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K-GROUPS OF RINGS AND THE HOMOLOGY OF THEIR ELEMENTARY MATRIX GROUPS

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Abstract

Low dimensional algebraic K-groups of a commutative ring are described in terms of the homology of its elementary matrix group. This approach is prompted by recent successful computations of low-dimensional K-groups using group homology methods, and it builds on the identity $K_2(R) = H_2(ER)$.

The proofs use Hochschild-Serre spectral sequences supplied with a multiplicative structure derived from direct sum of matrices in the elementary matrix group $ER = \varinjlim_n E_n R$.

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0. Introduction

Group homology techniques have recently proved fruitful in complete calculations of the algebraic K-groups of commutative rings in dimensions less than five. The link comes from the well-known equalities

$$K_2(R) = H_2(ER), \qquad K_3(R) = H_3(StR),$$

where $K_*(R)$ is $\pi_*(BGLR^+)$, ER is the elementary matrix subgroup of GLR and StR is the Steinberg group [5]. In practice low dimensional (co-)homology of E_nR or of the full general linear group is first computed (e.g. [3]). This note studies the relationship between $K_i(R)$ and $H_i(ER)$ for $i = 3, 4$ and 5, with a straightforward application of Hochschild-Serre spectral sequences.

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Our main results are as follows.

THEOREM 1. *Modulo* 2-*primary torsion*,

(i) $K_3(R) = H_3(ER)$;

(ii) *there is an exact sequence*

$$K_4(R) \overset{H}{\rightarrowtail} H_4(ER) \twoheadrightarrow P^2(K_2(R)); \tag{1}$$

(iii) *if* $(K_2(R))_p$ *is finitely generated for each odd prime* p, *there is an exact sequence*

$$K_5(R) \overset{H}{\to} H_5(ER) \twoheadrightarrow K_3(R) \otimes K_2(R) \oplus \Lambda^2(K_2(R)_T). \tag{2}$$

Here, H: $K_i(R) \cong \pi_i(BER^+) \to H_i(BER^+) \cong H_i(ER)$ is, up to isomorphism, the Hurewicz map. If A is abelian and A_i is the subgroup of $A \otimes A$ generated by $\{a \otimes b + (-1)^i b \otimes a\colon a, b \in A\}$, $P^2(A) = A/A_1$ and $\Lambda^2(A) = A/A_0$. If p is a prime, A_p denotes the p-component of A, and A_T is the odd torsion subgroup of A.

When 2-primary torsion is included, we have the following estimates.

THEOREM 2. (i) *There is an exact sequence*

$$K_2(R)/2K_2(R) \overset{\iota}{\to} K_3(R) \overset{H}{\twoheadrightarrow} H_3(ER). \tag{3}$$

(ii) *There are exact sequences*

$$\begin{aligned} &K_3(R)/2K_3(R) \to \operatorname{Ker}(H\colon K_4(R) \to H_4(ER)) \twoheadrightarrow K \quad \textit{and} \\ &K_2(R)/2K_2(R) \to H_4(ER)/H(K_4(R)) \twoheadrightarrow P^2(K_2(R)) \end{aligned} \tag{4}$$

where K is some quotient of Ker(2: $K_2(R) \to K_2(R)$).

EXAMPLE. $K_2(\mathbf{Z}) = \mathbf{Z}/2$ and $K_3(\mathbf{Z}) = \mathbf{Z}/48$ [6]. Since the special linear group $SL_n\mathbf{Z}$ coincides with $E_n\mathbf{Z}$, by sequence (3) $H_3(SL\mathbf{Z})$ is either $\mathbf{Z}/24$ or $\mathbf{Z}/48$. Soulé derives the dual to the following commutative diagram ([7], [1] page 188):

$$\begin{array}{ccc} \mathbf{Z}/12 \oplus \mathbf{Z}/48 = H_3(St_3\mathbf{Z}) & \overset{0\oplus 2}{\to} & H_3(St\mathbf{Z}) \cong K_3(\mathbf{Z}) = \mathbf{Z}/48 \\ \downarrow & & \downarrow \\ \mathbf{Z}/12 \oplus \mathbf{Z}/12 = H_3(SL_3\mathbf{Z}) & \to & H_3(SL\mathbf{Z}) \end{array}$$

Hence $H_3(SL\mathbf{Z})$ must be $\mathbf{Z}/24$, so that in (3), ι is an injection.

On the other hand, when $k \geqslant 2$, $K_2(\mathbf{Z}/2^k) = \mathbf{Z}/2$ [2], yet $H_3(SL\mathbf{Z}/2^k) \cong K_3(\mathbf{Z}/2^k)$ ([1] page 74 implies the dual to this).

Section 1 sets out the main elements of the proof. Section 2 contains subsidiary arguments.

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1. Proofs of the theorems

Much of the content of the theorems is in the following proposition, a more general form of which is originally due to J. Whitehead. $H_*(N,2)$ denotes the homology of the Eilenberg-Mac Lane space, $K(N,2)$, of type $(N,2)$.

PROPOSITION 1. *There are exact sequences*

$$K_4(R) \xrightarrow{H} H_4(ER) \to H_4(K_2(R),2) \to K_3(R) \overset{H}{\twoheadrightarrow} H_3(ER), \tag{5}$$

$$K_5(R) \xrightarrow{H} H_5(StR) \to K_3(R)/2K_3(R) \to K_4(R) \overset{H}{\twoheadrightarrow} H_4(StR). \tag{6}$$

PROOF. Sequence (5) is (14.4) in [9] applied to the simply connected space BER^+, with the identification $H_i(BER^+) = H_i(ER)$ (or see Table (10)); (6) is a special case of a well-known exact sequence associated to a 2-connected space ([9] page 81). $H_*(BStR^+)$ is identified with $H_*(StR)$ and $\pi_i(BStR^+)$ with $K_i(R)$, $i \geqslant 3$.

Proposition 1 is extended to Theorems 1 and 2 by studying Hochschild-Serre spectral sequences associated to the central extension $K_2(R) \rightarrowtail StR \overset{\phi}{\twoheadrightarrow} ER$, or more precisely, to the related fibrations

$$BK_2(R) \to BStR^+ \xrightarrow{\phi^+} BER^+ \tag{7}$$

and

$$BStR^+ \xrightarrow{\phi^+} BER^+ \xrightarrow{\psi} K(K_2(R),2). \tag{8}$$

Henceforth, abbreviate $K_2(R)$ as N, and for any group G identify $H_*(BG^+)$ with $H_*(G)$, and ϕ_*^+ with ϕ_*.

Low dimensional E^2_{**} terms in the spectral sequence

$$H_*(K(N,2); H_*(StR)) \Rightarrow H_*(ER) \tag{9}$$

are depicted below. The computations of $H_i(N,2)$ are in [4] Sections 21 and 22. ((5) can be read from this table, after appropriate identifications and use of the Hurewicz epimorphism $K_4(R) \to H_4(StR)$.)

$\Gamma(N) = H_4(N,2)$ is Whitehead's gamma group. The product Δ: $N \otimes N = H_2(N,2) \otimes H_2(N,2) \to \Gamma(N)$ has cokernel $N/2N$ ([4] Section 18). Its image is $P^2(N)$ ([9] Sections 5 and 6 covers the finitely generated case; for the general case use the fact that the functors Γ, P^2 and $\otimes \mathbf{Z}/2$ all commute with direct limits). $H_5(N,2)$ is a torsion group; if $(N = H_2(N,2))_p$ is finitely generated for any odd prime p, then the isomorphism $\mathrm{Tor}(N,N)_p \cong N_p \otimes N_p$ induces an isomorphism $(H_5(N,2))_p \cong \Lambda^2(N_p)$ ([4] 22.1).

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$H_*(StR)$						
5	H_5					
4	H_4	0				
3	H_3	0	$N \otimes H_3$			
2	0	0	0	0		
1	0	0	0	0	0	
0	$\mathbf{Z}$	0	N	0	$\Gamma(N)$	$H_5(N, 2)$
	0	1	2	3	4	5

$H_*(N, 2)$

The remainder of this section draws on Table (10), Proposition 1, and the following three propositions (the proofs of which constitute Section 2) to derive the theorems.

PROPOSITION 2. *$H_4(N,2)/\psi_* H_4(ER)$ is a quotient of $N/2N$.*

PROPOSITION 3. *$H_5(ER)$ has a summand $H_3(ER) \otimes H_2(ER)$ such that $H_5(ER)/\phi_* H_5(StR) \cong (H_3(ER) \otimes H_2(ER)) \oplus \psi_* H_5(ER)$.*

PROPOSITION 4. *$H_5(N,2)/\psi_* H_5(ER)$ is a quotient of $(\varinjlim_q H_{3+q}(n,q) \cong \operatorname{Ker}(N \xrightarrow{2} N))$, the third integral homology group of the Eilenberg-Mac Lane spectrum $K(N)$.*

For $i \geqslant 3$ there is a commutative diagram

$$
\text{(11)} \qquad
\begin{array}{ccccccc}
 & \pi_i(BStR^+) & \underset{\sim}{\rightarrow} & \pi_i(BER^+) & \cong K_i(R) \\
 & \downarrow H & & \downarrow H & \\
H_i(StR) \equiv & H_i(BStR^+) & \overset{\phi_*}{\rightarrow} & H_i(BER^+) & \equiv H_i(ER).
\end{array}
$$

When $i = 3$, the left vertical map is the Hurewicz isomorphism. When $i = 4$, (6) implies that it is an isomorphism off 2-primary torsion. Hence in these cases ϕ_* can be identified with the Hurewicz map $K_i(R) \rightarrow H_i(ER)$.

From Table (10) read the exact sequence

$$H_4(N,2)/\psi_* H_4(ER) \rightarrowtail H_3(StR) \overset{\phi_*}{\twoheadrightarrow} H_3(ER).$$

Identify Ker ϕ_* using Proposition 2, and ϕ_*, as above, to get part (i) of Theorems 1 and 2.

Table (10) and Proposition 3 together yield the exact sequence:

$$\text{(12)}\quad H_5(StR) \overset{\phi_*}{\to} H_5(ER)/H_2(ER) \otimes H_3(ER) \overset{\psi_*}{\to} H_5(N,2) \to H_4(StR)$$
$$\overset{\phi_*}{\to} H_4(ER) \twoheadrightarrow \operatorname{Im} \psi_*.$$

Off 2-primary torsion, $\psi_* H_4(ER) = H_4(N,2) \cong P^2(N)$, and $\psi_* H_5(ER) = H_5(N,2)$ (Propositions 2 and 4). Sequence (1) is the right portion of (12) after ϕ_* is identified with the Hurewicz homomorphism. Observe from (6) that H: $K_5(R) \to H_5(StR)$ is a surjection off 2-primary torsion. Therefore (2) is the left portion of (12) in the case that $H_5(N,2)$ is an exterior algebra. This completes the proof of Theorem 1.

The first sequence in Theorem 2 (ii) is the join of the epimorphism $K_3(R) \otimes \mathbf{Z}/2 \twoheadrightarrow \operatorname{Ker}(H\colon K_4(R) \twoheadrightarrow H_4(StR))$ of sequence (6), and the epimorphism $\operatorname{Ker}(K_2(R) \overset{2}{\to} K_2(R)) \twoheadrightarrow \operatorname{Ker}(\phi_*\colon H_4(StR) \to H_4(ER))$ of Proposition 4. Since $H\colon K_4(R) \to H_4(StR)$ is onto, $\phi_* H_4(StR) \equiv H(K_4(R))$; use (12) to conclude that $H_4(ER)/H(K_4(R)) \cong \operatorname{Im} \psi_*$. The second of the sequences (4) is thus equivalent to Proposition 2.

2. Proofs of Propositions 2, 3 and 4

Wagoner [8] defines a direct sum to be a group for which $[G,G]$ is perfect, and which has an operation $\oplus\colon G \times G \to G$ such that for any finite sets $\{g_1,\dots,g_n\} \subset G$ and $\{h_1,\dots,h_n\} \subset [G,G]$, and for any $g_0 \in G$, there exist $\bar{g}, g \in G$ and $h \in [G,G]$ with $g(1 \oplus g_i)g^{-1} = \bar{g}(g_i \oplus 1)\bar{g}^{-1} = g_i$ and $g_0 h_i g_0^{-1} = hh_ih^{-1}$, $1 \leqslant i \leqslant n$. StR and ER are examples of direct sum groups under the "interleaving" operation defined on the generators of StR to be $x_{ij} \oplus x_{mn} = x_{2i,2j}x_{2m+1,2n+1}$. He claims that if $f\colon G \to H$ is a homomorphism of direct sum groups which respects the sum, then $f_*\colon BG^+ \to BH^+$ is an H-map between H-spaces with operation induced by the direct sum. Thus the fibrations (7) and (8) can be considered as fibrations of H-spaces. The associated spectral sequences with coefficients in a ring are therefore bigraded differential algebras. Finally, Loday ([5] 1.4.1) shows that the inclusion $K_2(R) \to StR$ is a homomorphism of direct sum groups. That is, the direct sum operation on the kernel of ϕ: $StR \to ER$ coincides with the abelian group sum. So the ring structure on $H_*(N, i)$ induced by the direct sum is the usual one for Eilenberg-Mac Lane spaces.

With multiplicative structure, the spectral sequences are easily manipulated to prove the Propositions 2, 3 and 4. Note that each of these holds for 2-primary torsion also.

PROOF OF PROPOSITION 2. First consider the spectral sequence (9), low dimensional E^2_{**} terms of which are depicted in (10). Because there is an epimorphism ψ_*: $H_2(Er) \to H_2(N,2)$, compatible with the products, $\psi_*|H_4(ER)$ maps onto $\mathrm{Im}(H_2(N,2) \otimes H_2(N,2))$ in $H_4(N,2)$. This product is, up to isomorphism, the product map Δ: $N \otimes N \to \Gamma(N)$ which has cokernel $N/2N$. Hence $H_4(N,2)/\psi_* H_4(ER)$ is a quotient of $N/2N$. □

PROOF OF PROPOSITION 3. Consider the spectral sequence $H_*(ER; H_*(N)) \Rightarrow H_*(StR)$. The Künneth formula yields a split injection $N \otimes H_3(ER) \rightarrowtail H_3(ER; N)$. The composite $H_2(ER) \otimes H_2(ER) \to H_5(ER) \xrightarrow{d^2_{5,0}} H_3(ER; N)$ has image $N \otimes H_3(ER)$ and therefore may be used to split $H_2(ER) \otimes H_3(ER)$ from $H_5(ER)$ or indeed from $\mathrm{coker}(\phi_*$: $H_5(StR) \to H_5(ER))$.

Return now to the spectral sequence (9). From Table (10), there is an exact sequence $H_4(N,2) \xrightarrow{d^4_{4,0}} H_3(StR) \twoheadrightarrow H_3(ER)$, and hence an exact sequence.

$$H_2(N,2) \otimes H_4(N,2) \xrightarrow{1 \otimes d^4} H_2(N,2) \otimes H_3(StR) \twoheadrightarrow H_2(N,2) \otimes H_3(ER).$$

Because of the multiplicative structure, $d^4_{6,0}|H_2(N,2) \otimes H_4(N,2)$ is $1 \otimes d^4_{4,0}$, so that $E^\infty_{2,3}$ is a quotient of $H_2(N,2) \otimes H_3(ER)$. However, the product $X = H_2(ER) \otimes H_3(ER)$ is represented in $E^\infty_{2,0} \otimes E^\infty_{0,3} = E^\infty_{2,3}$ or terms of lower filtration degree. These are $E^\infty_{1,4} = 0$, and $E^\infty_{0,5} = \phi_* H_5(StR)$. By the previous paragraph $X \cap \mathrm{Im}\,\phi_* = 0$. Thus X is represented in $E^\infty_{2,3}$, and $E^\infty_{2,3} \cong N \otimes H_3(ER)$. An inspection of (10) then shows that $H_5(ER)/(\phi_* H_5(StR) \oplus E^\infty_{2,3}) = E^\infty_{5,0} \cong \mathrm{Im}\,\psi_*$.

PROOF OF PROPOSITION 4. In the commutative diagram below the maps χ are the well-known natural isomorphisms (e.g. [4], Section 12), and the maps ρ are induced by the homology product. The fact that $H_3(N,2) = 0$ has been used to simplify the bottom row. The left vertical isomorphism is obvious.

(13)

$$\begin{array}{ccccc}
\mathrm{Tor}(H_2(ER), H_2(ER)) & \underset{\cong}{\xrightarrow{\chi}} & \dfrac{H_5(ER \oplus ER)}{\sum_{i=0,1} H_{2+i}(ER) \otimes H_{3-i}(ER)} & \xrightarrow{\rho} & \dfrac{H_5(ER)}{\sum_{0,1} H_{2+i}(ER) \otimes H_{3-i}(ER)} \\
\cong \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* \\
\mathrm{Tor}(H_2(N,2), H_2(N,2)) & \underset{\cong}{\xrightarrow{\chi}} & H_5(K(N,2) \times K(N,2)) & \xrightarrow{\rho} & H_5(N,2)
\end{array}$$

By [4] 22.1 and 22.2, $H_5(N,2)/\mathrm{Im}\,\rho$ is isomorphic to Coker(Δ: ${}_2N \otimes {}_2N \to \Gamma({}_2N)) \cong {}_2N$, where ${}_2N = \mathrm{Ker}(N \xrightarrow{2} N)$. (Moreover, [4] 22.1 and 28.1 imply that this is the surjective image of $H_5(N,2)$ in $\varinjlim_q H_{3+q}(N,q)$.) From the diagram (13), we see that $H_5(N,2)/\psi_* H_5(ER)$ must be a quotient of ${}_2N$.

References

[1] J. Aisbett, E. Lluis-Puebla, V. Snaith and C. Soulé, 'On $K_*(\mathbf{Z}/n)$ and $K_*(\mathbf{F}_q[t]/(t^2))$', *Memoirs Amer. Math. Soc.*, to appear.

[2] K. Dennis and M. Stein, 'K_2 of discrete valuation rings,' *Adv. in Math.* **18** (1975), 182–238.

[3] L. Evens and E. Friedlander, 'On $K_*(\mathbf{Z}/p^2\mathbf{Z})$ and related homology groups,' *Trans. Amer. Math. Soc.* **270** (1982), 1–46.

[4] S. Eilenberg and S. Mac Lane, 'On the groups $H(\pi, n)$,' II, *Annals of Math.* **60** (1954), 49–139.

[5] J. -L. Loday, 'K-théorie algébrique et représentations de groupes,' *Ann. Sci. École Norm. Sup.*, (4) **9** (1976), 309–377.

[6] R. Lee and R. Szczarba, 'The group $K_3(\mathbf{Z})$ is cyclic of order 48, *Ann. of Math.* **104** (1976), 31–60.

[7] C. Soulé, 'Cohomology of $SL_3\mathbf{Z}$,' *Topology* **17** (1978), 1–22.

[8] J. Wagoner, 'Delooping classifying spaces in algebraic K-theory,' *Topology* **11** (1972), 349–370.

[9] J. Whitehead, 'A certain exact sequence,' *Ann. of Math.* **52** (1950), 51–95.

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