Glasgow Math. J. **46** (2004) 239–257. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001740. Printed in the United Kingdom

THE BRAUER GROUP OF THE DIHEDRAL GROUP

G. CARNOVALE

Dipartimento di Matematica Pura ed Applicata, via Belzoni 7, I-35131 Padua, Italy
e-mail: carnoval@math.unipd.it

and J. CUADRA

Universidad de Almería, Dept. Álgebra y análisis matemático, E-04120 Almería, Spain
e-mail: jcdiaz@ual.es

(Received 20 February, 2003; accepted 15 May, 2003)

Abstract. Let p^m be a power of a prime number p, $\mathbb{D}_{p^m}$ be the dihedral group of order $2p^m$ and k be a field where p is invertible and containing a primitive $2p^m$-th root of unity. The aim of this paper is computing the Brauer group $BM(k, \mathbb{D}_{p^m}, R_z)$ of the group Hopf algebra of $\mathbb{D}_{p^m}$ with respect to the quasi-triangular structure R_z arising from the group Hopf algebra of the cyclic group $\mathbb{Z}_{p^m}$ of order p^m, for z coprime with p. The main result states that $BM(k, \mathbb{D}_{p^m}, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ when p is odd and when $p = 2$, $BM(k, \mathbb{D}_{2^m}, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$.

2000 *Mathematics Subject Classification.* 16W30, 16H05, 16K50.

Introduction. Let k be a ring with unity and H be a Hopf algebra over k with bijective antipode. In [**2**] S. Caenepeel, F. Van Oystaeyen, and Y. H. Zhang defined the Brauer group of the Hopf algebra H, denoted by $BQ(k, H)$, consisting of Brauer equivalence classes of H-Azumaya algebras. The Brauer group $BQ(k, H)$ generalizes to arbitrary Hopf algebras the Brauer-Long group of a commutative and cocommutative Hopf algebra introduced in [**10**]. Thus the class of Hopf algebras with a Brauer group theory is enlarged. In particular, it makes sense to think about the Brauer group of the group Hopf algebra of a non abelian group. For G a finite abelian group the Brauer-Long group of the Hopf algebra kG, denoted by $BD(k, G)$ and studied in [**9**], was proposed as a generalization of previous existing Brauer groups of graded algebras like the Brauer-Wall group [**20**] or the Brauer group $B_\phi(k, G)$ of G-graded algebras with respect to a pairing $\phi : G \times G \to k$. See [**5**], [**6**], [**7**]. The Brauer group $BD(k, G)$ contains these other Brauer groups as subgroups.

In the generalization proposed in [**2**], the Brauer group $B_\phi(k, G)$ may be recognized as the Brauer group of a coquasi-triangular Hopf algebra; see [**3**, Lemma 1.2]. For a coquasi-triangular Hopf algebra (H, r) the Brauer group $BQ(k, H)$ contains a subgroup $BC(k, H, r)$ consisting of classes of H^{op}-comodule algebras with H-action stemming from the coquasi-triangular structure r. Dually, if (H, R) is a quasi-triangular Hopf algebra, $BQ(k, H)$ contains a subgroup $BM(k, H, R)$ consisting of classes of H-module algebras with H-coaction arising from the quasi-triangular structure R.

Let n be a nonnegative integer, let k be a field containing a primitive n-th root of unity ω and such that n is invertible in k. In this paper we study the Brauer group $BM(k, \mathbb{D}_n, R_z)$ of the group Hopf algebra of the dihedral group given by

$\mathbb{D}_n = \langle g, h : g^n = h^2 = 1, gh = hg^{n-1} \rangle$ with respect to the quasi-triangular structures

$$R_z = \frac{1}{n}\left(\sum_{0\le l,m<n} \omega^{-lm} g^l \otimes g^{zm}\right), \quad (0 \le z \le n-1)$$

for z coprime with n. These quasi-triangular structures arise from the quasi-triangular structure on the group Hopf algebra $k\mathbb{Z}_n$. For $n = p^m$ a power of a prime number p a concrete description of $BM(k, \mathbb{D}_n, R_z)$ is given. It is proved in Theorem 3.5 that $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ if p is odd and $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ if $p = 2$. Here $Br(k)$ denotes the usual Brauer group of k and $k^{\cdot}/k^{\cdot 2}$ is the multiplicative group of k modulo squares. For the case $p = 2$ the assumption that $\omega = \theta^2$ for a primitive $2n$-root of unity $\theta \in k$ is needed.

The underlying idea in our study of $BM(k, \mathbb{D}_n, R_z)$ is to relate it to the Brauer groups $BM(k, \mathbb{Z}_2, R_0)$ and $BM(k, \mathbb{Z}_n, R_z)$ which belong to the theory of the Brauer-Long group and describe $BM(k, \mathbb{D}_n, R_z)$ from the knowledge of them. The cases n odd and n even are different and need to be treated separately. The inclusion map $i : \mathbb{Z}_n \to \mathbb{D}_n$ induces a group homomorphism $i^* : BM(k, \mathbb{D}_n, R_z) \to BM(k, \mathbb{Z}_n, R_z)$. It is shown in Theorem 2.10 that $Ker(i^*) \cong k^{\cdot}/k^{\cdot 2}$ when n is odd and $Ker(i^*) \cong k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$ when n is even. Any $[\beta] \in k^{\cdot}/k^{\cdot 2}$ and $\bar{a} \in \mathbb{Z}_2$ is represented in $Ker(i^*)$ by the algebra $A(\beta, \omega^a)$. As an algebra $A(\beta, \omega^a)$ is the 2×2 matrix algebra $M_2(k)$ and the $\mathbb{D}_n$-action is defined by letting g and h act by conjugation by the elements

$$u = \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix},$$

respectively. The algebra $C(1) = k\langle \delta : \delta^n = 1 \rangle$ with g-action given by $g \cdot \delta = \omega^{z^{-1}} \delta$ is $\mathbb{Z}_n$-Azumaya. The class of $C(1)$ in $BM(k, \mathbb{Z}_n, R_z)$ lies in the image of i^* since the g-action may be extended to a $\mathbb{D}_n$-action by setting $h \cdot \delta = \omega^r \delta^{n-1}$ for $0 \le r \le n-1$. With this $\mathbb{D}_n$-action $C(1)$ is $\mathbb{D}_n$-Azumaya. When n is odd the isomorphism class of this $\mathbb{D}_n$-module algebra is independent of r while when n is even there are exactly two inequivalent $\mathbb{D}_n$-Azumaya algebra structures on $C(1)$ depending on the parity of r (Proposition 2.12). If k is algebraically closed and n is a power of a prime p not dividing z it is known that $BM(k, \mathbb{Z}_n, R_z) \cong \mathbb{Z}_2$ and it is generated by $[C(1)]$. From these facts it is derived that $BM(k, \mathbb{D}_n, R_z) \cong BM(k, \mathbb{Z}_n, R_z) \cong \mathbb{Z}_2$ if p is odd (Corollary 2.13), and $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if $p = 2$ (Corollary 2.16).

This result is used to determine $BM(k, \mathbb{D}_n, R_z)$ for k arbitrary by going to its algebraic closure $\overline{k}$. The inclusion map $\iota : k \to \overline{k}$ induces a group homomorphism $\iota_* : BM(k, \mathbb{D}_n, R_z) \to BM(\overline{k}, \mathbb{D}_n, R_z)$. When n is odd the kernel of ι_* is the subgroup $BAz(k, \mathbb{D}_n, R_z)$ consisting of classes of $BM(k, \mathbb{D}_n, R_z)$ containing a representative element which is classically Azumaya. It is shown in Proposition 3.2 that $Ker(\iota_*) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$. The group $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(\beta, 1)$ for $[\beta] \in k^{\cdot}/k^{\cdot 2}$. When $n = 2q$ with q even, $Ker(\iota_*) \cong k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$. For q odd, $Ker(\iota_*)$ is isomorphic to the direct product of $k^{\cdot}/k^{\cdot 2}$ and the group extension $k^{\cdot}/k^{\cdot 2} \times_{\{-,-\}} Br(k)$ where $\{-,-\} : k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \to Br(k)$ is the 2-cocycle mapping $([a],[b])$ to $[\{a,b\}]$. Here $\{a,b\}$ denotes the quaternion algebra generated by x, y subject to the relations $x^2 = a$, $y^2 = b$ and $xy = -yx$. In both cases the first copy of $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(\beta, 1)$ and the second copy is represented by the algebra $A(t)$ defined as follows: for $[t] \in k^{\cdot}/k^{\cdot 2}$, $A(t) = M_2(k)$ as an algebra and the $\mathbb{D}_n$-action is given by h

acting trivially and g acting by conjugation by

$$u = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

When n is a power of a prime number the map ι_* is surjective and split and its image commutes with $Ker(\iota_*)$ (Theorem 3.5).

1. Preliminaries. Throughout k will be a field and H a finite dimensional Hopf algebra over k. For general facts on Hopf algebras and related notions we refer the reader to [**8**], [**14**], and [**17**]. In this section we recall the construction of the Brauer group $BM(k, H, R)$ of a finite dimensional quasi-triangular Hopf algebra (H, R) over a field k; see [**2**], [**3**].

Let (H, R) be a quasi-triangular Hopf algebra with quasi-triangular structure $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$. Any left H-module algebra A is naturally endowed with a standard right H-comodule algebra structure

$$\rho : A \to A \otimes H^{op},\ a \mapsto \sum \big(R^{(2)} \cdot a\big) \otimes R^{(1)}. \tag{1}$$

The *braided product* $A\#B$ of two left H-module algebras A, B is again a left H-module algebra and it is defined as follows: as a vector space $A\#B = A \otimes B$, with multiplication and H-action defined by

$$(a\#b)(a'\#b') = \sum a\big(R^{(2)} \cdot a'\big)\#\big(R^{(1)} \cdot b\big)b',$$
$$h \cdot (a \otimes b) = \sum \big(h_{(1)} \cdot a\big) \otimes \big(h_{(2)} \cdot b\big),$$

for all $a, a' \in A$, $b, b' \in B$, $h \in H$. The H-*opposite* algebra $\overline{A}$ of a left H-module algebra A is equal to A as a left H-module but with multiplication given by

$$\overline{a} * \overline{b} = \sum \overline{\big(R^{(2)} \cdot b\big)\big(R^{(1)} \cdot a\big)},$$

for all $\overline{a}, \overline{b} \in \overline{A}$. For a finite dimensional left H-module M, the endomorphism algebra $End_k(M)$ becomes a left H-module algebra with H-action

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f\big(S\big(h_{(2)}\big) \cdot m\big),$$

for all $h \in H, f \in End_k(M)$, and $m \in M$, where S denotes the antipode of H. Similarly, the usual opposite algebra $End_k(M)^{op}$ becomes a left H-module algebra with H-action

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f\big(S^{-1}\big(h_{(1)}\big) \cdot m\big),$$

for all $h \in H, f \in End_k(M)^{op}$, and $m \in M$.

A finite dimensional left H-module algebra A is called H-*Azumaya* if the following two left H-module algebra maps are isomorphisms:

$$F : A\#\overline{A} \to End_k(A),\ F(a\#\overline{b})(c) = \sum a\big(R^{(2)} \cdot c\big)\big(R^{(1)} \cdot b\big),$$
$$G : \overline{A}\#A \to End_k(A)^{op},\ G(\overline{a}\#b)(c) = \sum \big(R^{(2)} \cdot a\big)\big(R^{(1)} \cdot c\big)b,$$

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

for all $a, b, c \in A$ and $\overline{a}, \overline{b} \in \overline{A}$. Let $Az(H, R)$ denote the set of isomorphism classes of H-Azumaya algebras. The following equivalence relation in $Az(H, R)$ is introduced: two H-Azumaya module algebras A, B are called *Brauer equivalent*, denoted by $A \sim B$, if there are two finite dimensional left H-modules M, N such that $A\#End(M) \cong B\#End(N)$ as left H-module algebras. The quotient set $BM(k, H, R) = Az(H, R)/\sim$ turns out to be a group with product induced by the braided product; that is, for $[A], [B] \in BM(k, H, R), [A][B] = [A\#B]$. The inverse of $[A]$ is $[\overline{A}]$ and the identity element is $[k]$. Note that for a finite dimensional left H-module M, $End(M)$ is a representative element of $[k]$. The group $BM(k, H, R)$ is called the *Brauer group of H with respect to the quasi-triangular structure R.*

The Brauer group $BM(k, H, R)$ has a functorial behaviour at the field level and at the Hopf algebra level. Any field homomorphism $f : k \to k'$ induces a group homomorphism $f_* : BM(k, H, R) \to BM(k', H \otimes_k k', R_{k'})$ by mapping the class $[A]$ into the class $[A \otimes_k k']$. Any quasi-triangular map $\chi : (H, R) \to (H', R')$ induces a group homomorphism $\chi^* : BM(k, H', R') \to BM(k, H, R),\ [A] \mapsto [A]$ by pulling back the action of H' on A along the map χ.

For a coquasi-triangular Hopf algebra (H, r) a dual construction of the Brauer group holds; one considers right H^{op}-comodule algebras and uses the coquasi-triangular structure in order to define a braiding, braided product, H-opposite algebras, and H-Azumaya algebras. The group obtained in this way is denoted by $BC(k, H, r)$ and it is called the *Brauer group of H with respect to the coquasi-triangular structure r*. For a quasi-triangular Hopf algebra (H, R), H^* is a coquasi-triangular Hopf algebra with coquasi-triangular structure r induced on H^* by R. Then $BM(k, H, R) \cong BC(k, H^*, r)$. If H is commutative and cocommutative, then r induces a pairing ϕ on H and the Brauer group $BC(k, H, r)$ is isomorphic to the Brauer group $B_\phi(k, H)$ of ϕ-Azumaya algebras. See [**3**, Lemma 1.1], [**4**, p. 329], [**7**], [**15**] for more details.

Let $(D(H), \mathcal{R})$ be the Drinfel'd double of H equipped with its canonical quasi-triangular structure $\mathcal{R}$. The Brauer group $BQ(k, D(H), \mathcal{R})$ is usually denoted by $BQ(k, H)$ and it is called the *Brauer group of H*. If H admits a quasi-triangular structure R, then $BM(k, H, R)$ is a subgroup of $BQ(k, H)$. Similarly, if (H, r) is a coquasi-triangular structure, then $BC(k, H, r)$ is a subgroup of $BQ(k, H)$. All these Brauer groups are particular cases of Brauer groups of a braided monoidal category. See [**19**].

When H is the group algebra $H = kG$ of some group G we denote $BM(k, H, R)$ by $BM(k, G, R)$.

2. The Brauer group $BM(k, \mathbb{D}_n, R)$. From now on k is a field containing a primitive $2n$-th root of unity θ and n is invertible in k. Let $k^{\cdot}$ denote the multiplicative group of k. Consider the dihedral group $\mathbb{D}_n = \langle g, h : g^n = h^2 = 1, gh = hg^{n-1} \rangle$. We identify $\mathbb{Z}_n$ with $\langle g \rangle$. The quasi-triangular structures on $k\mathbb{D}_n$ were studied in [**21**]. It is proved in [**21**, Proposition 3.2] that *for $n \neq 4$, $(k\mathbb{D}_n, R)$ is a quasi-triangular Hopf algebra if and only if $(k\mathbb{Z}_n, R)$ is quasi-triangular.* For $n = 4$ there are more quasi-triangular structures arising from the subgroups $\langle h, g^2 \rangle$, $\langle hg, g^2 \rangle$ which are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quasi-triangular structures on $k\mathbb{Z}_n$ are computed in [**16**, p. 219]. These are of the form,

$$R_z = \frac{1}{n}\left(\sum_{0 \leq l, m < n} \omega^{-lm} g^l \otimes g^{zm}\right),$$

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

for $0 \leq z \leq n-1$, where ω is a primitive n-th root of unity. Let $i : k\mathbb{Z}_n \to k\mathbb{D}_n$ be the inclusion map and $p : k\mathbb{D}_n \to k\mathbb{Z}_2$, $g \mapsto \bar{0}$, $h \mapsto \bar{1}$ be the canonical projection map. We have quasi-triangular maps,

$$(k\mathbb{Z}_n, R_z) \xrightarrow{\ i\ } (k\mathbb{D}_n, R_z) \xrightarrow{\ p\ } (k\mathbb{Z}_2, R_0),$$

where $R_0 = 1 \otimes 1$ is the trivial quasi-triangular structure on $k\mathbb{Z}_2$. The functorial behaviour of the Brauer group $BM(k, -)$ yields a sequence

$$BM(k, \mathbb{Z}_2, R_0) \xrightarrow{\ p^*\ } BM(k, \mathbb{D}_n, R_z) \xrightarrow{\ i^*\ } BM(k, \mathbb{Z}_n, R_z).$$

We describe explicitly these homomorphisms. Any $\mathbb{D}_n$-Azumaya algebra is a $\mathbb{Z}_n$-Azumaya algebra by forgetting the action of h. Indeed, *a $\mathbb{D}_n$-module algebra is $\mathbb{D}_n$-Azumaya if and only if it is $\mathbb{Z}_n$-Azumaya*. This is due to the fact that the quasi-triangular structures on $k\mathbb{Z}_n$ and $k\mathbb{D}_n$ are the same. Thus we get a map $i^* : BM(k, \mathbb{D}_n, R_z) \to BM(k, \mathbb{Z}_n, R_z)$, $[A] \mapsto [A]$ but with the latter A considered as a $\mathbb{Z}_n$-module algebra. Similarly, *any $\mathbb{Z}_2$-Azumaya module algebra is a $\mathbb{D}_n$-Azumaya module algebra* via p, and we have a map $p^* : BM(k, \mathbb{Z}_2, R_0) \to BM(k, \mathbb{D}_n, R_z)$, $[A] \mapsto [A]$.

The rest of this section is devoted to studing the above sequence. Let us first note that for the case $z = 0$, i.e., $R_0 = 1 \otimes 1$, the Brauer group $BM(k, \mathbb{D}_n, R_0)$ is already known. It consists of classes of $\mathbb{D}_n$-module algebras which are classically Azumaya. By [**10**, Theorem 1.12], $BM(k, \mathbb{D}_n, R_0) \cong Br(k) \times H^2(\mathbb{D}_n, k)$ where $H^2(\mathbb{D}_n, k)$ is the second cohomology group of $\mathbb{D}_n$ with values in k. We will concentrate on the case $z \neq 0$ and we will describe $BM(k, \mathbb{D}_n, R_z)$ in terms of $BM(k, \mathbb{Z}_n, R_z)$ and $BM(k, \mathbb{Z}_2, R_0)$. These two groups belong to the classical theory of the Brauer group of an abelian group. See [**4**], [**9**], [**10**]. The Brauer group $BM(k, \mathbb{Z}_2, R_0) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$. See [**10**, Theorem 1.12]. The Brauer group $BM(k, \mathbb{Z}_n, R_z)$ is just the group $B_{\phi_z}(k, \mathbb{Z}_n)$ of ϕ_z-Azumaya algebras with $\phi_z : \mathbb{Z}_n \times \mathbb{Z}_n \to k$ being the pairing induced by R_z. See [**3**, Lemma 1.2], [**4**, pp. 329, 341, 434]. For this description we have identified $k\mathbb{Z}_n$ and $(k\mathbb{Z}_n)^*$ as Hopf algebras. The Brauer group $B_{\phi_z}(k, \mathbb{Z}_n)$ was first defined by Child, Garfinkel and Orzech in [**5**] and it can be described by an exact sequence due to Childs. See [**6**] .

Recall that the action of a Hopf algebra H on an algebra A is called *inner* if there is a convolution invertible linear map $\pi : H \to A$ such that

$$h \cdot a = \sum \pi\left(h_{(1)}\right) a \pi^{-1}\left(h_{(2)}\right),$$

for all $h \in H$, $a \in A$. The action is called *strongly inner* if π may be chosen as an algebra map. The Skolem-Noether Theorem for Hopf algebras claims that the action of any Hopf algebra on a classically Azumaya algebra is inner. See [**12**]. The following lemma will be very useful in the sequel.

LEMMA 2.1. *Let (H, R) be a quasi-triangular Hopf algebra and A be a matrix algebra which is an H-Azumaya module algebra. Then $[A]$ is trivial in $BM(k, H, R)$ if and only if the action of H on A is strongly inner.*

Proof. This is proved in [**18**, Lemma 2] for the Drinfel'd double of a Hopf algebra with its canonical quasi-triangular structure. The same proof works for any quasi-triangular Hopf algebra. □

PROPOSITION 2.2. *Let A be a $\mathbb{D}_n$-module algebra which is classically Azumaya. The following statements hold.*

(i) *A contains a subalgebra generated by* u, v *subject to the relations* $u^n = \alpha$, $v^2 = \beta$, $uv = \gamma v u^{n-1}$, *with* $\alpha, \beta, \gamma \in k^{\cdot}$ *satisfying* $\gamma^n \alpha^{n-2} = 1$.

(ii) *The action of* $\mathbb{D}_n$ *on* A *is strongly inner if and only if there are* $s, t \in k^{\cdot}$ *such that* $\alpha = t^n$, $\beta = s^2$ *and* $\gamma = (t^{-1})^{n-2}$.

(iii) *If* $n = 2q$ *is even, then the action of* $\mathbb{D}_n$ *on* A *is strongly inner if and only if there are* $s, t \in k^{\cdot}$ *such that* $\alpha = t^n$, $\beta = s^2$ *and* $\gamma^q = \alpha^{1-q}$.

Proof. (i) Since A is classically Azumaya, the Skolem-Noether Theorem yields that the action of $\mathbb{D}_n$ on A is inner. Let $\pi \in Hom_k(k\mathbb{D}_n, A)$ be a convolution invertible map such that $\sigma \cdot a = \pi(\sigma) a \pi^{-1}(\sigma)$ for all $\sigma \in \mathbb{D}_n$. As σ is a group-like element, $\pi^{-1}(\sigma) = \pi(\sigma)^{-1}$.

Let $u = \pi(g)$ and $v = \pi(h)$. Then $a = 1 \cdot a = g^n \cdot a = u^n a (u^{-1})^n$ for all $a \in A$. Since A is central, there is $\alpha \in k^{\cdot}$ such that $u^n = \alpha$. Similarly, $v^2 = \beta$ for some $\beta \in k^{\cdot}$. From the equalities,

$$(gh) \cdot a = g \cdot (h \cdot a) = uvav^{-1}u^{-1},$$
$$(gh) \cdot a = (hg^{n-1}) \cdot a = h \cdot (g^{n-1} \cdot a) = vu^{n-1}a(u^{-1})^{n-1}v^{-1},$$

we deduce that there exists $\gamma \in k^{\cdot}$ satisfying $uv = \gamma v u^{n-1}$. Multiplying this latter equality on the left by u^{n-1} we get $\alpha v = \gamma^n v u^{n(n-1)} = \gamma^n \alpha^{n-1} v$. Hence $\gamma^n \alpha^{n-2} = 1$.

(ii) Assume that the action of $\mathbb{D}_n$ on A is strongly inner, and let $\zeta : k\mathbb{D}_n \to A$ be a convolution invertible algebra map such that $\sigma \cdot a = \zeta(\sigma) a \zeta(\sigma)^{-1}$ for all $\sigma \in \mathbb{D}_n$, $a \in A$. The elements $\bar{u} = \zeta(g)$ and $\bar{v} = \zeta(h)$ satisfy

$$\bar{u}^n = 1, \qquad \bar{v}^2 = 1, \qquad \bar{u}\bar{v} = \bar{v}\bar{u}^{n-1}.$$

Since $g \cdot a = uau^{-1} = \bar{u} a \bar{u}^{-1}$ for all $a \in A$, there is an element $t \in k^{\cdot}$ such that $u = t\bar{u}$. Then, $\alpha = u^n = t^n \bar{u}^n = t^n$. Similarly, there is $s \in k^{\cdot}$ such that $v = s\bar{v}$, and $\beta = s^2$. Now, $\gamma s t^{n-1} \bar{v} \bar{u}^{n-1} = \gamma v u^{n-1} = uv = ts\bar{u}\bar{v} = ts\bar{v}\bar{u}^{n-1}$. Therefore, $\gamma = (t^{-1})^{n-2}$.

Conversely, suppose that $\alpha = t^n$, $\beta = s^2$, and $\gamma = (t^{-1})^{n-2}$ for some $s, t \in k^{\cdot}$. Define

$$\zeta(g) = \frac{1}{t} u, \qquad \zeta(h) = \frac{1}{s} v,$$

and extend it to an algebra map from $\mathbb{D}_n$ into A. This map is well-defined and gives the same action as π.

(iii) If the action of $\mathbb{D}_n$ is strongly inner, then from (ii) we obtain

$$\alpha^{1-q} = (t^{2q})^{1-q} = (t^{-1})^{2q(q-1)} = \gamma^q.$$

Conversely, if $\alpha = t^n$, $\beta = s^2$ and $\gamma^q = \alpha^{1-q}$, then

$$\alpha = (\alpha\gamma)^q, \qquad \gamma = ((\alpha\gamma)^{-1})^{q-1}.$$

By part (ii) it is enough to show that $\alpha\gamma$ is a square in k. Since $\alpha = t^{2q} = (\alpha\gamma)^q$ there exists a q-th root of unity $\xi = \theta^{4r}$ for some r such that $\alpha\gamma = \xi t^2 = (\theta^{2r} t)^2$. Hence the statement. (iii) holds. □

REMARK 2.3. The elements u, v of Proposition 2.2 (i) are unique up to scalar multiples. The subalgebra generated by them is completely determined by the $\mathbb{D}_n$-action and we will call it the *induced subalgebra on A by the* $\mathbb{D}_n$*-action.* If we

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

take different generators u' and v', then $u' = tu$ and $v' = sv$ for some nonzero scalars t and s and the corresponding constants will be $\alpha' = t^n\alpha$, $\beta' = s^2\beta$ and $\gamma' = (t^{-1})^{n-2}\gamma$.

The set $G = \{(\alpha, \gamma) \in k^{\cdot} \times k^{\cdot} : \gamma^n\alpha^{n-2} = 1\}$ is a group with multiplication induced from $k^{\cdot} \times k^{\cdot}$. We introduce the following equivalence relation on G. Two elements $(\alpha, \gamma), (\alpha', \gamma') \in G$ are equivalent, denoted by $(\alpha, \gamma) \sim (\alpha', \gamma')$, if there is $t \in k^{\cdot}$ such that $\alpha' = t^n\alpha$ and $\gamma' = (t^{-1})^{n-2}\gamma$. The quotient set $\mathcal{G} = G/\sim$ is a group. Any $\mathbb{D}_n$-module algebra which is classically Azumaya has associated a unique invariant $([\beta], [(\alpha, \gamma)]) \in k^{\cdot}/k^{\cdot 2} \times \mathcal{G}$.

REMARK 2.4. Note from the proof of Proposition 2.2 that the action of g is strongly inner if and only if α is a n-th power in k and that in this case one can always choose u and v such that $u^n = 1$ and $uv = \gamma vu^{-1}$ with $\gamma^n = 1$.

LEMMA 2.5. (i) *If n is odd, then $\mathcal{G}$ is trivial.*
(ii) *If n is even, then $\mathcal{G} \cong k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$.*

Proof. (i) We only need to show that if n is odd we can always find $t \in k^{\cdot}$ such that $\alpha = t^n$ and $\gamma = t^{2-n}$. Since $\gamma^n\alpha^n = \alpha^2$ this is equivalent to $\alpha = t^n$ and $\alpha\gamma = t^2$. As $(2, n) = 1$, there exist integers a and b for which $1 = 2a + nb$. Then $\alpha = \alpha^{2a}\alpha^{nb} = (\alpha\gamma)^{an}\alpha^{bn}$ and $\alpha\gamma = (\alpha\gamma)^{2a}(\alpha\gamma)^{nb} = (\alpha\gamma)^{2a}\alpha^{2b}$ so that we may take $t = \alpha^{a+b}\gamma^a$.

(ii) Suppose that $n = 2q$ and let $[(\alpha, \gamma)] \in \mathcal{G}$. From $\gamma^n\alpha^{n-2} = 1$, it follows that $\gamma^q\alpha^{q-1} = \pm 1$. It may be checked that the map

$$\Phi : \mathcal{G} \to k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2, [(\alpha, \gamma)] \mapsto ([\gamma\alpha], \gamma^q\alpha^{q-1})$$

is an isomorphism. □

COROLLARY 2.6. *With notation as in Proposition 2.2* (i)*, for n odd we can always choose u such that $u^n = 1$ and $uv = vu^{n-1}$.*

Any $\mathbb{D}_n$-module algebra A becomes a $\mathbb{Z}_n$-comodule algebra with comodule structure as in (1) for the quasi-triangular structure R_z. Hence A is a $\mathbb{Z}_n$-graded algebra. An element $a \in A$ has degree r, denoted by $deg(a) = r$, if $\rho(a) = a \otimes g^r$. Equivalently, $g^z \cdot a = \omega^r a$. If A, B are $\mathbb{D}_n$-module algebras, then the multiplication in the braided product $A\#B$ is given by

$$(a\#b)(a'\#b') = aa'\#\big(g^{deg(a')} \cdot b\big)b' \tag{2}$$

for homogeneous $a, a' \in A$ and $b, b' \in B$.

LEMMA 2.7. *Let A, B be $\mathbb{D}_n$-module algebras and let B be a classically Azumaya algebra such that g acts strongly innerly on it. Then, $A\#B \cong A \otimes B$ as $\mathbb{D}_n$-module algebras. In particular, if A and B are both classically Azumaya with a strongly inner g-action, $A\#B$ is again so.*

Proof. The proof is inspired by [**9**, Lemma 2.2]. Since the action of g is strongly inner on the Azumaya algebra B there exists $u_B \in B$ with $g \cdot b = u_B b u_B^{-1}$ for every $b \in B$ and $u_B^n = 1$. Similarly, there exists $v_B \in B$ such that $h \cdot b = v_B b v_B^{-1}$ for every $b \in B$ with $u_B v_B = \gamma v_B u_B^{-1}$ and $\gamma^n = 1$. Let $\zeta = \theta^r \in k^{\cdot}$ be a $2n$-th root of unity for which $\zeta^2 = \gamma$. We check that the map

$$\Phi\colon A\#B \to A \otimes B,\ a\#b \mapsto a \otimes \zeta^{\deg(a)} u_B^{-\deg(a)} b,$$

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

for $a \in A$ homogeneous, is a $\mathbb{D}_n$-module algebra isomorphism. For $a, a' \in A$ homogeneous, and $b, b' \in B$,

$$\begin{aligned}\Phi((a\#b)(a'\#b')) &= \Phi\big(aa'\#\big(g^{\deg(a')}\cdot b\big)b'\big)\\ &= aa'\otimes\zeta^{\deg(a)+\deg(a')}u_B^{-\deg(a)}u_B^{-\deg(a')}\big(u_B^{\deg(a')}bu_B^{-\deg(a')}\big)b'\\ &= \big(a\otimes\zeta^{\deg(a)}u_B^{-\deg(a)}b\big)\big(a'\otimes\zeta^{\deg(a')}u_B^{-\deg(a')}b'\big)\\ &= \Phi(a\#b)\Phi(a'\#b').\end{aligned}$$

So the map Φ is an algebra homomorphism and it is clearly bijective because u_B is invertible. The inverse $\Phi^{-1}: A\otimes B \to A\#B$ is defined as $\Phi^{-1}(a\otimes b) = a\#\zeta^{-\deg(a)}u_B^{\deg(a)}b$ for $a \in A$ homogeneous and $b \in B$. We next show that Φ is a $\mathbb{D}_n$-module isomorphism. Notice that the action of g does not change the degree of an element in A and the action of h maps elements of a given degree into elements of opposite degree. Then, we have

$$\begin{aligned}g\cdot\Phi(a\#b) &= g\cdot\big(a\otimes\zeta^{\deg(a)}u_B^{-\deg(a)}b\big)\\ &= \big(g\cdot a\otimes\zeta^{\deg(a)}u_Bu_B^{-\deg(a)}bu_B^{-1}\big)\\ &= \big(g\cdot a\otimes\zeta^{\deg(g\cdot a)}u_B^{-\deg(g\cdot a)}g\cdot b\big)\\ &= \Phi(g\cdot(a\#b)),\\ h\cdot\Phi(a\#b) &= (h\cdot a)\otimes\zeta^{\deg(a)}v_Bu_B^{-\deg(a)}bv_B^{-1}\\ &= (h\cdot a)\otimes\zeta^{\deg(a)}\gamma^{-\deg(a)}u_B^{\deg(a)}(h\cdot b)\\ &= (h\cdot a)\otimes\zeta^{-\deg(a)}u_B^{\deg(a)}(h\cdot b)\\ &= (h\cdot a)\otimes\zeta^{\deg(h\cdot a)}u_B^{-\deg(h\cdot a)}(h\cdot b)\\ &= \Phi(h\cdot(a\#b)).\end{aligned}$$

To prove the last statement of the lemma, assume that A is also a classically Azumaya algebra with a strongly inner g-action, and let u_A, v_A be generators of the induced subalgebra such that $u_A^n = 1$. Then $A\#B \cong A\otimes B$ is again classically Azumaya and $u := \Phi^{-1}(u_A\otimes u_B)$ satisfies $g\cdot(a\#b) = u(a\#b)u^{-1}$ for every $a \in A$ and $b \in B$ and $u^n = 1\#1$. □

COROLLARY 2.8. *The subset $BAz^g(k, \mathbb{D}_n, R_z)$ of classes in $BM(k, \mathbb{D}_n, R_z)$ that can be represented by an Azumaya algebra with strongly inner g-action is an abelian subgroup of $BM(k, \mathbb{D}_n, R_z)$. If n is odd, $BAz^g(k, \mathbb{D}_n, R_z)$ coincides with $BAz(k, \mathbb{D}_n, R_z)$, the subgroup of $BM(k, \mathbb{D}_n, R_z)$ of elements which can be represented by an Azumaya algebra.*

Proof. The last statement follows by Corollary 2.6. □

LEMMA 2.9. *If $[A]$ in $BM(k, \mathbb{D}_n, R_z)$ can be represented by a classically Azumaya algebra A, then all other representatives will be also classically Azumaya. Moreover, with notation as in Remark 2.3, we may associate to $[A]$ the invariant $([\beta_A], [(\alpha_A, \gamma_A)]) \in k^{\cdot}/k^{\cdot 2}\times\mathcal{G}$ and this assignment does not depend on the representative of $[A]$.*

Proof. If B is any other representative of the class $[A]$ then there are $\mathbb{D}_n$-modules P and Q such that $A\#End(P)\cong B\#End(Q)$. Using Lemma 2.7,

$$A\otimes End(P)\cong A\#End(P)\cong B\#End(Q)\cong B\otimes End(Q).$$

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

Therefore $B \otimes End(Q)$ is classically Azumaya. Then the algebra B is also Azumaya because it is the centralizer of $End(Q)$ in a classically Azumaya algebra. This gives the first statement. We prove the second one. By Lemma 2.7, $u_{A\#End(P)} = \Phi^{-1}(u_A \otimes u_{End(P)})$ and $v_{A\#End(P)} = \Phi^{-1}(v_A \otimes v_{End(P)})$ are generators for the induced subalgebra of $A\#End(P)$. Similarly for $B\#End(Q)$. Since the $\mathbb{D}_n$-action on $End(P)$ and $End(Q)$ is strongly inner, then

$$\begin{aligned} \alpha_{A\#End(P)} &= \alpha_A \alpha_{End(P)} = \alpha_A t^n, & \alpha_{B\#End(Q)} &= \alpha_B \alpha_{End(Q)} = \alpha_B t'^n, \\ \beta_{A\#End(P)} &= \beta_A \beta_{End(P)} = \beta_A s^2, & \beta_{B\#End(Q)} &= \beta_B \beta_{End(Q)} = \beta_B s'^2, \\ \gamma_{A\#End(P)} &= \gamma_A \gamma_{End(P)} = \gamma_A t^{2-n}, & \gamma_{B\#End(Q)} &= \gamma_B \gamma_{End(Q)} = \gamma_B t'^{2-n}, \end{aligned}$$

for some $t, t', s, s' \in k^{\cdot}$. By Remark 2.3, there are $\tilde{s}, \tilde{t} \in k^{\cdot}$ such that $\alpha_A t^n = \tilde{t}^n \alpha_B t'^n$, $\beta_A s^2 = \tilde{s}^2 \beta_B s'^2$ and $\gamma_A t^{2-n} = \tilde{t}^{2-n} \gamma_B t'^{2-n}$. Hence the second statement is proved. □

THEOREM 2.10. *There are exact sequences of groups,*

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} BM(k, \mathbb{Z}_n, R_z), \tag{3}$$

for n odd and

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} BM(k, \mathbb{Z}_n, R_z), \tag{4}$$

for n even.

Proof. The kernel of i^* is given by elements which can be represented by a matrix algebra with a strongly inner g-action. Therefore it is a subgroup of the abelian group $BAz^g(k, \mathbb{D}_n, R_z)$. Let A be a representative of an element in $Ker(i^*)$. Its induced subalgebra is generated by u_A, v_A such that $u_A^n = 1$, $v_A^2 = \beta_A$ and $u_A v_A = \gamma_A v_A u_A^{-1}$, for $\beta_A \in k^{\cdot}$ and $\gamma_A \in k^{\cdot}$ an n-th root of unity. For n odd we can always make sure that $\gamma_A = 1$ by Corollary 2.6. For $n = 2q$ even, $\gamma_A^q = \pm 1$. In light of Lemma 2.9, the maps $Inv_o\colon Ker(i^*) \to k^{\cdot}/k^{\cdot 2}$, $[A] \mapsto [\beta_A]$ for n odd, and $Inv_e\colon Ker(i^*) \to k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$, $[A] \mapsto ([\beta_A], \gamma_A^q)$ for $n = 2q$ even are well defined. We check that they are group homomorphisms. If A, B are in $Ker(i^*)$ and have induced subalgebras generated by u_A, v_A and u_B, v_B respectively, then by Lemma 2.7, the induced subalgebra of $A\#B$ is generated by $u = \Phi^{-1}(u_A \otimes u_B)$ and $v = \Phi^{-1}(v_A \otimes v_B)$. Hence $v^2 = \beta_A \beta_B$ and $uv = \gamma_A \gamma_B v u^{-1}$. The injectivity follows by Lemma 2.1, Remark 2.4 and Proposition 2.2 (ii), (iii).

Finally we prove the surjectivity of these two maps. Let γ be an n-th root of unity. Consider the matrix algebra $A(\beta, \gamma) = M_2(k)$. Let

$$u = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that $u^n = 1$, $v^2 = \beta$ and $uv = \gamma v u^{-1}$. Thus the conjugation by u and v give to A a $\mathbb{D}_n$-module algebra structure. Consider the $\mathbb{Z}_n$-action induced by restriction. Since $A(\beta, \gamma)$ is classically Azumaya and it has a $\mathbb{Z}_n$-trivial graded center, it is $\mathbb{Z}_n$-Azumaya. Hence $A(\beta, \gamma)$ is $\mathbb{D}_n$-Azumaya. Clearly, if n is odd, $Inv_o(A(\beta, \gamma)) = [\beta]$ and if $n = 2q$ is even $Inv_e(A(\beta, \gamma)) = ([\beta], \gamma^q)$. Hence both maps are surjective. □

REMARK 2.11. The Brauer group $BM(k, \mathbb{Z}_n, R_z)$ may be identified with the Brauer group $B_{\phi_z}(k, \mathbb{Z}_n)$ where $\phi_z : \mathbb{Z}_n \times \mathbb{Z}_n \to k$, $(g^i, g^j) \mapsto \omega^{zij}$ is the pairing induced by the

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

quasi-triangular structure R_z, [**3**, Lemma 1.2]. When $n = p^m$ is a power of a prime number p with p invertible in k, k containing a primitive $2n$-th root of unity and ϕ_z is non-degenerate (equivalently, z is coprime with n), the multiplication rules of $B_{\phi_z}(k, \mathbb{Z}_n)$ are known. See [**4**, Corollary 13.12.36]. As a set $B_{\theta_z}(k, \mathbb{Z}_n) = \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot n} \times Br(k)$. The product is given by

$$(\pm, S, A)(+, S', A') = (\pm, SS', AA'|S'\#S|),$$
$$(\pm, S, A)(-, S', A') = (\mp, S^{-1}S', AA'|S'\#S^{-1}|).$$

We identify these rules in $B_{\phi_z}(k, \mathbb{Z}_n)$; see [**1**, p. 235]. For $\alpha \in k^{\cdot}$, the algebra $C(\alpha) = k\langle\delta : \delta^n = \alpha\rangle$ with $\mathbb{Z}_n$-action given by $g \cdot \delta = \omega^{z^{-1}}\delta$ is $\mathbb{Z}_n$-Azumaya. The symbol $-$ is represented by $[C(1)]$. Each $[\alpha] \in k^{\cdot}/k^{\cdot n}$ is viewed in $B_{\phi_z}(k, \mathbb{Z}_n)$ as $[C(\alpha)\#(k\mathbb{Z}_n)^*]$. For $[\alpha], [\beta] \in k^{\cdot}/k^{\cdot n}$, the braided product $C(\alpha)\#C(\beta)$ is an Azumaya algebra. See [**11**, Proposition 2.1], [**4**, p. 359]. By $|C(\alpha)\#C(\beta)|$ we denote the underlying algebra. It is generated by two elements x, y subject to the relations $x^n = \alpha, y^n = \beta, yx = \omega^{z^{-1}}xy$. The Brauer group $Br(k)$ is embedded as usual as the subgroup of ordinary Azumaya algebras with trivial $\mathbb{Z}_n$-action. In particular, if k is algebraically closed, $BM(k, \mathbb{Z}_n, R_z) \cong \mathbb{Z}_2$ and it is generated by $[C(1)]$.

By the Remark above, if k is algebraically closed and n is a power of a prime p not dividing z, then the exact sequences (3), (4) in Theorem 2.10 become

$$1 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} \mathbb{Z}_2 \tag{5}$$

for n odd and

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} \mathbb{Z}_2 \tag{6}$$

for n even. In this setting $BM(k, \mathbb{D}_n, R_z)$ is thus always an abelian group. In particular, for n odd, we can prove that $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2$ by showing that it is nontrivial. The even case is slightly more complicated. We will prove that $BM(k, \mathbb{D}_{2^m}, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by showing that i^* is surjective and split. For this purpose, we study all possible lifts of the $\mathbb{Z}_n$-action on $C(\alpha)$ to a $\mathbb{D}_n$-action.

In the sequel we shall assume that z is coprime with n and we shall denote by s the inverse of z modulo n.

Proposition 2.12. *Consider the algebra $C(\alpha) = k\langle\delta : \delta^n = \alpha\rangle$ with $\mathbb{Z}_n$-action given by $g \cdot \delta = \omega^s\delta$. Then, $C(\alpha)$ is $\mathbb{D}_n$-Azumaya if and only if there is $\lambda \in k^{\cdot}$ such that $\lambda^n\alpha^{n-2} = 1$. In this case, $h \cdot \delta = \lambda\delta^{n-1}$. Furthermore* (i) *and* (ii) *hold.*

(i) *If n is odd all possible lifts of the $\mathbb{Z}_n$-action give isomorphic $\mathbb{D}_n$-module algebras.*

(ii) *If $n = 2q$, there are either* 0 *or* 2 *possible isomorphism classes of lifts of the $\mathbb{Z}_n$-action according to the existence of a λ as above. Two lifts corresponding to λ and λ' are isomorphic if and only if $\lambda^q = (\lambda')^q$.*

Proof. From [**11**, p. 442], $C(\alpha)$ is $\mathbb{Z}_n$-Azumaya. Recall that an algebra is $\mathbb{D}_n$-Azumaya if and only if it is $\mathbb{Z}_n$-Azumaya. So it is enough to check whether we can give $C(\alpha)$ a $\mathbb{D}_n$-module algebra structure. It is easy to see that for $\lambda, \alpha \in k^{\cdot}$ satisfying $\lambda^n\alpha^{n-2} = 1$, the action given by $g \cdot \delta = \omega^s\delta,\ h \cdot \delta = \lambda\delta^{n-1}$ makes $C(\alpha)$ into a $\mathbb{D}_n$-module algebra.

Conversely, the h-action on $C(\alpha)$ maps eigenvectors of the g-action of eigenvalue ω^t into eigenvectors of eigenvalue ω^{-t}. As s is coprime with n, the eigenspaces for the

g-action are 1-dimensional. Thus, necessarily $h\cdot\delta=\lambda\delta^{n-1}$. From the equations

$$\delta=h^2\cdot\delta=h\cdot(h\cdot\delta)=h\cdot(\lambda\delta^{n-1})=\lambda(h\cdot\delta)^{n-1}=\lambda(\lambda\delta^{n-1})^{n-1}=\lambda^n\delta^{(n-1)^2}$$
$$=\lambda^n\alpha^{n-2}\delta,$$

it follows that $\lambda^n\alpha^{n-2}=1$.

For $\lambda\in k^{\cdot}$ such that $\lambda^n\alpha^{n-2}=1$ let $C_\lambda(\alpha)$ denote the lift of $C(\alpha)$ with $h\cdot\delta=\lambda\delta^{n-1}$. Consider two lifts $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$. Then $(\lambda')^n=\lambda^n$, so that $\lambda'=\zeta\lambda$ for an n-th root of unity $\zeta=\omega^r$ for some integer r. It is easy to check that if $r=2t$ is even, then the map $\Psi\colon C_\lambda(\alpha)\to C_{\lambda'}(\alpha),\ \delta\mapsto\omega^t\delta$ is a $\mathbb{D}_n$-module algebra isomorphism.

(i) For n odd, we can always make sure that r is even.

(ii) For $n=2q$ even, r is even if and only if $\lambda^q=(\lambda')^q$. Hence if $\lambda^q=(\lambda')^q$, then $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$ are isomorphic as $\mathbb{D}_n$-module algebras. Conversely, suppose now that $\Psi: C_\lambda(\alpha)\to C_{\lambda'}(\alpha)$ is an isomorphism of $\mathbb{D}_n$-module algebras. Then $\Psi(\delta)=\omega^r\delta$ for some r because $(s,n)=1$ and $\delta^n=\alpha$. Since the elements $\Psi(h\cdot\delta)=\lambda'\omega^{-t}\delta^{n-1}$ and $h\cdot\Psi(\delta)=\omega^t\lambda\delta^{n-1}$ coincide, it follows that $\lambda'=\omega^{2t}\lambda$. Therefore $\lambda^q=\lambda'^q$. □

For n a power of an odd prime number and k algebraically closed the computation of $BM(k,\mathbb{D}_n,R_z)$ derives from the sequence (5) and Proposition 2.12 (i).

Corollary 2.13. *Let $n=p^m$ for an odd prime p and let k be algebraically closed. Then, for every z not divisible by p, $BM(k,\mathbb{D}_n,R_z)\cong\mathbb{Z}_2$. The non trivial element is* $[C_1(1)]$.

For n a power of 2 and k algebraically closed more work is needed to compute $BM(k,\mathbb{D}_n,R_z)$.

Proposition 2.14. *Let $n=2q$ and let $C_\lambda(\alpha)$, $C_{\lambda'}(\alpha)$ be as above. Then we have $[C_{\lambda'}(\alpha)]=[C_\lambda(\alpha)]$ in $BM(k,\mathbb{D}_n,R_z)$ if and only if $\lambda^q=\lambda'^q$.*

Proof. If $\lambda^q=\lambda'^q$, we know from Proposition 2.12 (ii) that $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$ are indeed isomorphic. Conversely, suppose that $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$ represent the same element in $BM(k,\mathbb{D}_n,R_z)$, and let P,Q be two $\mathbb{D}_n$-modules such that $C_\lambda(\alpha)\#End(P)\cong C_{\lambda'}(\alpha)\#End(Q)$ as $\mathbb{D}_n$-module algebras. It follows from Lemma 2.7 that $C_\lambda(\alpha)\otimes End(P)\cong C_{\lambda'}(\alpha)\otimes End(Q)$ as $\mathbb{D}_n$-module algebras. Then the centres $C_\lambda(\alpha)\otimes k$ and $C_{\lambda'}(\alpha)\otimes k$ of these two algebras are isomorphic as $\mathbb{D}_n$-module algebras. By Proposition 2.12 (ii), $\lambda^q=\lambda'^q$. □

From now on the algebra $C_1(1)$ will be denoted by $C_{\bar 0}(1)$ both for n even or odd. For n even, $C_{\bar 1}(1)$ will denote $C_{\omega^s}(1)$.

Lemma 2.15. *With notation as above, the classes $[C_{\bar 0}(1)]$ (n even or odd), $[C_{\bar 1}(1)]$ and $[C_{\bar 0}(1)\#C_{\bar 1}(1)]$ have all order 2 in the corresponding $BM(k,\mathbb{D}_n,R_z)$. Moreover, $[C_{\bar 0}(1)]$ commutes with $[C_{\bar 1}(1)]$.*

Proof. As the braided product of $\mathbb{D}_n$-module algebras coincides with the braided product of $\mathbb{Z}_n$-module algebras, the algebra $C_{\bar a}(1)\#C_{\bar b}(1)$ is a matrix algebra ([**11**, Proposition 2.4]) with strongly inner g-action. We prove that the $\mathbb{D}_n$-action on $C_{\bar a}(1)\#C_{\bar b}(1)$ for $a,b=0,1$ is strongly inner if and only if $a=b$. Let δ,η denote generators of $C(1)$. Let $u=\zeta(\delta^{n-1}\#\eta)$ with ζ an n-th (respectively $2n$-th) root of unity for n odd (respectively even) for which $\zeta^2=\omega^s$. By induction, $u^r=\zeta^{2r-r^2}\delta^{n-r}\#\eta^r$, so that $u^n=1$. It may be checked that the g-action on $C_{\bar a}(1)\#C_{\bar b}(1)$ is given by conjugation

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

by u. The h-action on $C_{\bar{a}}(1)$ and $C_{\bar{b}}(1)$ is defined by

$$h\cdot\delta^j=\omega^{saj}\delta^{-j},\qquad h\cdot\eta^j=\omega^{sbj}\eta^{-j}.$$

Let

$$v=\begin{cases}\dfrac{1}{n}\displaystyle\sum_{i,j=0}^{n-1}\zeta^{ij}\delta^i\#\eta^j & \text{if } n \text{ is odd,}\\ \dfrac{1}{q}\displaystyle\sum_{i,j=0}^{q-1}\omega^{-sai-sbj+2sij}\delta^{2i}\#\eta^{2j} & \text{if } n=2q.\end{cases}$$

We claim that the element v satisfies $v^2=1$ and $h\cdot(\delta^i\#\eta^j)=v(\delta^i\#\eta^j)v^{-1}$. We prove it for $n=2q$; the odd case is proved similarly.

$$\begin{aligned}v^2&=\frac{1}{q^2}\sum_{i,j=0}^{q-1}\sum_{l,m=0}^{q-1}\omega^{-sa(i+l)-sb(j+m)+2sij+2slm+4sjl}\delta^{2(i+l)}\#\eta^{2(j+m)}\\&=\frac{1}{q^2}\sum_{r,t=0}^{q-1}\sum_{l,m=0}^{q-1}\omega^{-sar-sbt+2sr(t-m)+2slt}\delta^{2r}\#\eta^{2t}\\&=\frac{1}{q^2}\sum_{r,t=0}^{q-1}\omega^{-sar-sbt+2str}\left(\sum_{l,m=0}^{q-1}\omega^{-2srm+2stl}\right)\delta^{2r}\#\eta^{2t}\\&=1\#1.\end{aligned}$$

In order to prove that the h-action is conjugation by v we show that $v(\delta^i\#\eta^j)=\omega^{sai+sbj}(\delta^{-i}\#\eta^{-j})v$. We do so for the even case. The odd case is done similarly.

$$\begin{aligned}v(\delta^i\#\eta^j)&=\frac{1}{q}\sum_{l,m=0}^{q-1}\omega^{-sal-sbm+2slm+2sim}\delta^{2l+i}\#\eta^{2m+j}\\&=\frac{1}{q}\sum_{l'=i}^{q-1+i}\sum_{m'=j}^{q-1+j}\omega^{-sal'-sbm'+sai+sbj+2sl'm'-2sl'j}\delta^{2l'-i}\#\eta^{2m'-j}\\&=\omega^{sai+sbj}(\delta^{n-i}\#\eta^{n-j})\left(\frac{1}{q}\sum_{l'=0}^{q-1}\sum_{m'=0}^{q-1}\omega^{-sal'-sbm'+2sl'm'}\delta^{2l'}\#\eta^{2m'}\right)\\&=\omega^{sai+sbj}(\delta^{n-i}\#\eta^{n-j})v,\end{aligned}$$

where in the second equality the limits of the sums are reduced modulo q if necessary. Hence, for n odd, $[C_{\bar{0}}(1)]^2=1$ because $v^2=1\#1$ is a square in k. For $n=2q$ we still have to compute γ^q where γ is defined as usual. Using the commutation rules for v and $\delta^i\#\eta^j$ and the expression of powers of u we find that

$$vu^{n-1}=\zeta^{-3}v(\delta\#\eta^{n-1})=\zeta^{-3}\omega^{s(a-b)}(\delta^{n-1}\#\eta)v=\omega^{-2s}\omega^{s(a-b)}uv.$$

Thus $\gamma=\omega^{2s}\omega^{s(b-a)}$. Hence $\gamma^q=1$ if and only if $a=b$. It follows that $[C_{\bar{0}}(1)]^2=[C_{\bar{1}}(1)]^2=1$ while $[C_{\bar{0}}(1)\#C_{\bar{1}}(1)]=[C_{\bar{0}}(1)][C_{\bar{1}}(1)]\neq 1$. The algebra $C_{\bar{0}}(1)\#C_{\bar{1}}(1)$ is a matrix algebra with strongly inner g-action and so $[C_{\bar{0}}(1)\#C_{\bar{1}}(1)]$ is in the kernel of i^*.

Its image through the map Inv_e of Theorem 2.10 is $([1], -1)$. A similar argument applies to $[C_{\bar{1}}(1)\# C_{\bar{0}}(1)] = [C_{\bar{1}}(1)][C_{\bar{0}}(1)]$. Since Inv_e is injective, both classes coincide. □

COROLLARY 2.16. *Let k be algebraically closed, $n = 2^m$ and let z be odd. Then $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is generated by $[C_{\bar{0}}(1)]$ and $[C_{\bar{1}}(1)]$.*

Proof. By Lemma 2.15 the map i^* in sequence (4) is surjective and split. Hence $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $[C_{\bar{0}}(1)]$ and $[C_{\bar{1}}(1)]$. □

3. The map ι_*. In this section we study the Brauer group $BM(k, \mathbb{D}_n, R_z)$ when the field k is not necessarily algebraically closed. Let $\bar{k}$ denote the algebraic closure of k. The inclusion map $\iota : k \to \bar{k}$ induces a group homomorphism $\iota_* : BM(k, \mathbb{D}_n, R_z) \to BM(\bar{k}, \mathbb{D}_n, R_z)$, $[A] \mapsto [A \otimes_k \bar{k}]$. We describe the kernel of ι_*.

LEMMA 3.1. *If n is odd there is an exact sequence*

$$1 \longrightarrow BAz(k, \mathbb{D}_n, R_z) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} BM(\bar{k}, \mathbb{D}_n, R_z),$$

where $BAz(k, \mathbb{D}_n, R_z) = BAz^g(k, \mathbb{D}_n, R_z)$ is the set consisting of classes of $BM(k, \mathbb{D}_n, R_z)$ represented by classically Azumaya algebras.

If $n = 2q$ is even, then $Ker(\iota_*)$ consists of classically Azumaya algebras with α, γ in the induced subalgebra satisfying $\gamma^q\alpha^{q-1} = 1$.

Proof. The kernel of ι_* consists of classes of $\mathbb{D}_n$-Azumaya algebras $[A]$ such that $[A \otimes_k \bar{k}]$ becomes Brauer-trivial in $BM(\bar{k}, \mathbb{D}_n, R_z)$. Hence $A \otimes_k \bar{k}$ is a matrix algebra over $\bar{k}$ with strongly inner $\mathbb{D}_n$-action, and consequently, an Azumaya algebra over $\bar{k}$. But it is well known that A is Azumaya over k if and only if $A_{\bar{k}} = A \otimes_k \bar{k}$ is Azumaya over $\bar{k}$.

If n is odd, then $[A] \in BAz(k, \mathbb{D}_n, R_z)$. Conversely, for n odd and A a $\mathbb{D}_n$-Azumaya module algebra which is classically Azumaya, $A \otimes_k \bar{k}$ is Azumaya over $\bar{k}$. But the only Azumaya algebras over an algebraically closed field are matrix algebras. Moreover, from Proposition 2.2, the $\mathbb{D}_n$-action on $A \otimes_k \bar{k}$ is strongly inner since $\bar{k}$ is algebraically closed. Then $A \otimes_k \bar{k}$ is Brauer-trivial in $BM(\bar{k}, D_n, R_z)$ by Lemma 2.1.

If $n = 2q$ and $[A] \in Ker(\iota_*)$, then $A_{\bar{k}} = A \otimes_k \bar{k}$ is a matrix algebra over $\bar{k}$. So A is Azumaya over k. The induced subalgebra B on $A_{\bar{k}}$ is generated by u and v such that $u^n = \alpha$ and $uv = \gamma vu^{n-1}$ with $\alpha, \gamma \in \bar{k}$ satisfying $\gamma^q\alpha^{q-1} = 1$ by Proposition 2.2. On the other hand, $B = B' \otimes_k \bar{k}$ where B' is the induced subalgebra on A. Let u', v' be the generators of B'. The elements u, v in B must be scalar multiples of u', v'. If $u = tu'$ and $v = sv'$ for some $s, t \in \bar{k}$, then $\alpha' = t^n\alpha$ and $\gamma' = (t^{-1})^{n-2}\gamma$, so that

$$\gamma'^q\alpha'^{q-1} = (t^{2-n})^q\gamma^q(t^n)^{q-1}\alpha^{q-1} = (t^{q-1})^{2-n}t^{2-n}(t^{q-1})^{n-2}(t^{q-1})^2\gamma^q\alpha^{q-1} = \gamma^q\alpha^{q-1}.$$

Conversely, if A is a $\mathbb{D}_n$-Azumaya module algebra which is classically Azumaya and satisfying $\gamma^q\alpha^{q-1} = 1$, then $A \otimes_k \bar{k}$ is Brauer trivial in $BM(\bar{k}, \mathbb{D}_n, R_z)$ because $\bar{k}$ is algebraically closed. □

PROPOSITION 3.2. (i) *For n odd, $BAz^g(k, \mathbb{D}_n, R_z) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$.*
(ii) *For n even, $BAz^g(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$.*

Proof. We know from Corollary 2.8 that $BAz^g(k, \mathbb{D}_n, R_z)$ is abelian. The assignment $\tau : BAz^g(k, \mathbb{D}_n, R) \to Br(k)$ which maps $[A]$ into $[A]$ by forgetting the $\mathbb{D}_n$-action is a group homomorphism by Lemma 2.7. Moreover, any k-Azumaya algebra may

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

be endowed with the trivial $\mathbb{D}_n$-action becoming clearly $\mathbb{D}_n$-Azumaya. Thus the map so defined splits τ. Hence $BAz^g(k, \mathbb{D}_n, R_z) \cong Br(k) \times Ker(\tau)$. As in the proof of Theorem 2.10 we can show that $Ker(\tau) \cong k^\cdot/k^{\cdot 2}$ for n odd, and $Ker(\tau) \cong k^\cdot/k^{\cdot 2} \times \mathbb{Z}_2$ for n even. In both cases $Ker(\tau)$ is represented by the classes of the algebras $A(\beta, \gamma)$ for $\beta \in k^\cdot$ and γ an n-th root of unity. □

For $a, b \in k^\cdot$ let $\{a, b\}$ denote the quaternion algebra generated by x, y such that $x^2 = a$, $y^2 = b$ and $xy = -yx$. Since this algebra is also generated by x and $\theta^q bxy^{-1}$, we have that $\{a, b\} = \{a, ab\}$. When $b = 1$, $\{a, 1\}$ is a matrix algebra. For more details on these algebras see [**11**], [**13**, Section 15].

For any $t \in k^\cdot$ let $A(t)$ denote the $\mathbb{D}_n$-module algebra constructed in the following way: as an algebra $A(t) = M_2(k)$, and the $\mathbb{D}_n$-action is given by h acting trivially and g acting as conjugation by

$$u = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

Lemma 3.3. *With $A(t)$ as above and $n = 2q$ even, the following assertions hold.*

(i) *$A(t)$ is a $\mathbb{D}_n$-Azumaya module algebra.*

(ii) *$A(t) \cong A(tr^2)$ as $\mathbb{D}_n$-module algebras for any $r \in k^\cdot$.*

(iii) *If q is even, then $A(t)\#A(r) \cong M_2(k) \otimes A(tr)$ as $\mathbb{D}_n$-module algebras where $M_2(k)$ has trivial $\mathbb{D}_n$-action. If q is odd, then $A(t)\#A(r) \cong \{t, r\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras where $\{t, r\}$ has trivial $\mathbb{D}_n$-action;*

(iv) *$[A(t)]$ belongs to $Ker(\iota_*)$ and it has order two.*

Proof. (i) We show that $A(t)$ is a $\mathbb{Z}_n$-Azumaya algebra and so a $\mathbb{D}_n$-Azumaya algebra. We observe that since $u^2 = t$ and since z is odd in this case, the action of g^z is again conjugation by u. Therefore

$$g^z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & tc \\ t^{-1}b & a \end{pmatrix}.$$

There are only elements of degrees 0 and q in $A(t)$, so that $A(t)$ is in fact $\mathbb{Z}_2$-graded. The elements of degree 0 (even elements) and the elements of degree q (odd elements) are given by matrices of the form

$$\begin{pmatrix} a & tc \\ c & a \end{pmatrix}, \qquad \begin{pmatrix} a & -tc \\ c & -a \end{pmatrix},$$

respectively. It is easy to check that the graded center of $A(t)$ is k, and consequently, $A(t)$ is $\mathbb{Z}_n$-Azumaya.

(ii) The elements

$$x = \theta^q \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

are generators for $A(t)$. These satisfy $x^2 = t$, $y^2 = 1$, $xy = -yx$ and $g \cdot x = -x$, $g \cdot y = -y$. For $r \in k^\cdot$, the isomorphism of $\mathbb{D}_n$-module algebras from $A(t)$ to $A(tr^2)$ is given by mapping x into rx and y into y.

(iii) Let $M, M' \in A(t)$ and $N, N' \in A(r)$ be homogeneous. From (2),

$$(M\#N)(M'\#N') = MM'\#\big(g^{deg(M')} \cdot N\big)N'.$$

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

As we saw in (i), $deg(M')$ is equal to 0 or q. If q is even, then the action by g^q is trivial. Thus $A(t)\#A(s) = A(t) \otimes A(s)$. Let x, y be generators for $A(t)$ and x', y' generators for $A(r)$ as in (7). Let

$$X = x\#y',\ \ Y = y\#y',\ \ Z = 1\#y',\ \ W = \theta^q(xy\#x').$$

A computation shows that these elements satisfy the following relations:

$$\begin{array}{ll} X^2 = t,\ \ Y^2 = 1,\ \ XY = -YX, & Z^2 = 1,\ W^2 = tr,\ ZW = -WZ,\\ XZ = ZX,\ \ XW = WX, & YZ = ZY,\ \ YW = WY,\\ g \cdot X = X,\ \ g \cdot Y = Y, & g \cdot Z = -Z,\ \ g \cdot W = -W. \end{array}$$

This yields that $A(t) \otimes A(r) \cong \{t, 1\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras with $\{t, 1\}$ having trivial g-action. Since $\{t, 1\} \cong M_2(k)$ as algebras, the statement follows.

Assume now that q is odd. Then the action by g^q is the same as the action by g. Thus $g^q \cdot N = (-1)^{deg(N)}N$. The product takes the form

$$(M\#N)(M'\#N') = MM'\#(-1)^{deg(M')deg(N)}NN'. \tag{8}$$

Let $X = \theta^q(xy\#1),\ \ Y = \theta^q(x\#x'),\ \ Z = 1\#y'$ and $W = \theta^q(xy\#x')$. Using the formula (8), it may be checked that

$$\begin{array}{ll} X^2 = t,\ \ Y^2 = tr,\ \ XY = -YX, & Z^2 = 1,\ W^2 = tr,\ ZW = -WZ,\\ XZ = ZX,\ \ XW = WX, & YZ = ZY,\ \ YW = WY,\\ g \cdot X = X,\ \ g \cdot Y = Y, & g \cdot Z = -Z,\ \ g \cdot W = -W. \end{array}$$

From these relations, $A(t)\#A(r) \cong \{t, tr\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras. Notice now that $\{t, tr\} \cong \{t, r\}$ as algebras.

(iv) The elements $\alpha_{A(t)}, \beta_{A(t)}$, and $\gamma_{A(t)}$ of the induced subalgebra on $A(t)$ are $\alpha_{A(t)} = t^q$, $\beta_{A(t)} = 1$ and $\gamma_{A(t)} = t^{1-q}$. As $\gamma^q_{A(t)}\alpha^{q-1}_{A(t)} = 1$ and $A(t)$ is a matrix algebra, $[A(t)]$ belongs to $Ker(\iota_*)$.

The algebra $A(t)\#A(t)$ is classically Azumaya since it belongs to $Ker(\iota_*)$. Moreover, it has strongly inner $\mathbb{D}_n$-action. Note that $u_{A(t)\#A(t)} = u_{A(t)}\#u_{A(t)}$ and $v_{A(t)\#A(t)} = 1$ because $u_{A(t)}$ has degree 0 and the h-action is trivial on $A(t)$. From this, $\alpha_{A(t)\#A(t)} = t^n$, $\beta_{A(t)\#A(t)} = 1$ and $\gamma_{A(t)\#A(t)} = t^{2-n}$. If q is even, then $A(t)\#A(t) \cong M_2(k) \otimes A(t^2)$, and so $A(t)\#A(t)$ is a matrix algebra. If q is odd, then $A(t)\#A(t) \cong \{t, t^2\} \otimes A(t^2)$. But $\{t, t^2\} \cong \{t, 1\}$ and $\{t, 1\}$ is a matrix algebra. Hence, in this case also $A(t)\#A(t)$ is a matrix algebra. Finally, Lemma 2.1 implies that $[A(t)\#A(t)]$ is trivial. □

The map $\{-, -\} : k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \to Br(k),\ \ ([a], [b]) \mapsto [\{a, b\}]$ is a 2-cocycle, see [**13**, p. 146]. Let $k^{\cdot}/k^{\cdot 2} \times_{\{-,-\}} Br(k)$ denote the extension of $k^{\cdot}/k^{\cdot 2}$ and $Br(k)$ by this cocycle.

THEOREM 3.4. *With notation as above*

$$Ker(\iota_*) \cong \begin{cases} k^{\cdot}/k^{\cdot 2} \times Br(k) & \textit{for } n \textit{ odd},\\ k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) & \textit{for } n = 2q,\ \ q \textit{ even},\\ k^{\cdot}/k^{\cdot 2} \times (k^{\cdot}/k^{\cdot 2} \times_{\{-,-\}} Br(k)) & \textit{for } n = 2q,\ \ q \textit{ odd}. \end{cases}$$

Proof. For n odd, Lemma 3.1 and Corollary 2.8 establish that $Ker(\iota_*) = BAz^g(k, \mathbb{D}_n, R_z)$. Now Proposition 3.2 (i) applies. The case n even is more complicated and requires a different argument. The elements of $Ker(\iota_*)$ may all be represented by classically Azumaya algebras with α, γ in the induced subalgebra satisfying $\gamma^q\alpha^{q-1} = 1$, by Lemma 3.1.

Suppose that $n = 2q$ is even. Let $[A] \in Ker(\iota_*)$. Then A is classically Azumaya and the elements α_A, β_A, and γ_A in the induced subalgebra satisfy $\gamma_A^q\alpha_A^{q-1} = 1$. For $t_A = (\alpha_A\gamma_A)^{-1}$, the algebra $A\#A(t_A)$ represents an element of $Ker(\iota_*)$ because A and $A(t_A)$ do. Hence it is classically Azumaya. Moreover, it has strongly inner g-action because $u_{A\#A(t_A)} = u_A\#u_{A(t_A)}$ and $u^n_{A\#A(t_A)} = u^n_A\#u^n_{A(t_A)} = \alpha_A(\alpha_A\gamma_A)^{-q} = 1$ (the degree of u in the induced subalgebra is always zero). Thus $[A\#A(t_A)] \in BAz^g(k, \mathbb{D}_n, R_z)$. By Proposition 3.2,

$$[A\#A(t_A)] = [A(\beta, \gamma)][|A\#A(t_A)|] \in Ker(\iota_*),$$

where $[\beta] \in k^{\cdot}/k^{\cdot 2}$, γ is an n-th root of unity, and $|A\#A(t_A)|$ denotes the underlying algebra of $A\#A(t_A)$ with trivial action. By Lemma 2.9 we obtain $[\gamma] = [\gamma_{A\#A(t_A)}]$ and $\gamma^q = 1$. By the proof of Theorem 2.10, $[A(\beta, \gamma)] = [A(\beta, 1)]$ so we may assume that the g-action on the right hand side is trivial and that the braided product of the representative of elements of the right hand side with $A(t_A)$ is trivial. Hence

$$[A] = [A(\beta, 1) \otimes |A\#A(t_A)| \otimes A(t_A)],$$

where both representatives are classically Azumaya. By Lemma 2.9, $[\beta] = [\beta_A] \in k^{\cdot}/k^{\cdot 2}$. Thus the three classes $[A(\beta_A, 1)]$, $[A(t_A)]$ and $|A\#A(t_A)|$ are uniquely determined by $[A]$.

Assume that q is even. We prove that the map

$$\begin{aligned} \Psi : Ker(\iota_*) &\longrightarrow k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \\ [A] &\mapsto ([\beta_A], [(\alpha_A\gamma_A)^{-1}], [|A\#A((\alpha_A\gamma_A)^{-1})|]) \end{aligned}$$

is an isomorphism. We first check that it is well defined. Assume that $[A] = [B]$ in $Ker(\iota_*)$. Let $t_A = (\alpha_A\gamma_A)^{-1}$ and $t_B = (\alpha_B\gamma_B)^{-1}$. By Lemma 2.9 and Lemma 2.5, $[\beta_A] = [\beta_B]$ and $[t_A] = [t_B]$ in $k^{\cdot}/k^{\cdot 2}$. By Lemma 3.3 (ii), $A(t_A) \cong A(t_B)$. Then $[A\#A(t_A)] = [B\#A(t_B)]$ in $BM(k, \mathbb{D}_n, R_z)$. There are finite dimensional $\mathbb{D}_n$-modules P, Q such that

$$(A\#A(t_A))\#End(P) \cong (B\#A(t_B))\#End(Q)$$

as $\mathbb{D}_n$-module algebras. Since $End(P)$, $End(Q)$ are classically Azumaya with strongly inner g-action, from Lemma 2.7 it follows that

$$(A\#A(t_A)) \otimes End(P) \cong (B\#A(t_B)) \otimes End(Q)$$

as algebras. Hence $[|A\#A(t_A)|] = [|B\#A(t_B)|]$ in $Br(k)$. This proves that Ψ is well-defined. Secondly, we show that Ψ is a group homomorphism. Let $[A], [B] \in Ker(\iota_*)$ and assume that

$$\begin{aligned} &[A] = [A(\beta_A, 1)][|A\#A(t_A)|][A(t_A)], \quad [B] = [A(\beta_B, 1)][|B\#A(t_B)|][A(t_B)], \\ &[A\#B] = [A(\beta_{A\#B}, 1)][|(A\#B)\#A(t_{A\#B})|][A(t_{A\#B})]. \end{aligned}$$

Observe that when q is even $[A(t_A)]$ commutes with $[A(t_B)]$ in light of Lemma 3.3, $[A(t_A)]$ commutes with the elements $[A(\beta, 1)]$ and with the elements of $Br(k)$ since these

https://doi.org/10.1017/S0017089504001740 Published online by Cambridge University Press

have trivial g-action. This implies that $[B][A(t_A)] = [A(t_A)][B]$. Then

$$\begin{aligned}[A\#B][A(t_{A\#B})] &= [A][B][A(t_A)][A(t_B)] \\ &= [A][A(t_A)][B][A(t_B)] \\ &= [(A\#A(t_A))\#(B\#A(t_B))] \\ &= [(A\#A(t_A)) \otimes (B\#A(t_B))],\end{aligned}$$

where in the last equality we have used Lemma 2.7 since the g-action on $B\#A(t_B)$ is strongly inner. Hence

$$[|(A\#B)\#A(t_{A\#B})|] = [|(A\#A(t_A))| \otimes |(B\#A(t_B))|]$$

in $Br(k)$. Using all the preceding facts, we have

$$\begin{aligned}[A\#B] &= [A][B] \\ &= [A(\beta_A, 1)][|A\#A(t_A)|][A(t_A)][A(\beta_B, 1)][|B\#A(t_B)|][A(t_B)] \\ &= [A(\beta_A, 1)][A(\beta_B, 1)][|A\#A(t_A)|][|B\#A(t_B)|][A(t_A)][A(t_B)] \\ &= [A(\beta_A, 1)\#A(\beta_B, 1)][|A\#A(t_A)| \otimes |B\#A(t_B)|][A(t_A)\#A(t_B)] \\ &= [A(\beta_A\beta_B, 1)][|(A\#A(t_A))|][|(B\#A(t_B))|][A(t_At_B)], \end{aligned} \tag{9}$$

where in the last equality we have used Lemma 3.3 (iii) and Theorem 2.10. Finally we show that Ψ is bijective. It is clearly surjective since to any $([\beta], [\lambda], [D]) \in k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ we can associate $[A(\lambda^{-1}) \otimes A(\beta, 1) \otimes |D|] \in Ker(\iota_*)$. To prove the injectivity, let $[A] \in Ker(\Psi)$. Then β_A, t_A are squares and $|A\#A(t_A)|$ is a matrix algebra. Thus $[A] = [A(1, 1)][|M_m(k)|][A(s^2)]$ for some $m \in \mathbb{N}$ and $s \in k^{\cdot}$ such that $t_A = s^2$. Then $[A]$ is represented by a matrix algebra with strongly inner $\mathbb{D}_n$-action. Lemma 2.1 implies that $[A]$ is trivial.

For q odd, the same proof works but we have to modify the multiplication on $k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$. With notation as in (9), for q odd we have by Lemma 3.3 $A(t_A)\#A(t_B) \cong \{t_A, t_B\} \otimes A(t_At_B)$. Then

$$[|(A\#B)\#A(t_{A\#B})|] = [|(A\#A(t_A))| \otimes |(B\#A(t_B))| \otimes \{t_A, t_B\}].$$

Notice that $[B][A(t_A)] = [A(t_A)][B]$ is true in this case because $\{t_A, t_B\} \cong \{t_B, t_A\}$. □

THEOREM 3.5. *Let p be a prime number not dividing z, $m \in \mathbb{N}$, and $n = p^m$. Let k be a field containing a primitive 2n-th root of unity and let n be invertible in k. Then*

$$BM(k, \mathbb{D}_n, R_z) \cong \begin{cases} k^{\cdot}/k^{\cdot 2} \times Br(k) \times \mathbb{Z}_2 & \text{if } p \text{ is odd,} \\ k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

Proof. By Corollary 2.13, Corollary 2.16, Lemma 3.1 and Theorem 3.4 we have exact sequences

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times Br(k) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} \mathbb{Z}_2$$

for p odd and

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} \mathbb{Z}_2 \times \mathbb{Z}_2$$

for $p = 2$.

Let $C_{\bar{a}}(1)_{\overline{k}} = C_{\bar{a}}(1) \otimes_k \overline{k}$ for $a = 0, 1$. The nontrivial element of the latter term in the first exact sequence is represented by $C_{\bar{0}}(1)_{\overline{k}}$. The latter term in the second exact sequence is given by the group generated by $[C_{\bar{a}}(1)_{\overline{k}}]$ with $a = 0, 1$. Hence ι_* is surjective in both cases. Mapping $[C_{\bar{a}}(1)_{\overline{k}}]$ to $[C_{\bar{a}}(1)]$ we obtain a group homomorphism in light of Lemma 2.15, that splits ι_*. Then $BM(k, \mathbb{D}_n, R_z)$ is a semidirect product of $k^{\cdot}/k^{\cdot 2} \times Br(k)$ and $\mathbb{Z}_2$ for n odd and a semidirect product of $k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ for n even. If n is odd, since the elements representing $BAz(k, \mathbb{D}_n, R_z)$ have trivial g-action, the braided product of such an element and $C_{\bar{0}}(1)$ is just the usual tensor product. Thus the elements of $BAz(k, \mathbb{D}_n, R_z)$ commute with $[C(1)_{\bar{0}}]$ and we have the direct product decomposition for $BM(k, \mathbb{D}_n, R_z)$. If n is even the elements representing the first copy of $k^{\cdot}/k^{\cdot 2}$ and those representing $Br(k)$ have trivial g-action. Hence they commute with the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The second copy of $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(t)$ defined in the proof of Theorem 3.4, with $\mathbb{Z}_n$-grading inducing a $\mathbb{Z}_2$-grading, which we will denote by $\deg'$. Let δ be the generator of $C(1)$ and let $M, N \in A(t)$ with M homogeneous. By formula (2),

$$(\delta^i \# M)(\delta^j \# N) = \delta^{i+j} \# (g^{j \bmod 2} \cdot M)N = (-1)^{(j \bmod 2)\deg'(M)} \delta^{i+j} \# MN.$$

Thus $C_{\bar{a}}(1) \# A(t) \cong C_{\bar{a}}(1) \otimes_2 A(t)$. Here $\otimes_2$ denotes the $\mathbb{Z}_2$-graded tensor product. Similarly,

$$\begin{aligned}(M\#\delta^i)(N\#\delta^j) &= MN\#\left(g^{\deg(N)} \cdot \delta^i\right)\delta^j \\ &= \omega^{siq \deg'(N)} MN\#\delta^{i+j} \\ &= (-1)^{(i \bmod 2)\deg'(N)} MN\#\delta^{i+j}.\end{aligned}$$

Since $A(t) \otimes_2 C_{\bar{a}}(1) \cong C_{\bar{a}}(1) \otimes_2 A(t)$ as $\mathbb{D}_n$-module algebras, $[A(t)]$ commutes with $[C_{\bar{a}}(1)]$ for $a = 0, 1$. Therefore the kernel of ι_* commutes with $\mathbb{Z}_2 \times \mathbb{Z}_2$ and we are done. □

Acknowledgments. Most of this work was carried out while the second named author had a post-doc position at the University of Antwerp (UIA) supported by the *Noncommutative Geometry* (NOG) programme of the ESF. This work was completed during a visit of the first named author to UIA supported by the Progetto Giovani Ricercatori CPDG017575 of the Italian Ministero dell'Università e della Ricerca Scientifica and the University of Padua.

The authors thank Prof. Fred Van Oystaeyen and Y. H. Zhang for many helpful discussions, and Prof. A. Masuoka and M. Wakui for sending the reference [**21**].

REFERENCES

1. M. Beattie and S. Caenepeel, The Brauer-Long group of $\mathbb{Z}/\mathbb{Z}_{p^t}$-dimodule algebras, *J. Pure Appl. Algebra* **60** (1989), 219–236.

2. S. Caenepeel, F. Van Oystaeyen and Y. H. Zhang, Quantum Yang-Baxter module algebras, *K-theory* **8** (1994), 231–255.

3. S. Caenepeel, F. Van Oystaeyen and Y. H. Zhang, The Brauer group of Yetter-Drinfel'd module algebras, *Trans. Amer. Math. Soc.* **349** (1997), 3737–3771.

4. S. Caenepeel, *Brauer groups, Hopf algebras and Galois theory*, K-Monographs in Mathematics (Kluwer Academic Publishers, 1998).

5. L. N. Childs, G. Garfinkel and M. Orzech, The Brauer group of graded Azumaya algebras, *Trans. Amer. Math. Soc.* **175** (1973), 299–326.

6. L. N. Childs, The Brauer Group of graded Azumaya algebras II: graded Galois extensions, *Trans. Amer. Math. Soc.* **204** (1975), 137–160.

7. M. A. Knus, *Algebras graded by a group*, in *Category theory, homology theory and their applications* II, P. Hilton (Ed), Lecture Notes in Mathematics No. 92 (Springer-Verlag, 1969), 117–133.

8. L. A. Lambe and D. E. Radford, *Introduction to the quantum Yang-Baxter equation and quantum groups: an algebraic approach* (Kluwer Academic Publishers, 1997).

9. F. W. Long, A generalization of the Brauer group of graded algebras. *Proc. London Math. Soc. (3)* **29** (1974), 237–256.

10. F. W. Long, The Brauer group of dimodule algebras, *J. Algebra* **31** (1974), 559–601.

11. F. W. Long, Generalized Clifford algebras and dimodule algebras, *J. London Math. Soc. (2)* **13** (1976), 438–442.

12. A. Masuoka, Coalgebra actions on Azumaya algebras, *Tsukuba J. Math.* **14** (1990), 107–112.

13. J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies (Princeton University Press, 1971).

14. S. Montgomery, *Hopf algebras and their actions on rings*, CBMS No. 28 (Amer. Math. Soc., 1993).

15. M. Orzech, Brauer groups of graded algebras, in *Brauer groups (Proc. Conf., Northwestern Univ. Evanston, Illinois, 1975)*, Lecture Notes in Mathematics No. 549 (Springer-Verlag, 1976).

16. D. E. Radford, On Kauffman's knot invariants arising from finite dimensional Hopf algebras, in *Advances in Hopf algebras*, J. Bergen and S. Montgomery, Eds, Lecture Notes in Pure Appl. Math. No. 158 (Marcel-Dekker, 1994), 205–266.

17. M. E. Sweedler, *Hopf algebras* (Benjamin, 1969).

18. F. Van Oystaeyen and Y. H. Zhang, *Embedding the Hopf automorphism group into the Brauer group*, *Canad. Math. Bull.* **41** (1998), 359–367.

19. F. Van Oystaeyen and Y. H. Zhang, The Brauer group of a braided monoidal category, *J. Algebra* **202** (1998), 96–128.

20. C. T. C. Wall, Graded Brauer groups, *J. Reine Angew. Math.* **213** (1964), 187–199.

21. M. Wakui, On the quasi-triangular structures of the group Hopf algebra of a non-abelian group, preprint.