Proceedings of the Edinburgh Mathematical Society (2002) **45**, 547–548 ©
DOI:10.1017/S0013091502000287 Printed in the United Kingdom

A DIMENSION-DEPENDENT MAXIMAL INEQUALITY

ROMAN SHVIDKOY

Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA (shvidkoy@math.missouri.edu)

(Received 26 February 2002)

Abstract In this short note we show that $\sup\{\|M_\nu\| : \nu \text{ is a measure on } \mathbb{R}^n\}$, where $\|M_\nu\|$ denotes the centred Hardy–Littlewood maximal operator, depends exponentially on n.

Keywords: Hardy–Littlewood maximal operator; maximal inequality

AMS 2000 *Mathematics subject classification:* Primary 42B25

1. Statement of the problem

Let ν be a σ-finite measure on the Borel subsets of $\mathbb{R}^n$. Define the Hardy–Littlewood centred maximal operator associated with ν by

$$M_\nu f(x) = \sup_{r>0} \left(\frac{1}{\nu(B_r(x))} \right) \int_{B_r(x)} |f| \, \mathrm{d}\nu, \quad x \in \mathbb{R}^n.$$

It was proved in [**1**, **2**] that

$$\|M_\nu f\|_{L_p(\mathbb{R}^n,\nu)} \leqslant C \|f\|_{L_p(\mathbb{R}^n,\nu)}, \quad 1 < p < \infty,$$

where C does not depend on ν. We present a simple construction showing that C depends exponentially on n. This answers the question posed in [**2**, **3**].

2. Construction

Claim 2.1. *There is an absolute constant $\alpha > 1$ such that one can find $[\alpha^n]$ points $x_1, x_2, \dots, x_{[\alpha^n]}$ on the Euclidean sphere S^{n-1} such that*

$$\|x_i - x_j\| > 1, \quad i \neq j.$$

The maximal value of α is immaterial. A simple argument based on volume estimates yields $\alpha \geqslant \mathrm{e}^{(\pi/6)^2/2}$.

Let us fix $x_1, x_2, \dots, x_{[\alpha^n]}$ as in the claim and put

$$\nu = \delta_{\{0\}} + \sum_i \delta_{\{x_i\}}.$$

Define $f = \delta_{\{0\}}$. Then $\|f\|_{L_p(\mathbb{R}^n,\nu)} = 1$. On the other hand,

$$(M_\nu f)(x_i) \geqslant \frac{1}{\nu(B_1(x_i))} \int_{B_1(x_i)} |f| \, \mathrm{d}\nu = \tfrac{1}{2}, \quad i = 1, \dots, [\alpha^n].$$

Hence, $\|M_\nu f\|_{L_p(\mathbb{R}^n,\nu)} \geqslant \frac{1}{2}[\alpha^n]$.

This is the end of the construction.

References

1. R. Fefferman, Strong differentiation with respect to measures, *Am. J. Math.* **103** (1981), 33–40.
2. L. Grafakos and J. Kinnunen, Sharp inequalities for maximal functions associated with general measures, *Proc. R. Soc. Edinb.* A **128** (1998), 717–723.
3. I. E. Verbitsky, A dimension-free Carleson measure inequality, in *Complex analysis, operators, and related topics*, pp. 393–398 (Birkhäuser, Basel, 2000).

https://doi.org/10.1017/S0013091502000287 Published online by Cambridge University Press