset $\mathfrak{N} \subseteq \mathfrak{M}$ such that $x \in (\mathfrak{N}; \mathscr{X})$.*

Proof. The conclusion is immediate if $\mathfrak{M}$ is finite. We proceed by induction on the cardinality of $\mathfrak{M}$ and assume henceforth that $\mathfrak{M}$ is infinite. Then $\mathfrak{M} = \bigcup \mathfrak{M}_i$, where the $\mathfrak{M}_i$ are totally ordered by inclusion and each $\mathfrak{M}_i$ is of cardinality less than that of $\mathfrak{M}$. Now $\mathfrak{M} \subseteq \bigcup (\mathfrak{M}_i; \mathscr{X})$, and

$$\bigcup (\mathfrak{M}_i; \mathscr{X}) \in \mathscr{X}.$$

Hence, $(\mathfrak{M}; \mathscr{X}) \subseteq \bigcup (\mathfrak{M}_i; \mathscr{X})$. Then for some i, $x \in (\mathfrak{M}_i; \mathscr{X})$, and by the induction hypothesis, $x \in (\mathfrak{N}; \mathscr{X})$ for some finite $\mathfrak{N} \subseteq \mathfrak{M}_i$.

THEOREM VII. *Let $\mathscr{X}$ be a conservative system of ideals in the commutative ring $\mathfrak{A}$. There exists a set Σ of finite links in $\mathfrak{A}$ such that the set of ideals admitting the links of Σ is $\mathscr{X}$.*

Proof. Let $\{x_1, \ldots, x_n; x\} \subseteq \mathfrak{A}$. Using maps whose range is a single element one may express, by means of a finite link, the condition $x_1, \ldots, x_n \in \mathfrak{a}$ implies $x \in \mathfrak{a}$. Let Σ be the set of such links for all subsets $\{x_1, \ldots, x_n; x\}$, $n = 1, 2, \ldots$, of $\mathfrak{A}$ such that $x \in (\{x_1, \ldots, x_n\}; \mathscr{X})$. Clearly, every ideal of $\mathscr{X}$ admits the links of Σ. Conversely, let $\mathfrak{a}$ admit these links. If $x \in (\mathfrak{a}; \mathscr{X})$, then by Lemma XV there exists a finite set $\mathfrak{N} \subseteq \mathfrak{a}$ such that $x \in (\mathfrak{N}; \mathscr{X})$. Hence, there is a link in Σ which requires $x \in \mathfrak{a}$. Then $\mathfrak{a} = (\mathfrak{a}; \mathscr{X}) \in \mathscr{X}$.

A link will be called *simple* if its only antecedent is the identity map.

Remark. A link $(\lambda; \mu)$ with but one antecedent is equivalent to a set of simple links. For one can construct maps ν_i, $i \in \mathscr{I}$, of $\mathfrak{A}$ into $\mathfrak{A}$ such that the $\nu_i x$ yield the distinct values of $\mu\lambda^{-1}x$ for any $x \in \mathfrak{A}$ for which $\mu\lambda^{-1}x$ exists, and are 0 for all other $x \in \mathfrak{A}$. The ν_i furnish the consequents of a set of simple links equivalent to $(\lambda; \mu)$.

It is not possible to obtain all conservative systems by means of sets of simple links.

Counterexample. Let $\mathfrak{A}$ be a unique factorization domain but not a principal ideal domain. Let $\mathscr{X}$ be the set of principal ideals of $\mathfrak{A}$. Then $\mathscr{X}$ is conservative (indeed, divisible). Let μ be the consequent of a simple link Λ such that $\mathscr{X}$ admits Λ. For $x \in \mathfrak{A}$, $(x) \in \mathscr{X}$ and, hence, $\mu x \in (x)$. It follows that every ideal of $\mathfrak{A}$ admits Λ so that $\mathscr{X}$ cannot be the set of ideals admitting a set of simple links. If $\mathfrak{A}$ contains a radical ideal which is not principal, $\mathscr{P}(\mathscr{X})$ is not determined by any system of simple links. $\mathscr{X}$ is the set of ideals $\mathfrak{a}$ admitting the links which express the requirement that for each $a, b \in \mathfrak{A}$, $a, b \in \mathfrak{a}$ implies $d(a, b) \in \mathfrak{a}$, where $d(a, b)$ denotes the greatest common divisor of a and b.

6. Examples. Throughout this section and hereafter, η denotes the identity map of $\mathfrak{A}$ onto $\mathfrak{A}$.

(1) *Homogeneous ideals.* Let $\mathfrak{A}$ be a graded commutative ring. If $x \in \mathfrak{A}$, we write $x = \sum_{i=-\infty}^{\infty} x^{(i)}$, where $x^{(i)}$ is the homogeneous component of x of degree i, and all but a finite number of the $x^{(i)}$ are 0. If $x \neq 0$, we also write $x = \sum_{i}^{r} {}_1 x_i$, where the x_i are the non-zero homogeneous components of x arranged in order of increasing degree. Let h_i, $i = 0, \pm 1, \pm 2, \ldots$, be the map defined by $h_i x = x^{(i)}$. Let m be the map defined by $mx = x_1$ for all $x \neq 0$, $m(0) = 0$.

The set $\mathscr{H}$ of homogeneous ideals of $\mathfrak{A}$ is a conservative system. $\mathscr{H}$ consists of all ideals admitting the link $(\eta; m)$. Alternatively, $\mathscr{H}$ consists of all ideals admitting the links $(\eta; h_i)$, $i = 0, \pm 1, \pm 2, \ldots$.

It will be shown that $\mathscr{H} = \mathscr{P}\mathscr{D}(\mathscr{H})$. Let $\mathfrak{a} \in \mathscr{H}$, $xy \in \mathfrak{a}$, $x = \sum_{i=1}^{r} x_i$, $y = \sum_{i=1}^{s} y_i$. Then $x_1 y_1 = m(xy) \in \mathfrak{a}$. Suppose that it has been shown that $x_1 y_1^{j}, \ldots, x_j y_1^{j} \in \mathfrak{a}$, $1 \leqq j < r$. Let z be the homogeneous component of xy which is of the same degree as $x_{j+1} y_1$. Then $z - x_{j+1} y_1$ is either 0 or a sum of terms $x_a y_b$ with $a \leqq j$. Hence $(z - x_{j+1} y_1) y_1^{j} \in \mathfrak{a}$. Since $z \in \mathfrak{a}$, $x_{j+1} y_1^{j+1} \in \mathfrak{a}$. Hence, by induction, $x_i y_1^{r} \in \mathfrak{a}$, $i = 1, \ldots, r$. Then $xy_1^{r} \in \mathfrak{a}$.

Let $z \in \mathfrak{A}$. It must be shown that $\mathfrak{a}_z \in \mathscr{H}$. That is, if $v \in \mathfrak{a}_z$, say $v = \sum_{i=1}^{t} v_i$, and $vz^h \in \mathfrak{a}$, then it must be shown that each $v_i \in \mathfrak{a}_z$. This amounts to showing that for some positive integer k each $v_i z^k \in \mathfrak{a}$. If $z = 0$, this is trivial. If z has but one homogeneous component, it is true with $k = h$, since the $v_i z^h$ are then the homogeneous components of vz^h. We proceed by induction on the number of homogeneous components of z. Since $z_1^{h} = (z^h)_1$, it follows from the result of the preceding paragraph that there is a positive integer t such that $vz_1^{t} \in \mathfrak{a}$. Then also each $v_i z_1^{t} \in \mathfrak{a}$. Let $w = z - z_1$. Then $vw^{t+h} \in \mathfrak{a}$, since when w^{t+h} is expanded in powers of z and z_1 each term has either z_1^{t} or z^h as a factor. Then $v \in \mathfrak{a}_w$. Since w has fewer homogeneous components than z, it follows from the

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induction hypothesis that there is a positive integer n such that each $v_i w^n \in \mathfrak{a}$. Then each $v_i z^{n+t} \in \mathfrak{a}$, completing the proof.

It follows from the preceding result that $\mathscr{P}(\mathscr{H})$ consists precisely of the radical homogeneous ideals. Since the radical of a homogeneous ideal is homogeneous, we also have $\mathscr{H} = \mathscr{C}(\mathscr{H})$. It is not true in general that $\mathscr{H} = \mathscr{D}(\mathscr{H})$. An inductive argument from the criterion of Lemma XIV for membership in $\mathscr{D}(\mathscr{H})$ shows that an ideal $\mathfrak{a}$ of $\mathfrak{A}$ is in $\mathscr{D}(\mathscr{H})$ if and only if, given $vw \in \mathfrak{a}$, $v = \sum_{i=1}^{r} v_i$, $w = \sum_{i=1}^{s} w_i$, then each $v_i w_j \in \mathfrak{a}$. Let $\mathfrak{A}$ be the polynomial ring $\mathfrak{K}[x, y, u, v]$, $\mathfrak{K}$ a field. Use the usual grading. Let

$$\mathfrak{a} = (x^2u, yv, x^2v + u^2y).$$

Then $\mathfrak{a} \in \mathscr{H}$. Since $(x^2 + y)(u^2 + v) \in \mathfrak{a}$, $u^2y \notin \mathfrak{a}$, the preceding criterion shows that $\mathfrak{a} \notin \mathscr{D}(\mathscr{H})$.

Theorems I and II yield the standard results on the representation of a radical homogeneous ideal as an intersection of homogeneous prime ideals. As has already been indicated, Theorem III is inadequate to give the standard representation of a homogeneous ideal in terms of homogeneous primary ideals. This is provided by Theorem IX below.

(2) *Extensions*. Let $\mathfrak{A}$ possess an identity element 1, and let $\mathfrak{B}$ be a unitary over-ring. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{A}$, and consider the set $\mathscr{Y}$ consisting of those ideals of $\mathfrak{B}$ whose intersection with $\mathfrak{A}$ is $\mathfrak{p}$, and of $\mathfrak{B}$ itself. Then $\mathscr{Y}$ is a conservative system. Let $\mathfrak{q}$ be a radical ideal of $\mathscr{Y}$. Is $\mathfrak{q}$ the intersection of prime ideals of $\mathfrak{B}$ whose intersection with $\mathfrak{A}$ is $\mathfrak{p}$? That is, is $\mathfrak{q} \in \mathscr{P}(\mathscr{Y})$? This will be so if $\mathfrak{q}{:}x \in \mathscr{Y}$ for all $x \in \mathfrak{B}$. We find at once the well-known criterion: $st \in \mathfrak{q}$, $s \in \mathfrak{A}$, $t \in \mathfrak{B}$ implies either $s \in \mathfrak{p}$ or $t \in \mathfrak{q}$.

(3) *Multiplicatively closed sets*. Let $\dot{\mathfrak{J}}$ be a multiplicatively closed subset of $\mathfrak{A}$. An ideal $\mathfrak{a} \in \mathscr{S}$ is called a $\dot{\mathfrak{J}}$-ideal if $xy \in \mathfrak{a}$, $x \in \dot{\mathfrak{J}}$, implies $y \in \mathfrak{a}$. Let $\mathscr{X}$ denote the set of $\dot{\mathfrak{J}}$-ideals. The implications defining $\mathscr{X}$ may be regarded as a set of links, and we see at once that $\mathscr{X}$ is conservative and $\mathscr{X} = \mathscr{D}(\mathscr{X})$. Hence, every radical $\dot{\mathfrak{J}}$-ideal is an intersection of prime $\dot{\mathfrak{J}}$-ideals. The definition and this result were stated by Robinson (**13**, p. 432).

(4) *M-rings*. A set m_i, $i \in \mathscr{I}$, of maps of $\mathfrak{A}$ into $\mathfrak{A}$ is said to have a multiplication theorem if there exist elements $c_{ijk} \in \mathfrak{A}$, $i, j, k \in \mathscr{I}$, such that for $x, y \in \mathfrak{A}$, $i \in \mathscr{I}$,

$$m_i(xy) = \sum_{j,k\in\mathscr{I}} c_{ijk}(m_j x)(m_k y),$$

where only a finite number of the c_{ijk} are different from 0 for given i.

An *M-ring* is a commutative ring together with a set m_i, $i \in \mathscr{I}$, of linear maps with a multiplication theorem. An ideal of the ring admitting the links $(\eta; m_i)$, $i \in \mathscr{I}$, is called an *M-ideal*. The theory of M-rings has been developed by Kreimer (**7**).

If $\mathfrak{A}$ is a graded ring in which the homogeneous components of negative degree are 0, then $\mathfrak{A}$ is an M-ring with the set of maps $M = h_i, i = 0, 1, \ldots,$ as defined in (1) above, and the M-ideals of $\mathfrak{A}$ are the homogeneous ideals. Other important examples of M-rings will now be considered.

(5) *Derivations.* Let $D = \{d_i, i \in \mathscr{I}\}$ be a set of derivations of $\mathfrak{A}$ into $\mathfrak{A}$. Then $\mathfrak{A}$ and D constitute a *differential ring*, and the ideals of $\mathfrak{A}$ admitting the links $(\eta; d_i), i \in \mathscr{I}$, are called *differential ideals.* Let $\mathscr{X}$ denote the set of differential ideals. Then $\mathscr{X} = \mathscr{P}\mathscr{D}(\mathscr{X})$. For if $\mathfrak{a} \in \mathscr{X}$, $xy \in \mathfrak{a}$, and d is one of the d_i, then $x^2dy = xd(xy) - xydx \in \mathfrak{a}$. It follows from a remark following the definition of partially divisible that $\mathscr{P}(\mathscr{X})$ consists of all radical differential ideals. Then every radical differential ideal is an intersection of prime differential ideals. If $\mathfrak{A}$ contains the rationals, then $\mathfrak{A}$ is a *Ritt algebra* (**5**, p. 12). It is easily shown that if $\mathfrak{A}$ is a Ritt algebra, then the radical of a differential ideal is a differential ideal, so that $\mathscr{X} = \mathscr{C}(\mathscr{X})$. On the other hand, let $\mathfrak{A}$ be the polynomial ring $\mathfrak{K}[x]$, where $\mathfrak{K}$ is a field of positive characteristic p, and let D consist of the derivation d of $\mathfrak{K}[x]$ over $\mathfrak{K}$ such that $dx = 1$. Then $\mathfrak{a} = (x^p) \in \mathscr{X}$, but $\sqrt{\mathfrak{a}} = (x) \notin \mathscr{X}$. Hence, in this case,

$$\mathscr{C}(\mathscr{X}) \subset \mathscr{P}\mathscr{D}(\mathscr{X}) = \mathscr{X}.$$

(6) *Higher derivations.* Let $D_0 = \eta, D_1, \ldots, D_m$ be additive maps of $\mathfrak{A}$ into $\mathfrak{A}$ such that $D_i(xy) = \sum_{j=0}^{i}(D_jx)(D_{i-j}y); x, y \in \mathfrak{A}, i = 0, 1, \ldots, m$. Then the D_i constitute a higher derivation of rank m of $\mathfrak{A}$ into $\mathfrak{A}$. A higher derivation of infinite rank is defined similarly. An ideal of $\mathfrak{A}$ is said to admit the higher derivation if it admits all the links (η, D_i).

Let a set of higher derivations (not necessarily all of the same rank) be given in $\mathfrak{A}$. Let $\mathscr{X}$ denote the set of ideals of $\mathfrak{A}$ admitting all these higher derivations. Then $\mathscr{X}$ is conservative, and an inductive proof with steps resembling the proof in (5) shows that $\mathscr{X} = \mathscr{P}\mathscr{D}(\mathscr{X})$. Hence, $\mathscr{P}(\mathscr{X})$ consists of all the radical ideals of $\mathscr{X}$, and every radical ideal of $\mathscr{X}$ is an intersection of prime ideals of $\mathscr{X}$. It has been shown by Hamara (**4**) that if the higher derivations are of infinite rank, then $\mathscr{X} = \mathscr{C}(\mathscr{X})$.

(7) *Difference rings.* Let $T = \{t_i, i \in \mathscr{I}\}$ be a set of homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}$. Let $\mathscr{J} \subseteq \mathscr{I}$. Let $\mathscr{X}$ consist of those ideals admitting the links $(\eta; t_i), i \in \mathscr{I}$, and let $\mathscr{Y}$ consist of those ideals of $\mathscr{X}$ which also admit the links $(t_j; \eta), j \in \mathscr{J}$. We call the pair $\mathfrak{A}$, T a *difference ring.* The ideals of $\mathscr{X}$ are called *difference ideals*, and the ideals of $\mathscr{Y}$ are said to be *reflexive* in the $t_j, j \in \mathscr{J}$. If $\mathscr{J} = \mathscr{I}$, then the ideals of $\mathscr{Y}$ are called *reflexive* difference ideals. (The ideals of $\mathscr{X}$, but not necessarily those of $\mathscr{Y}$, constitute a set of M-ideals.)

Remark. From the present, more general approach, a slight conflict with earlier terminology is natural. Ritt (**10**) (see also (**3**)) used the term perfect difference ideals to designate $\mathscr{P}(\mathscr{Y})$ with $\mathscr{I} = \mathscr{J}$, and the term complete difference ideals to designate $\mathscr{C}(\mathscr{X})$.

Let $\mathscr{I} = \mathscr{J}$. The following criterion is due essentially to Ritt and Raudenbush (**12**): *A radical ideal* $\mathfrak{a} \in \mathscr{Y}$ *is in* $\mathscr{P}(\mathscr{Y})$ *if and only if for each* t_i, $i \in \mathscr{I}$, $xt_ix \in \mathfrak{a}$ *implies* $x \in \mathfrak{a}$.

Proof. Let $\mathfrak{a} \in \mathscr{P}(\mathscr{Y})$, $t \in T$. Then $xtx \in \mathfrak{a}$ implies $(tx)^2 \in \mathfrak{a}$ by Lemma XIV. Since $\mathfrak{a}$ is radical and reflexive, $x \in \mathfrak{a}$. Now suppose that $\mathfrak{a}$ satisfies the stated condition. We must show that $xy \in \mathfrak{a}$ implies $xty \in \mathfrak{a}$, and that $xty \in \mathfrak{a}$ implies $xy \in \mathfrak{a}$. Let $xy \in \mathfrak{a}$. Let $u = xty$. Then $utu = ht(xy)$, where $h = xt^2y$. Hence $utu \in \mathfrak{a}$, $u \in \mathfrak{a}$. On the other hand, let $xty \in \mathfrak{a}$. By the preceding case, $txty = t(xy) \in \mathfrak{a}$. Since $\mathfrak{a}$ is reflexive, $xy \in \mathfrak{a}$. From the preceding condition we obtain, at once, the following criterion: *A difference ideal* $\mathfrak{a}$ *is in* $\mathscr{P}(\mathscr{Y})$ (*with* $\mathscr{I} = \mathscr{J}$) *if and only if for each* $t \in T$, $x \in \mathfrak{A}$, *the presence in* $\mathfrak{a}$ *of a product* $(x)^{k_0}(tx)^{k_1}\ldots(t^mx)^{k_m}$ *for some non-negative integers* $m, k_0, \ldots, k_m$, *implies* $x \in \mathfrak{a}$. This criterion reduces to the usual one for a difference ideal to be perfect in the special cases heretofore studied, and Theorem II yields the previously known decomposition theorem.

Ritt has shown that in certain difference rings, in particular in polynomial difference rings over a field, $\mathscr{C}(\mathscr{X})$ does not satisfy C-1; see (**10**, p. 689; **3**, p. 106). For such rings, one has at once that $\mathscr{X} \supset \mathscr{C}(\mathscr{X}) \supset \mathscr{D}(\mathscr{X})$. Since a polynomial difference ring contains an ideal $\mathfrak{a}$ such that $\mathfrak{a} \in \mathscr{X}$, $\mathfrak{a} \neq (1)$, and $(\mathfrak{a}; \mathscr{P}(\mathscr{X})) = (1)$, not all ideals of $\mathscr{X}$ are even intersections of ideals of $\mathscr{C}(\mathscr{X})$ in this case. It is easy to see that one may substitute $\mathscr{Y}$ for $\mathscr{X}$ in all the above. Since the radical of an ideal of $\mathscr{Y}$ is in $\mathscr{Y}$, $\mathscr{P}\mathscr{D}(\mathscr{Y}) \subseteq \mathscr{C}(\mathscr{Y})$. The following example shows that this inclusion may be proper. (Contrast with the differential case.)

Let $\mathfrak{A}$ be the ordinary polynomial difference ring $\mathfrak{K}\{y, z\}$, $\mathfrak{K}$ a field. T consists of a single isomorphism t, and we denote t^mx by x_m. Let $\mathscr{Y}$ be the set of reflexive difference ideals of $\mathfrak{K}$. Let $\mathfrak{a}$ be the ideal of $\mathfrak{A}$ generated by the monomials $(y_iz_j)^2$ and y_iz_i, $i, j = 0, 1, \ldots$. Then $\mathfrak{a} \in \mathscr{Y}$. Let

$$\mathfrak{b} = (y, y_1, \ldots), \quad \mathfrak{c} = (z, z_1, \ldots).$$

Then $\sqrt{\mathfrak{a}} = \mathfrak{b} \cap \mathfrak{c}$, and $\mathfrak{b}$ and $\mathfrak{c}$ are prime ideals of $\mathscr{Y}$. Hence $\mathfrak{a} \in \mathscr{C}(\mathscr{Y})$. Since $y^rz_1 \notin \mathfrak{a}$, $r = 0, 1, \ldots$, $\mathfrak{a} \notin \mathscr{P}\mathscr{D}(\mathscr{Y})$.

(8) *D-rings.* Let $T = \{t_i, i \in \mathscr{I}\}$ be a set of homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}$, $\mathscr{D} = \{d_i, i \in \mathscr{I}'\}$, a set of derivations of $\mathfrak{A}$ into $\mathfrak{A}$, and let $\mathscr{J} \subseteq \mathscr{I}$. This structure will be called a *D-ring* if:

(a) $\mathscr{I}$ and $\mathscr{I}'$ are finite,

(b) the t_i are isomorphisms,

(c) the d_i and t_j commute among themselves and with each other.

The set $\mathscr{X}$ of D-ideals of $\mathfrak{A}$ consists of those ideals which are differential ideals for the derivations d_i and difference ideals for the isomorphisms t_i reflexive in the t_j, $j \in \mathscr{J}$. D-rings form a natural setting for the abstract study of difference-differential equations, and with some further specializations have formed the subject matter of most work in difference and differential algebra. Conditions (a) and (c) are used to obtain theorems showing $\mathscr{P}(\mathscr{X})$ to be Noetherian in important cases. It is probable that (c) could be replaced by weaker conditions on the commutators, but this has not been explored. Such conditions would certainly be appropriate for some analytic problems. It has been usual to assume (b), but this is unimportant, at least if $\mathscr{I} = \mathscr{J}$.

It will now be shown that analytic situations make it natural to study subsets of $\mathscr{X}$ admitting additional links. One of several possible examples will be given. None has been studied in any depth.

Let $\mathfrak{F}$ be the field of functions meromorphic in a strip of the complex plane parallel to the real axis. Let d denote the derivative in the analytic sense and t the isomorphism defined by $tf(z) = f(z + 1)$. Then $df(z) = 0$ implies $tf(z) = f(z)$. We extend $\mathfrak{F}$ to a difference-differential polynomial ring $\mathfrak{A} = \mathfrak{F}\{y\}$ in the usual way (cf. **3**, p. 64 or **11**, p. 2) and continue to use d and t to denote the derivation and the isomorphism of $\mathfrak{A}$, respectively. We wish to consider manifolds of the D-ideals of $\mathfrak{A}$. It is natural to restrict attention to manifolds whose solutions lie in rings for which an implication like the preceding one is valid. Certainly, meromorphic solutions will lie in such rings. Let $\mathfrak{p}$ be a prime D-ideal of $\mathfrak{A}$. Then the desired implication will hold in the ring $\mathfrak{A}/\mathfrak{p}$ if and only if $\mathfrak{p}$ admits the link $(d; t - \eta)$. It will hold in the quotient field of $\mathfrak{A}/\mathfrak{p}$ if and only if for each $x, y \in \mathfrak{A}$, $xdy - ydx \in \mathfrak{p}$ implies $xty - ytx \in \mathfrak{p}$. These requirements can easily be expressed by a set of links.

(9) *Restricted manifolds.* In studying manifolds of polynomials over a field $\mathfrak{K}$ one may wish to restrict attention to solutions lying in a given subset of a certain extension of $\mathfrak{K}$. For example, if $\mathfrak{K}$ is the rational field, one may wish to consider only integral or only real solutions. Since each solution is determined to within equivalence by a homomorphism of the polynomial ring onto an integral domain, and hence by the prime ideal which is the kernel of the homomorphism, the restriction is described by giving certain subsets of the set of prime ideals of the polynomial ring. One may also wish to impose the condition that the ideals of the manifolds studied belong to a certain conservative system. We are thus led to the following generalization.

Let $\mathfrak{A}$ be a ring, $\mathscr{X}$ a conservative system of ideals in $\mathfrak{A}$ such that $\mathscr{P}(\mathscr{X})$ is Noetherian, $\mathscr{Q}$ a set of prime ideals of $\mathfrak{A}$ such that $\mathfrak{A} \in \mathscr{Q}$. Let $\mathscr{Y}$ consist of those ideals which are intersections of ideals of $\mathscr{Q}$, and let $\mathscr{Z}$ consist of $\mathfrak{A}$ and the members of $\mathscr{Y} \cap \mathscr{P}(\mathscr{X})$. Then $\mathscr{Z}$ is a Noetherian perfect conservative system. Every ideal of $\mathscr{Z}$ can be expressed uniquely as the irredundant intersection of finitely many prime ideals of $\mathscr{Z}$; however, these prime ideals need not be in $\mathscr{Q}$. If $\mathfrak{b} \in \mathscr{Z}$, we define the (restricted) *manifold* $\mathbf{M}(\mathfrak{b})$ of $\mathfrak{b}$ to be the set of ideals of $\mathscr{Q}$ containing $\mathfrak{b}$. If M is a manifold (that is, if $M = \mathbf{M}(\mathfrak{b})$ for some $\mathfrak{b} \in \mathscr{Z}$), then the ideal $\mathfrak{J}(M)$ of M is the intersection of the ideals constituting M. (Note that $\mathbf{M}(\mathfrak{b})$ may neither be included in nor include the set of ideals appearing in the irredundant representation of $\mathfrak{b}$ as an intersection of prime ideals.) It follows at once from the definitions that if $\mathfrak{b} \in \mathscr{Z}$, then $\mathfrak{b} = \mathfrak{J}[\mathbf{M}(\mathfrak{b})]$. If M is a manifold, say $M = \mathbf{M}(\mathfrak{b})$, then $\mathfrak{b} = \mathfrak{J}(M)$, therefore, $M = \mathbf{M}[\mathfrak{J}(M)]$.

Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_r \in \mathscr{Z}$, and let $\mathfrak{b} = \mathfrak{b}_1 \cap \ldots \cap \mathfrak{b}_r$. Then

$$\mathbf{M}(\mathfrak{b}) = \mathbf{M}(\mathfrak{b}_1) \cup \ldots \cup \mathbf{M}(\mathfrak{b}_r).$$

Conversely, let $M_1, \ldots, M_r$ be manifolds. Let $\mathfrak{b} = \mathfrak{J}(M_1) \cap \ldots \cap \mathfrak{J}(M_r)$. Since $M_i = \mathbf{M}[\mathfrak{J}(M_i)]$, $i = 1, \ldots, r$, it follows from the preceding remark that $\mathbf{M}(\mathfrak{b}) = M_1 \cup \ldots \cup M_r$. Hence, a union of finitely many manifolds is a manifold. It is easy to see that the intersection of manifolds is a manifold provided $\mathscr{Q} \subseteq \mathscr{X}$.

A manifold M is called *irreducible* if it is not the union of two proper submanifolds. A manifold M is irreducible if and only if $\mathfrak{J}(M)$ is prime. For suppose, first, that M is reducible, say $M = M_1 \cup M_2$, where $M_1 \neq M$, $M_2 \neq M$. Then $\mathfrak{J}(M) = \mathfrak{J}(M_1) \cap \mathfrak{J}(M_2)$. Since $M_1 = \mathbf{M}[\mathfrak{J}(M_1)]$ and $M = \mathbf{M}[\mathfrak{J}(M)]$, we see that $\mathfrak{J}(M_1) \neq \mathfrak{J}(M)$. Similarly, $\mathfrak{J}(M_2) \neq \mathfrak{J}(M)$. Choose $x \in \mathfrak{J}(M_1)$, $x \notin \mathfrak{J}(M)$, and $y \in \mathfrak{J}(M_2)$, $y \notin \mathfrak{J}(M)$. Then $xy \in \mathfrak{J}(M)$. Hence, $\mathfrak{J}(M)$ is not prime. Next, suppose that $\mathfrak{J}(M)$ is not prime. Then $\mathfrak{b} = \mathfrak{J}(M) = \mathfrak{b}_1 \cap \ldots \cap \mathfrak{b}_r$, the $\mathfrak{b}_i \in \mathscr{Z}$, $\mathfrak{b}_i \neq \mathfrak{b}$, $r > 1$; and

$$M = \mathbf{M}(\mathfrak{b}) = \mathbf{M}(\mathfrak{b}_1) \cup \ldots \cup \mathbf{M}(\mathfrak{b}_r).$$

We cannot have $\mathbf{M}(\mathfrak{b}) = \mathbf{M}(\mathfrak{b}_i)$ for any i, since this would imply that $\mathfrak{b} = \mathfrak{J}[\mathbf{M}(\mathfrak{b})] = \mathfrak{J}[\mathbf{M}(\mathfrak{b}_i)] = \mathfrak{b}_i$. Hence, M is reducible.

If $M = M_1 \cup \ldots \cup M_r$, where the M_i are irreducible manifolds no one of which is contained in any other, then the M_i are called a set of *irreducible components* of M. Now, starting with an arbitrary manifold M, we may write $\mathfrak{J}(M)$ as an irredundant intersection of prime ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$ of $\mathscr{Z}$. It follows that $\mathbf{M}(\mathfrak{b}_1), \ldots, \mathbf{M}(\mathfrak{b}_r)$ are a set of irreducible components of M. They are unique, since if $M = M_1' \cup \ldots \cup M_s'$, where the M_i' are irreducible manifolds no one of which is contained in any other, then $\mathfrak{J}(M) = \mathfrak{J}(M_1') \cap \ldots \cap \mathfrak{J}(M_s')$, and it follows at once from earlier remarks that the ideals $\mathfrak{J}(M_i')$ are prime ideals no one of which is contained in any other. Hence, they coincide when suitably ordered with $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$. It follows that the M_i' and the M_i coincide when suitably ordered. *Every manifold is the irredundant union of a uniquely determined finite set of irreducible components.*

(10) *Bi-ideals.* We introduce a second commutative ring $\mathfrak{B}$ with a conservative system $\mathscr{Y}$ of ideals. Let ψ be a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. We call a pair $[\mathfrak{a}, \mathfrak{b}]$, $\mathfrak{a} \in \mathscr{P}(\mathscr{X})$, $\mathfrak{b} \in \mathscr{P}(\mathscr{Y})$, $\psi\mathfrak{a} \subseteq \mathfrak{b}$, a *bi-ideal* of $[\mathfrak{A}, \mathfrak{B}, \psi, \mathscr{X}, \mathscr{Y}]$. Let $\mathscr{U}$ denote the set of these bi-ideals. Robinson (**13**) introduced bi-ideals in order to give an abstract algebraic treatment of the initial value problem for differential equations.

Let $\alpha = [\mathfrak{a}, \mathfrak{b}] \in \mathscr{U}$. Then α is *prime* if $\mathfrak{a}$ and $\mathfrak{b}$ are prime, *proper* if $\mathfrak{b} \neq \mathfrak{B}$. Let $\alpha_i = [\mathfrak{a}_i, \mathfrak{b}_i]$, $i \in \mathscr{I}$, be members of $\mathscr{U}$. The *greatest lower bound* γ of the

α_i is defined to be $[\bigcap_{i\in\mathscr{I}}\mathfrak{a}_i, \bigcap_{i\in\mathscr{I}}\mathfrak{b}_i]$. We write $\gamma = \bigcap_{i\in\mathscr{I}}\alpha_i$. Of course, $\gamma \in \mathscr{U}$. If $\mathfrak{b}$ is an ideal of $\mathfrak{B}$, then $\mathfrak{b}^*$ will denote the set of $c \in \mathfrak{A}$ such that $\psi c \in \mathfrak{b}$. Then $\mathfrak{b}^*$ is an ideal, and if $\mathfrak{b}$ is prime, $\mathfrak{b}^*$ is prime.

Let $\alpha = [\mathfrak{a}, \mathfrak{b}] \in \mathscr{U}$, and let $\mathfrak{a}_i, i \in \mathscr{I}, \mathfrak{b}_j, j \in \mathscr{J}$, be the minimal prime ideals containing $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then $\mathfrak{b}_j^* \supseteq \mathfrak{a}, j \in \mathscr{J}$; so that we may define $i(j), j \in \mathscr{J}$, such that $i(j) \in \mathscr{I}$ and $a_{i(j)} \subseteq \mathfrak{b}_j^*$. By Theorem I, the $[\mathfrak{a}_{i(j)}, \mathfrak{b}_j]$ and $[\mathfrak{a}_i, \mathfrak{B}]$ are in $\mathscr{U}$. They are prime, and their intersection is α. Hence, *every bi-ideal is the greatest lower bound of a set of prime bi-ideals.* A proper bi-ideal is not necessarily the greatest lower bound of a set of proper prime bi-ideals (**13**, p. 444). However, we see at once from the decomposition just obtained that if α is proper and $\bar{\alpha} = [\bar{\mathfrak{a}}, \bar{\mathfrak{b}}]$ is the greatest lower bound of the set of all proper prime bi-ideals containing α, then $\bar{\mathfrak{b}} = \mathfrak{b}$. We call $\bar{\alpha}$ the *closure* of α.

A bi-ideal $[\mathfrak{a}, \mathfrak{b}]$ is regular if, given $x \in \mathfrak{A}$ such that $\psi(x;\mathscr{P}(\mathscr{X})) \subseteq \mathfrak{b}$, then $x \in \mathfrak{a}$. One sees at once that there is at most one regular bi-ideal with the second member $\mathfrak{b}$.

To obtain a satisfactory theory of regular bi-ideals we impose the following *regularity* conditions on $(\mathfrak{A}, \mathfrak{B}, \psi, \mathscr{X}, \mathscr{Y})$:

R-1: $\psi(0;\mathscr{P}(\mathscr{X})) \subseteq (0;\mathscr{P}(\mathscr{Y}))$;

R-2: Let $\mathfrak{b} \in \mathscr{P}(\mathscr{Y})$ and let $\mathfrak{a}_i, i \in \mathscr{I}$, be members of $\mathscr{P}(\mathscr{X})$ such that for each i, $\psi\mathfrak{a}_i \subseteq \mathfrak{b}$. Then $\psi(\bigcup_{i\in\mathscr{I}}\mathfrak{a}_i;\mathscr{P}(\mathscr{X})) \subseteq \mathfrak{b}$.

Let $\mathfrak{b} \in \mathscr{P}(\mathscr{Y})$. Let $\mathscr{X}'$ denote the set of all $\mathfrak{a} \in \mathscr{X}$ such that $\psi\mathfrak{a} \subseteq \mathfrak{b}$. Then $\mathscr{X}'$ is not empty by R-1. It follows from R-2 that the union $\mathfrak{b}'$ of the members of $\mathscr{X}'$ is in $\mathscr{X}'$. Then $[\mathfrak{b}', \mathfrak{b}]$ is a bi-ideal. If

$$x \in \mathfrak{A} \quad \text{and} \quad \psi(x;\mathscr{P}(\mathscr{X})) \subseteq \mathfrak{b},$$

then $(x;\mathscr{P}(\mathscr{X})) \in \mathscr{X}'$, and therefore $x \in \mathfrak{b}'$. Hence, $[\mathfrak{b}', \mathfrak{b}]$ is regular. We call $\mathfrak{b}'$ the *expansion* of $\mathfrak{b}$.

Remark. Robinson's definition (**13**) of regularity for the case that $\mathscr{P}(\mathscr{X})$ consists of those perfect differential ideals which are also $\dot{\mathfrak{J}}$-ideals, $\dot{\mathfrak{J}}$ a multiplicatively closed set, agrees with ours provided that (**13**, conditions 6.1 and 6.2) and the assumptions made throughout that paper hold; and R-1, R-2 are then valid. These facts follow from (**13**, 6.3 and 6.4). If $\mathscr{P}(\mathscr{X})$ consists of perfect differential ideals (i.e. $\dot{\mathfrak{J}} = \mathfrak{A} - 0$), they are obvious.

If $\mathfrak{b} \subseteq \mathscr{Y}$ is prime, then so is the expansion $\mathfrak{b}'$ of $\mathfrak{b}$.

Proof. Let $xy \in \mathfrak{b}'$. Let $\mathfrak{u} = (x;\mathscr{P}(\mathscr{X}))$, $\mathfrak{v} = (y;\mathscr{P}(\mathscr{X}))$. By Lemma VI, $\mathfrak{u} \cap \mathfrak{v} = (xy;\mathscr{P}(\mathscr{X})) \subseteq \mathfrak{b}'$. Hence $\psi\mathfrak{u} \cap \psi\mathfrak{v} \subseteq \mathfrak{b}$. Since $\mathfrak{b}$ is prime, either $\psi\mathfrak{u} \subseteq \mathfrak{b}$ or $\psi\mathfrak{v} \subseteq \mathfrak{b}$. The regularity of $[\mathfrak{b}', \mathfrak{b}]$ implies either $x \in \mathfrak{b}'$ or $y \in \mathfrak{b}'$.

It is evident that the greatest lower bound of a set of regular bi-ideals is regular. Let $\mathfrak{b} \in \mathscr{Y}$ and let $\mathfrak{b}_i, i \in \mathscr{I}$, be the minimal prime ideals containing $\mathfrak{b}$. Let $\mathfrak{b}_i'$ be the expansion of $\mathfrak{b}_i, i \in \mathscr{I}$. Then the $\mathfrak{b}_i'$ are prime. If $\mathfrak{b}' = \bigcap_{i\in\mathscr{I}}\mathfrak{b}_i'$, then $[\mathfrak{b}', \mathfrak{b}]$ is regular. Hence, *every regular bi-ideal is the greatest lower bound of a set of regular prime bi-ideals.*

7. Primary representations. Let $\mathscr{X}$ be a conservative system of ideals in $\mathfrak{A}$ such that $\mathscr{P}\mathscr{D}(\mathscr{X}) \subseteq \mathscr{C}(\mathscr{X})$, and let (η, f) be a simple link admitted by every ideal of X. Let $\mathfrak{a} \in \mathscr{X}$ be the intersection of primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of $\mathscr{X}$. Then $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$, as was pointed out in the first paragraph of § 2, so that if $xy \in \mathfrak{a}$, there exists an integer r such that $x^r fy \in \mathfrak{a}$. However, it also follows easily from the fact that the $\mathfrak{q}_i$ are in $\mathscr{C}(\mathscr{X})$ that there exists an integer s such that $x(fy)^s \in \mathfrak{a}$. Hence, we have a new necessary condition for an ideal of $\mathscr{X}$ to be an intersection of primary ideals of $\mathscr{X}$. Under rather special circumstances, this condition becomes sufficient.

THEOREM VIII. *Let $\mathscr{X}$ be a conservative system in the commutative ring $\mathfrak{A}$ such that:*

(1) *$\mathscr{X}$ consists of those ideals of $\mathfrak{A}$ admitting a set M of simple links with consequents m_i, $i \in \mathscr{I}$. Let $\mathscr{M}$ denote the set consisting of all products of the m_i and of the identity;*

(2) *$\mathscr{X}$ is additive, and $\mathscr{X} = \mathscr{P}\mathscr{D}(\mathscr{X}) = \mathscr{C}(\mathscr{X})$;*

(3) *For each $x \in \mathfrak{A}$, the ideal generated by the elements mx, $m \in \mathscr{M}$, is $(x;\mathscr{X})$ and has a finite basis;*

(4) *If $x \in \mathfrak{A}$, $m \in \mathscr{M}$, then for any positive integer h, mx^h is a linear combination with coefficients in $\mathfrak{A}$ of products of the form $(m^{(1)}x) \ldots (m^{(h)}x)$, where the $m^{(i)}$ are in $\mathscr{M}$;*

(5) *If $\mathfrak{a} \in \mathscr{X}$, $xy \in \mathfrak{a}$, $i \in \mathscr{I}$, there exists an integer r such that $x(m_i y)^r \in \mathfrak{a}$.* (This implies that if $m \in \mathscr{M}$, then $x(my)^s \in \mathfrak{a}$, for a suitable s.)

Then if $\mathfrak{a} \in \mathscr{X}$ is an intersection of finitely many primary ideals of $\mathfrak{A}$, $\mathfrak{a}$ is an intersection of finitely many primary ideals in $\mathscr{X}$.

Remark. If the m_i are linear, the first parts of (2) and (3) follow from the other conditions. Even in this case, (4) is more general than the requirement that $\mathfrak{A}$ be an M-ring.

Proof. Let $\mathfrak{P} \subseteq \mathfrak{A}$. If $\mathfrak{P}$ is not empty, the ideal generated by the elements ms, $m \in \mathscr{M}$, $s \in \mathfrak{P}$, is $(\mathfrak{P};\mathscr{X})$. To prove this, let Σ denote the set of subsets $\mathfrak{Q}$ of $\mathfrak{P}$ such that the ideal generated by the elements mt, $m \in \mathscr{M}$, $t \in \mathfrak{Q}$, is $(\mathfrak{Q}; X)$. Σ contains the one-element subsets of $\mathfrak{P}$ by (3). By C-2 and Zorn's lemma, Σ contains a maximal subset $\mathfrak{P}^*$. Let $s \in \mathfrak{P}$. Let $\mathfrak{a} = (s;\mathscr{X}) + (\mathfrak{P}^*;\mathscr{X})$. Then $\mathfrak{a} \in \mathscr{X}$, since $\mathscr{X}$ is additive. However, by (3), $\mathfrak{a}$ is the ideal generated by the mt, $m \in \mathscr{M}$, $t \in \mathfrak{P}^*$, and the ms, $m \in \mathscr{M}$. Hence $s \in \mathfrak{P}^*$, $\mathfrak{P}^* = \mathfrak{P}$.

Let $\mathfrak{a} \in \mathscr{X}$, and suppose that $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$, where the $\mathfrak{q}_i$ are primary ideals. Let $\mathfrak{q}_i^*$, $i = 1, \ldots, n$, denote the set of elements x of $\mathfrak{A}$ such that for each $m \in \mathscr{M}$, $mx \in \mathfrak{q}_i$. If $x \in \mathfrak{q}_i^*$, then $mx \in \mathfrak{q}_i^*$ for each $m \in \mathscr{M}$. Hence, $(\mathfrak{q}_i^*) = (\mathfrak{q}_i^*;\mathscr{X})$ by the result just proved. (Here, $(\mathfrak{q}_i^*)$ denotes, as usual, the ideal generated by $\mathfrak{q}_i^*$.) Evidently, $(\mathfrak{q}_i^*) \subseteq \mathfrak{q}_i$. These statements imply that if $x \in (\mathfrak{q}_i^*)$, $m \in \mathscr{M}$, then $mx \in \mathfrak{q}_i$. Hence, $\mathfrak{q}_i^* = (\mathfrak{q}_i^*)$, $i = 1, \ldots, n$. Evidently, $\mathfrak{a} \subseteq \mathfrak{q}_i^*$, $i = 1, \ldots, n$. Hence $\mathfrak{a} = \mathfrak{q}_1^* \cap \ldots \cap \mathfrak{q}_n^*$. It remains only to prove that the $\mathfrak{q}_i^*$ are primary.

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Let $xy \in \mathfrak{q}_i^*$, $x \notin \mathfrak{q}_i^*$. Let $m \in \mathscr{M}$. Let r be such that $x(my)^r \in \mathfrak{q}_i^*$. Then $x \in (\mathfrak{q}_i^*)_{my} \in \mathscr{X}$. There exists $m' \in \mathscr{M}$ such that $m'x \notin \mathfrak{q}_i$. Since

$$m'x \in (\mathfrak{q}_i^*)_{my},$$

$\mathfrak{q}_i$ does not contain $(\mathfrak{q}_i^*)_{my}$. Then certainly $\mathfrak{q}_i$ does not contain $(\mathfrak{q}_i)_{my}$. Since $\mathfrak{q}_i$ is primary, this implies that $my \in \sqrt{\mathfrak{q}_i}$. Hence, there exists for each $m \in \mathscr{M}$ a positive integer s such that $(my)^s \in \mathfrak{q}_i$. Since the ideal generated by the my, $m \in \mathscr{M}$, has a finite basis by (3), there exists a positive integer t such that if $m^{(1)}, \ldots, m^{(t)} \in \mathscr{M}$, then $\Pi_{i=1}^t m^{(i)}y \in \mathfrak{q}_i$. It follows from (4) that $my^t \in \mathfrak{q}_i$, $m \in \mathscr{M}$. Hence $y^t \in \mathfrak{q}_i^*$.

We investigate two special cases, the first of which is well known, and the second of which has recently been treated by Seidenberg (**16**).

COROLLARY I. *Let $\mathfrak{A}$ be a graded commutative ring. If a homogeneous ideal $\mathfrak{a}$ of $\mathfrak{A}$ is the intersection of finitely many primary ideals, then $\mathfrak{a}$ is the intersection of finitely many homogeneous primary ideals.*

Proof. Let $\mathscr{H}$ denote the conservative system consisting of the homogeneous ideals of $\mathfrak{A}$, and define m and the h_i as in example (1). $\mathscr{H}$ is an additive, conservative system, and it has already been shown that

$$\mathscr{H} = \mathscr{P}\mathscr{D}(\mathscr{H}) = \mathscr{C}(\mathscr{H}).$$

All the conditions except (5) are then readily verified, with

$$M = \{h_i, i = 0, \pm 1, \ldots\}.$$

To verify (5), let $\mathfrak{a} \in \mathscr{H}$, $xy \in \mathfrak{a}$, and let $y = \sum_{i=1}^s y_i$, in the notation of example (1). It is easy to see that (5) will follow if we can find a positive integer t such that $xy_i^t \in \mathfrak{a}$, $i = 1, \ldots, s$. Suppose that there exists a positive integer r such that $xy_i^r \in \mathfrak{a}$, $i = 1, \ldots, k$, where $1 \leqq k < s$. For $k = 1$, the existence of such an integer was shown in the course of proving that $\mathscr{H} = \mathscr{P}\mathscr{D}(\mathscr{H})$. Now $x(y - y_1 - \ldots - y_k)^{rk} \in \mathfrak{a}$, as one sees by expanding the left-hand side. However, $m(y - y_1 - \ldots - y_k)^{rk} = (y_{k+1})^{rk}$. Now applying again the result cited for the case $k = 1$, we find that there exists a positive integer r' such that $x(y_{k+1})^{r'} \in \mathfrak{a}$. Thus, the desired result follows by induction.

COROLLARY II (Seidenberg). *Let $\mathfrak{A}$ be a commutative ring with derivations d_i, $i \in \mathscr{I}$. If $\mathfrak{A}$ is Noetherian and a Ritt algebra, then every differential ideal of $\mathfrak{A}$ is an intersection of finitely many differential primary ideals.*

Proof. All the requirements of the theorem except (5) are readily verified for the set $\mathscr{X}$ of differential ideals using the discussion of example (5). It only remains to show that if $\mathfrak{a}$ is a differential ideal, $xy \in \mathfrak{a}$, and d one of the d_i, then there exists an integer r such that $y(dx)^r \in \mathfrak{a}$.

Since only the derivation d will concern us henceforth, we introduce the set $\mathscr{Y}$ of differential ideals closed under d, and note that $\mathfrak{a} \in \mathscr{X} \subseteq \mathscr{Y}$. We also introduce a differential polynomial ring $\mathfrak{B}$ in two indeterminates u, v over the field $\mathfrak{R}$ isomorphic to the rational field which is contained in $\mathfrak{A}$ by hypothesis. In both $\mathfrak{A}$ and $\mathfrak{B}$, derivatives will be indicated by subscripts, for example y_i for $d^i y$. Let $\mathscr{Z}$ be the set of differential ideals of $\mathfrak{B}$.

Let $\mathfrak{J}$ be the set of all differential polynomials of $\mathfrak{B}$ which yield members of $((xy);\mathscr{Y})$ on substituting $u = x, v = y$. Evidently, $\mathfrak{J}$ is an ideal, $\mathfrak{J} \in \mathscr{Z}$, and $((uv);\mathscr{Z}) \subseteq \mathfrak{J}$.

Since $\mathfrak{A}$ is Noetherian, there exists a non-negative integer k such that $(y;\mathscr{Y}) = (y, y_1, \ldots, y_k)$. Since $\mathscr{Y} = \mathscr{P}\mathscr{D}(\mathscr{Y})$, there exists a positive integer r such that $x^r y_i \in ((xy);\mathscr{Y}), i = 0, 1, \ldots, k$. Then actually $x^r y_i \in ((xy);\mathscr{Y}), i = 0, 1, \ldots$. It follows that $u^r v_i \in \mathfrak{J}, i = 0, 1, \ldots$.

From the work of Levi (see **9**, p. 562), it follows easily that

$$u_1{}^r v - k u^r v_r \in ((uv);\mathscr{Z}), \qquad k \in \mathfrak{R}.$$

(Note that $u^r v_r$ is the only "α-term" of signature $(r, 1)$ and weight r, and that $u_1{}^r v$ is a "β-term" of this weight and signature.) Applying the result of the preceding paragraph we find that $u_1{}^r v \in \mathfrak{J}$. Hence $x_1{}^r y \in \mathfrak{a}$.

Remark. If $\mathfrak{A}$ is a unique factorization domain and $\mathscr{X}$ the admissible system consisting of principal ideals of $\mathfrak{A}$, then also every ideal of $\mathscr{X}$ is an intersection of primary ideals of $\mathscr{X}$, but this case is not covered by Theorem VIII if $\mathscr{X} \neq \mathscr{S}$.

8. Comaximal representations in M-rings. Theorems IV and V furnish comaximal representations for certain ideals of $\mathscr{P}\mathscr{D}(\mathscr{X})$, where $\mathscr{X}$ is a conservative system. Ritt obtained such representations for $\mathscr{C}(\mathscr{X})$, where $\mathscr{X}$ is the set of difference ideals; and since we have seen that $\mathscr{C}(\mathscr{X}) \supset \mathscr{P}\mathscr{D}(\mathscr{X})$ is possible in this case, these theorems are not a generalization of Ritt's results. The following theorem generalizes Ritt's theorems on comaximal representations in either differential or difference rings. The notation follows example (4).

Definition. An M-ring is of *finite type* if each $i \in \mathscr{I}$ is contained in a finite subset $\mathscr{J}(i)$ of $\mathscr{I}$ such that for each $j \in \mathscr{J}(i)$ the c_{jkm} are 0 if either $k \notin \mathscr{J}(i)$ or $m \notin \mathscr{J}(i)$.

THEOREM IX. *Let $\mathfrak{A}$ be an M-ring of finite type with identity e. Let $\mathscr{X}$ be the set of M-ideals of $\mathfrak{A}$. If $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$ and $\mathfrak{b} = \sqrt{\mathfrak{a}}$ is the intersection of pairwise comaximal ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$, then there exist unique pairwise comaximal ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ in $\mathscr{C}(\mathscr{X})$ such that $\mathfrak{a} = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_r$ and $\mathfrak{b}_i = \sqrt{\mathfrak{a}_i}$, $i = 1, \ldots, r$.*

Proof. By Lemma IX there exist unique pairwise comaximal ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ such that $\mathfrak{a} = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_r$ and $\mathfrak{b}_i = \sqrt{\mathfrak{a}_i}, i = 1, \ldots, r$. From Lemma X and the fact that the $\mathfrak{b}_i$ are radical, each $\mathfrak{b}_i \in \mathscr{P}(\mathscr{X})$. It remains only to prove that each $\mathfrak{a}_i \in \mathscr{X}$. By Lemma VII it suffices to consider only the case $r = 2$.

There exists $x \in \mathfrak{a}_1, y \in \mathfrak{a}_2$ such that $x + y = e$. Since $x \in \mathfrak{b}_1, m_i x \in \mathfrak{b}_1$, $i \in \mathscr{I}$. Hence, for each $i \in \mathscr{I}$ there exists a positive integer $s = s(i)$ such

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that $(m_ix)^s \in \mathfrak{a}_1$. It follows that if $\mathscr{J}$ is a finite subset of $\mathscr{I}$, there exists a positive integer $t = t(\mathscr{J})$ such that every product of the form $\prod_{j=1}^t m_{i_j}x$, $i_j \in \mathscr{J}$, is in $\mathfrak{a}_1$.

Since $x^k - x^{k+1} = x^k y \in \mathfrak{a}$, $k = 1, 2, \ldots,$ it follows that $x - x^k \in \mathfrak{a}$, $k = 1, 2, \ldots$. Then $m_ix - m_ix^k \in \mathfrak{a}$, $i \in \mathscr{I}$, $k = 1, 2, \ldots$. For given $i \in \mathscr{I}$ let $\mathscr{J}$ be the finite subset of $\mathscr{I}$ to which i is assigned in the definition of M-ring of finite type. Define $t = t(\mathscr{J})$ as above. Then $m_ix - m_ix^t \in \mathfrak{a} \subseteq \mathfrak{a}_1$; however, m_ix^t can be expanded by the multiplication theorem to a sum of terms which are in $\mathfrak{a}_1$ by the choice of t. Hence, $m_ix \in \mathfrak{a}_1$.

Let $u \in \mathfrak{a}_1$. Then $uy \in \mathfrak{a}$. Let $i \in \mathscr{I}$. Then the multiplication theorem yields $m_i(uy) = \sum c_{ijk}(m_ju)(m_ky)$, where the sum is over those j and k in a finite subset of $\mathscr{I}$. For each k, $m_ky = m_ke - m_kx$, and $m_kx \in \mathfrak{a}_1$. Hence $\sum c_{ijk}(m_ju)(m_ke) \in \mathfrak{a}_1$. However, the multiplication theorem shows that the sum on the left is $m_i(ue) = m_iu$. Hence, $m_iu \in \mathfrak{a}_1$ for all $u \in \mathfrak{a}_1$, $i \in \mathscr{I}$.

THEOREM X. *Let $\mathfrak{A}$ be an M-ring of finite type with identity, and let $\mathscr{X}$ be the set of M-ideals of $\mathfrak{A}$. Let $\mathscr{P}(\mathscr{X})$ be Noetherian. If $\mathfrak{a} \in \mathscr{C}(\mathscr{X})$, $\mathfrak{a} \neq \mathfrak{A}$, then there is a uniquely determined set $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ of ideals of $\mathscr{C}(\mathscr{X})$ distinct from $\mathfrak{A}$ such that:*

(a) *The $\mathfrak{a}_i$ are pairwise comaximal;*

(b) *None of the $\mathfrak{a}_i$ is the intersection of two pairwise comaximal ideals distinct from $\mathfrak{A}$;*

(c) $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{a}_i$.

If $\mathfrak{a} \in \mathscr{P}\mathscr{D}(\mathscr{X})$, then each $\mathfrak{a}_i \in \mathscr{P}\mathscr{D}(\mathscr{X})$. If $\mathfrak{a} \in \mathscr{D}(\mathscr{X})$, then each $\mathfrak{a}_i \in \mathscr{D}(\mathscr{X})$.

Proof. The proof of (a), (b), (c), and uniqueness is similar to the proof of Theorem V. The final statements follow from Theorem V.

9. Concluding remarks. A set $\mathscr{X}$ of radical ideals of a commutative ring $\mathfrak{A}$ will be said to have the Krull property if $\mathscr{X}$ consists precisely of those ideals which are intersections of prime ideals of $\mathscr{X}$. If $\mathscr{X}$ has the Krull property, and if also $\mathscr{X}$ contains the minimal prime divisors of each ideal of $\mathscr{X}$, then $\mathscr{X}$ will be said to have the strong Krull property. Evidently, C-1 and divisibility are necessary conditions for $\mathscr{X}$ to have the Krull property. Theorem I shows that C-1, C-2, and divisibility are sufficient for the strong Krull property. However, neither of these sets of conditions is both necessary and sufficient for either the Krull property or the strong Krull property, and further elucidation of the situation is needed.

Example (1). Let $\mathfrak{A}$ be a ring containing an infinite ascending sequence of distinct prime ideals, and let $\mathscr{X}$ consist of the ideals of such a sequence. Then $\mathscr{X}$ has the strong Krull property, but does not satisfy C-2.

Example (2). Let $\mathfrak{A}$ be the ring of continuous real-valued functions on a closed interval I; and let J be a closed proper sub-interval of I. Let $\mathfrak{a}$ be the

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ideal of functions of $\mathfrak{A}$ which vanish on J, and let $\mathscr{X}$ be the set of all ideals $\mathfrak{a}:\mathfrak{T}$, $\mathfrak{T} \subseteq \mathfrak{A}$. Then $\mathscr{X}$ is divisible and satisfies C-1. For any $\mathfrak{T}$, $\mathfrak{a}:\mathfrak{T}$ consists of those functions vanishing at every point of $J - J'$, where J' is the set of points in J which are zeros of every member of $\mathfrak{T}$. Since $J - J'$ cannot consist of a single point, $\mathfrak{a}:\mathfrak{T}$ is not a proper prime ideal. Hence, $\mathscr{X}$ does not have the Krull property.

References

1. C. I. Andreian, *Anneaux différentiels*, Acad. Républ. Pop. Române Bul. Şti. Secţ. Şti. Mat. Fiz. *3* (1951), 319–332.

2. R. M. Cohn, *Linked ideals*, Proceedings of the International Mathematical Congress (International Congress of Mathematicians, Abstracts of short communications, Stockholm, 1962).

3. ——— *Difference algebra* (Interscience, New York, 1965).

4. O. Hamara, *Some topics in differential algebra*, Technical Report, Office of Naval Research; Research Grant number NONR (G)-00039-64.

5. I. Kaplansky, *An introduction to differential algebra* (Hermann, Paris, 1957).

6. E. R. Kolchin, *Le théorème de la base fini pour les polynômes différentiels*, Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 14e année: 1960/61, *Algèbre et théorie des nombres*, Fasc. 1 (Secrétariat mathématique, Paris, 1963).

7. H. F. Kreimer, *The foundations for an extension of differential algebra*, Trans. Amer. Math. Soc. *111* (1964), 482–492.

8. W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Bd. IV, Part III (J. Springer, Berlin, 1935).

9. H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. *51* (1942), 532–568.

10. J. F. Ritt, *Complete difference ideals*, Amer. J. Math. *63* (1941), 681–690.

11. ——— *Differential algebra*, Amer. Math. Soc. Colloq. Publ., Vol. XXXIII (Amer. Math. Soc., Providence, R.I., 1950).

12. J. F. Ritt and H. W. Raudenbush, Jr., *Ideal theory and algebraic difference equations*, Trans. Amer. Math. Soc. *46* (1939), 445–452.

13. A. Robinson, *Local differential algebra*, Trans. Amer. Math. Soc. *97* (1960), 427–456.

14. A. Robinson and S. Halfin, *Local partial differential algebra*, Trans. Amer. Math. Soc. *109* (1963), 165–180.

15. A. Seidenberg, *Some basic theorems in differential algebra (characteristic p, arbitrary)*, Trans. Amer. Math. Soc. *73* (1952), 174–190.

16. ——— *Differential ideals in rings of finitely generated type*, Amer. J. Math. *89* (1967), 22–42.

17. O. Zariski and P. Samuel, *Commutative algebra*, Vol. 1 (Van Nostrand, Princeton, N.J., 1958).

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