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A CHARACTERISATION OF C^*-ALGEBRAS

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1. Introduction

It is of some interest to the theory of locally convex *-algebras to know under what conditions such an algebra A is a pre-C^*-algebra (the topology of A can be described by a submultiplicative norm such that $\|x^*x\| = \|x\|^2, \forall x \in A$). We recall that a locally convex *-algebra is a complex *-algebra A with the structure of a Hausdorff locally convex topological vector space such that the multiplication is separately continuous, and the involution is continuous.

Allan [3] has studied the problem for normed algebras, showing that a unital Banach *-algebra is a C^*-algebra if and only if A is symmetric and the set $\mathscr{B}$ of all absolutely convex hermitian idempotent closed bounded subsets of A has a maximal element. Recalling that an element x of a locally convex algebra A is bounded if there exists $\lambda > 0$ such that the set $\{(\lambda x)^n : n \in \mathbb{N}\}$ is bounded, we find the following simple generalisation of Allan's result:

Proposition 0. *A unital locally convex *-algebra A is a C^*-algebra if and only if:*

(a) *A is sequentially complete;*

(b) *A is barreled;*

(c) *every element of A is bounded;*

(d) *A is symmetric;*

(e) *the family $\mathscr{B}$ of absolutely convex hermitian idempotent closed bounded subsets of A has a maximal element B_0.*

Proof. For any $x = x^*$ in A, we can find $\lambda > 0$ such that the closed absolutely convex hull of the set $\{(\lambda x)^n : n \in \mathbb{N}\}$ is contained in B_0. Thus B_0 is absorbing, hence a barrel. Thus B_0 is a bounded neighbourhood of 0, so the Minkowski functional $\|\cdot\|$ of B_0 is a norm on A defining the original topology, and $(A, \|\cdot\|)$ is a Banach *-algebra satisfying the conditions of Theorem 2 of Allan [3]. □

Another interesting characterisation of C^*-algebras is to be found in the Vidav–Palmer Theorem (Bonsall and Duncan [4]), which states that a unital Banach algebra can be given an involution which turns it into a C^*-algebra if and only if $A = H(A) + iH(A)$, where

$$H(A)=\{h\in A: f(h)\in\mathbb{R}\,\forall f\in A' \text{ s.t. } f(e)=\|f\|=1\}. \tag{1}$$

Unlike that of Allan, this approach does not lend itself so well to generalisation to the non-normed case. However Wood [12] has proved an analogue of the Vidav–Palmer Theorem, which uses the same principle of numerical range to characterise a particular class of locally convex *-algebras (which contains all C^*-algebras), the so-called complete semi-GB^*-algebras with hypocontinuous involution. A semi-GB^*-algebra with continuous involution is a GB^*-algebra in the sense of Dixon [6]. Thus the Vidav–Palmer Theorem can be generalised to characterise GB^*-algebras, if not C^*-algebras.

In this paper we shall find a new characterisation of pre-C*-algebras in terms of properties of the positive elements A^+ of the locally convex *-algebra A and the continuous positive linear functionals $P(A)$ on A. A^+ will be defined to be the closed algebraic cone, as used by Alcantara and Dubin [1] and other quantum field theorists; we shall not need to use the spectral theory of Allan [2]. The main property that we shall use (in the unital case) is an order-boundedness property concerning the absorption of positive elements by the order interval $[0,e]$. We shall prove that a unital locally convex *-algebra with identity e is a C^*-algebra if and only if:

(a) A is sequentially complete;

(b) A is barreled;

(c) A^+ is a normal cone for A;

(d) for $x\in A$ there exists $\lambda>0$ such that $e-\lambda x^*x\in A^+$,

and a similar characterisation for non-unital algebras will also be found.

Thanks go to Dr. D. A. Dubin for some interesting ideas, and for advice with the terminology and notation.

2. Introduction of notation

Let (A,t) be a locally convex *-algebra. A natural candidate for the cone of positive elements of A is given by A^+—the closed algebraic cone generated by the elements $\{x^*x: x\in A\}$. It is clear that $A^+\subseteq A_h$, the set of hermitian elements of A, and that A_h is a real Hausdorff locally convex space with the induced topology.

If A' is the topological dual of (A,t), we define the positive functionals $P(A)$ to be

$$P(A)=\{f\in A': f(x^*x)\geqq 0\ \forall x\in A\}. \tag{2}$$

Applying the Cauchy–Schwarz inequality and the Hahn–Banach Theorem, we see that:

Lemma 1.

(a) $f(x^*y)=\overline{f(y^*x)}$ $\quad f\in P(A),\quad x,y\in A$;

(b) $|f(x^*y)|^2\leqq f(x^*x)f(y^*y)$ $\quad f\in P(A),\quad x,y\in A$;

(c) *if* $x\in A_h$, *then* $x\in A^+$ *if and only if* $f(x)\geqq 0$ *for all* $f\in P(A)$.

Corollary 2. *$P(A)=\{0\}$ if and only if $A^+=A_h$.*

If A is unital with identity e, then A^+ is generating, every element of $P(A)$ is hermitian, and

$$|f(x)|^2 \leqq f(e)f(x^*x) \qquad f\in P(A), \quad x\in A, \tag{3}$$

so that $P(A)=\{0\}$ if and only if $-e\in A^+$ (cf. Ky Fan [**8**, Theorem 1]). In this paper we shall at the least assume that $P(A)$ separates points of A. This is equivalent to saying that $P(A)$ spans a dense linear subspace of the weak dual A'_σ of A, or that A^+ is a proper cone in A.

Lemma 3. *If A^+ is a proper cone in A, then $x\in A_h$ if and only if $f(x)\in\mathbb{R}$ for all $f\in P(A)$. Also $x^*x=0$ implies $x=0$.*

It will also be necessary sometimes to assume that $P(A)$ is generating. When (A,t) is barreled, this is equivalent to saying that A^+ is a normal cone in A (Schaefer [**11**, V. 3.4]).

3. Infrabarreled spaces

As in the proof of Proposition 0, we shall need to include a property akin to barreledness to characterise pre-C^*-algebras. However, although every Fréchet space (and hence every C^*-algebra) is barreled (Schaefer [**11**, II.7.1]), not every metrisable locally convex space is. For example (Schaefer [**11**, p. 70, Ex. 14]), if we let X denote the subspace of $C[0,1]$ consisting of functions f which vanish on a neighbourhood (depending on f) of 0, then X with the uniform norm is a pre-C^*-algebra which is not barreled. Thus, when characterising pre-C^*-algebras, barreledness is too strong a property.

Let X be a locally convex space. A barrel in X is called bound-absorbing if it absorbs every bounded subset of X, and we recall that X is called infrabarreled if every bound-absorbing barrel is a neighbourhood of 0. Every bornological space (and hence every pre-C^*-algebra) is infrabarreled. The relationship between bounded sets and null sequences found in I.5.3 of Schaefer [**11**] enables us to simplify the criterion for infrabarreledness to a form which we shall find more useful. We shall say that a barrel is null-absorbing if it absorbs every null sequence.

Lemma 4. *A barrel is null-absorbing if and only if it is bound-absorbing.*

Proof. If U is a barrel in X which is not bound-absorbing, let B be a bounded set not absorbed by U. Thus we can find a sequence (x_n) in B such that $x_n\notin n^2U$ for all $n\in\mathbb{N}$. Thus $(n^{-1}x_n)$ is a null sequence. If $n^{-1}x_n\in\lambda U$ for all $n\in\mathbb{N}$ and some $\lambda>0$, it would follow that $x_n\in n^2U$ for all $n\geqq\lambda$. This contradiction implies that $(n^{-1}x_n)$ is not absorbed by U, and so U is not null-absorbing. The converse follows since every null sequence is bounded. □

Corollary 5. *A locally convex space X is infrabarreled if and only if every null-absorbing barrel is a neighbourhood of* 0.

4. Properties of pre-C^*-algebras

In this section we shall list some properties which are common to all pre-C^*-algebras. In the next section we shall prove that these properties in fact characterise pre-C^*-algebras.

Let A be a pre-C^*-algebra, and let B be its C^*-algebra completion. If we define the positive elements B^+ and the positive linear functionals $P(B)$ of B as above, then $P(B)$ is generating (Sakai [**10**, Proposition 1.17.1]), so that B^+ is a normal cone in B, and:

Lemma 6.

(a) $A^+ = B^+ \cap A$;

(b) *if* $F \in P(B)$, *then* $F|_A \in P(A)$;

(c) *if* $f \in P(A)$, *we can find a unique element* $F \in P(B)$ *which extends* f.

Corollary 7. *$P(A)$ is generating.*

Finally, if A is unital with identity e, we obtain the following result:

Proposition 8. *If (x_n) is a null sequence in A, we can find $\lambda > 0$ such that $e - \lambda x_n^* x_n \in A^+$ for all $n \in \mathbb{N}$.*

Proof. We can find $K > 0$ such that $\|x_n^* x_n\| = \|x_n\|^2 \leqq K$ for all n, and hence $e - K^{-1} x_n^* x_n$ belongs to $B^+ \cap A = A^+$ for all n. □

Thus, if A is a unital pre-C^*-algebra, then A is a unital locally convex $*$-algebra such that:

(A) A is infrabarreled;

(B) $P(A)$ is generating;

(C) for any null sequence (x_n) in A, we can find $\lambda > 0$ such that $e - \lambda x_n^* x_n \in A^+$ for all $n \in \mathbb{N}$.

5. A characterisation of pre-C^*-algebras

We shall now show that the properties (A), (B), (C) characterise the pre-C^*-algebras over all unital locally convex $*$-algebras. Initially, however, we begin by weakening property (B). Let us therefore assume that (A, t) is a unital locally convex $*$-algebra which satisfies properties (A), (C) and

(B′) A^+ is a proper cone in A.

We need to find a bounded neighbourhood of 0 in A whose Minkowski functional defines a C^*-algebra norm on A. To this end we define the set

$$V = \{x \in A : e - x^* x \in A^+\}. \tag{4}$$

Theorem 9. *V is an idempotent barrel in A.*

Proof. For any $x \in A$, considering the null sequence (x_n) defined by $x_n = n^{-1}x$ shows us that V is absorbing. An elementary application of the Cauchy–Schwarz inequality (Lassner [9]) and Lemma 1 shows that

$$V = \bigcap_{f \in P(A)} \bigcap_{y \in A} \{x \in A : |f(y^*x)|^2 \leqq f(y^*y)f(e)\},$$

and hence V is absolutely convex and closed. Thus V is a barrel.

For any $y \in A$ and $f \in P(A)$ we can define $f_y \in P(A)$ by $f_y(x) = f(y^*xy)$ $(x \in A)$. If $x, y \in V$, then for any $f \in P(A)$ we have $f((xy)^*(xy)) = f_y(x^*x) \leqq f_y(e) = f(y^*y) \leqq f(e)$, so that $e - (xy)^*(xy) \in A^+$, and hence $xy \in V$. Thus V is idempotent. □

Corollary 10. *V is a neighbourhood of* 0 *in A, and so the Minkowski functional $\|x\| = \inf\{\lambda > 0 \colon x \in \lambda V\}$ $(x \in A)$ of V is a continuous submultiplicative seminorm on A.*

Proof. Condition (C) states precisely that V is null-absorbing. □

Proposition 11. *If $x \in A$ and $\lambda \geqq 0$, then $f(x^*x) \leqq \lambda^2 f(e)$ for all $f \in P(A)$ if and only if $\lambda \geqq \|x\|$. Thus*

$$|f(x)| \leqq f(e)\|x\| \qquad f \in P(A), \quad x \in A, \tag{5}$$

and so $\|\cdot\|$ is a norm on A, and every element of $P(A)$ is a continuous linear functional on the normed space $(A, \|\cdot\|)$.

Proof. $f(x^*x) \leqq \lambda^2 f(e)\, \forall f \in P(A) \Leftrightarrow f(x^*x) \leqq (\lambda+\varepsilon)^2 f(e)\, \forall f \in P(A), \forall \varepsilon > 0$

$$\Leftrightarrow e - [(\lambda+\varepsilon)^{-1}x]^*[(\lambda+\varepsilon)^{-1}x] \in A^+ \; \forall \varepsilon > 0$$

$$\Leftrightarrow x \in (\lambda+\varepsilon)V \; \forall \varepsilon > 0$$

$$\Leftrightarrow \|x\| \leqq \lambda + \varepsilon \Leftrightarrow \|x\| \leqq \lambda$$

(using Lemma 1). Thus $f(x^*x) \leqq \|x\|^2 f(e)$ for all $f \in P(A)$ and $x \in A$, and so (3) yields (5). Since A^+ is a proper cone, (5) implies that $\|\cdot\|$ is a norm. □

Proposition 12. $\|x^*x\| = \|x\|^2$ *for all $x \in A$, so that $(A, \|\cdot\|)$ is a pre-C^*-algebra.*

Proof. For any $f \in P(A)$ we see that

$$f((x^*x)^*(x^*x))^2 = f(x^*xx^*x)^2 \leqq f(x^*x)f(x^*xx^*xx^*x) \leqq \|x\|^2 f(e) f_{x^*x}(x^*x)$$

$$\leqq \|x\|^4 f(e) f_{x^*x}(e) = \|x\|^4 f(e) f(x^*xx^*x).$$

Thus $f((x^*x)^*(x^*x)) \leqq \|x\|^4 f(e)$ for all $f \in P(A)$, and so $\|x^*x\| \leqq \|x\|^2$. But (5) implies that $f(x^*x) \leqq \|x^*x\| f(e)$ for all $f \in P(A)$, so that $\|x\|^2 \leqq \|x^*x\|$. □

Thus, if (A,t) is a unital locally convex *-algebra satisfying (A), (B′), (C), then we can define a pre-C^*-algebra topology T on A which is coarser than t. Replacing (B′) by (B) enables us to sharpen the result.

Proposition 13. *If* (A,t) *satisfies properties* (A),(B),(C), *then* $T=t$, *so that* (A,t) *is a pre-*C^**-algebra.*

Proof. Since $t \geqq T$, every element of the norm dual $A^{\sim}$ of (A,T) belongs to A'. But $P(A)$ is generating, and every element of $P(A)$ belongs to $A^{\sim}$. Thus $A^{\sim}=A'$, and so $\tau(A,A') \geqq t \geqq T \geqq \sigma(A,A')$. Since V is the closed unit ball of (A,T), it is T-bounded, and hence t-bounded (Schaefer [**11**, IV.3.3]). Thus $T \geqq t$, and so $T=t$. □

Hence, summarising the results of the last two sections, we see that:

Theorem 14. *If A is a unital locally convex *-algebra, then A is a pre-*C^**-algebra if and only if it satisfies properties* (A),(B),(C).

The order-boundedness property (C) is fairly complicated. We might like to simplify it by replacing (C) by the property

(C′) for any $x \in A$, we can find $\lambda > 0$ such that $e - \lambda x^* x \in A^+$.

Examination of the proof of Theorem 9 shows that V is still an idempotent barrel, but is no longer necessarily null-absorbing. Thus, if we wish to replace property (C) by property (C′), we need to strengthen property (A).

Theorem 15. *If* (A,t) *is a unital locally convex *-algebra such that:*

(A′) *A is barreled;*

(B′) *A^+ is a proper cone in A;*

(C′) *for any $x \in A$ we can find $\lambda > 0$ such that $e - \lambda x^* x \in A^+$,*

then we can find a pre-C^**-algebra topology T on A which is coarser than t.*

Corollary 16. *A unital locally convex *-algebra (A,t) is a barreled pre-*C^**-algebra if and only if it satisfies properties* (A′),(B),(C′).

6. Algebras without identity

It would be useful to generalise the results of Section 5 to cover the case of algebras without identity. Evidently, property (C) would have to be changed, as it is explicitly dependent on an identity element e.

If A is a locally convex *-algebra without identity, we can form the unital algebra $A_e = A \oplus \mathbb{C}e$ in the usual way (Allan [**2**]), giving it the product topology. Our first important observation is that the property of infrabarreledness transfers from A to A_e. The proof is straightforward.

Proposition 17. *If A is infrabarreled, so is A_e.*

In order to relate $P(A)$ *to* $P(A_e)$, we recall (Hewitt and Ross [7]) that an element f of $P(A)$ is called extendable if it is hermitian and there exists $a \geqq 0$ such that

$$|f(x)|^2 \leqq af(x^*x) \qquad x \in A. \tag{6}$$

If $f \in P(A)$ is extendable we define

$$N(f) = \inf\{a \geqq 0 : |f(x)|^2 \leqq af(x^*x)\ \forall x \in A\}, \tag{7}$$

and notice that $f = 0$ if and only if $N(f) = 0$. It is well-known that if $f \in P(A)$, then there exists $F \in P(A_e)$ which extends f if and only if f is extendable. In this case we must have $F(e) \geqq N(f)$.

Let us now assume that every element of $P(A)$ is extendable.

Proposition 18. *If* $P(A)$ *is generating, so is* $P(A_e)$.

Proof. If $F \in A'_e$, then $F|_A \in A'$, so we can find g in the linear span of $P(A)$ which equals $F|_A$. Thus we can find G in the linear span of $P(A_e)$ such that $F|_A = g = G|_A$. Thus $F - G \in A'_e$ must be of the form $(F-G)(x+\lambda e) = \lambda\mu$ $(x \in A, \lambda \in \mathbb{C})$ for some $\mu \in \mathbb{C}$. For any $\alpha \geqq 0$ the element G_α of A'_e defined by $G_\alpha(x+\lambda e) = \alpha\lambda$ belongs to $P(A_e)$. Thus $F - G$, and hence F, belongs to the linear span of $P(A_e)$. □

If we introduce the following generalisations of properties (C) and (C′):

(GC) for any null sequence (x_n) in A we can find $\lambda > 0$ such that $\lambda f(x_n^* x_n) \leqq N(f)$ for all $n \in \mathbb{N}$ and $f \in P(A)$;

(GC′) for any $x \in A$ we can find $\lambda > 0$ such that $\lambda f(x^*x) \leqq N(f)$ for all $f \in P(A)$,

then simple calculations now show that:

Proposition 19. *If* A *satisfies* (GC), *then* A_e *satisfies* (C). *If* A *satisfies* (GC′), *then* A_e *satisfies* (C′).

Consequently, the results of Section 5 may be appealed to.

Theorem 20. *If* (A, t) *is a locally convex* **-algebra such that every element of* $P(A)$ *is extendable, then* A *is a pre-*C^**-algebra if and only if it satisfies properties* (A),(B),(GC), *and* A *is a barreled pre-*C^**-algebra if and only if it satisfies properties* (A′),(B),(GC′).

Finally, let us consider under what circumstances every element of $P(A)$ is extendable. Let us suppose that the algebra A possesses an approximate identity (e_α). We say that (e_α) is C^*-bounded if the net $(e_\alpha^* e_\alpha)$ is bounded. If A is a pre-C^*-algebra, then bounded and C^*-bounded approximate identities are the same.

Proposition 21. *If* A *possesses a* C^**-bounded approximate identity, or if* A *is barreled and possesses a bounded approximate identity, then every element of* $P(A)$ *is extendable.*

Proof. Let (e_α) be the approximate identity for A. Since $f(e_\alpha^* x) = \overline{f(x^* e_\alpha)}$ for all $f \in P(A)$, $x \in A$ and all α, it follows that every element of $P(A)$ is hermitian. For any

$f \in P(A)$, the net $(f(e_\alpha^* e_\alpha))$ is bounded. This is obvious if (e_α) is C^*-bounded. If A is barreled and (e_α) is bounded, it follows from the fact that the map $x \mapsto f(x^*x)^{1/2}$ is a continuous seminorm on A (Lassner [9]). Thus we can always find $a \geqq 0$ such that $f(e_\alpha^* e_\alpha) \leqq a$ for all α, and so $|f(e_\alpha^* x)|^2 \leqq f(e_\alpha^* e_\alpha) f(x^*x) \leqq af(x^*x)$ for all $x \in A$ and all α. Taking limits, we deduce that f is extendable. □

If A is a C^*-algebra, then A possesses a bounded approximate identity (Dixmier [5], 1.7.2). Thus we have the following results.

Theorem 22. *If A is a locally convex $*$-algebra such that:*

(a) *A is infrabarreled;*

(b) *$P(A)$ is generating;*

(c) *A has a C^*-bounded approximate identity;*

(d) *for every null sequence (x_n) in A we can find $\lambda > 0$ such that $\lambda f(x_n^* x_n) \leqq N(f)$ for all $f \in P(A)$ and $n \in \mathbb{N}$.*

then A is a pre-C^*-algebra.

Theorem 23. *A locally convex $*$-algebra A is a C^*-algebra if and only if:*

(a) *A is sequentially complete;*

(b) *A is barreled;*

(c) *A^+ is a normal cone in A;*

(d) *A has a bounded approximate identity;*

(e) *for any $x \in A$ we can find $\lambda > 0$ such that $\lambda f(x^*x) \leqq N(f)$ for all $f \in P(A)$.*

The version of Theorem 23 for unital algebras is as follows:

Theorem 24. *A unital locally convex $*$-algebra A is a C^*-algebra if and only if:*

(a) *A is sequentially complete;*

(b) *A is barreled;*

(c) *A^+ is a normal cone in A;*

(d) *for any $x \in A$ we can find $\lambda > 0$ such that $e - \lambda x^* x \in A^+$.*

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