Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1173–1185

QUASI-INJECTIVE AND QUASI-PROJECTIVE MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS

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The structure theory of hereditary noetherian prime (hnp) rings—in particular of Dedekind prime rings—has been recently developed by many authors including Eisenbud, Griffith, Michler and Robson; this theory extends some of the well-known results concerning commutative Dedekind domains. In this paper we study quasi-injective modules and quasi-projective modules over those (hnp) rings which are not right primitive and establish some results which extend the corresponding well-known results concerning commutative Dedekind domains. Let R be an (hnp) ring, which is not right primitive. In section 3, we firstly determine the structure of a generalized uniserial ring with homogeneous socle (Theorem 2); this theorem generalizes [**15**, Theorem 15]. With the help of Theorem 2, the structure of an indecomposable injective torsion right R-module is determined in Theorem 4. Theorem 6 gives a sufficient condition for the existence of a proper ideal A of R such that the generalized uniserial ring R/A has homogeneous right socle. Michler, in [**12**; **13**] determined the structure of a complete semi-perfect, hereditary noetherian prime ring. This structure is used to prove the following result, which generalizes the corresponding result due to Rangaswamy and Vanaja for Dedekind domains [**18**]: Let R be an (hnp) ring which is not right primitive and Q be its classical quotient ring. Then Q is quasi-projective right R-module if and only if $R = D_n$, where n is some positive integer and D is a local complete Dedekind domain (not necessarily commutative); further, in this case R is a Dedekind prime ring having $J(R)$ as its maximal ideal and Q is quasi-projective as a left R-module (Theorem 8).

2. Preliminaries. All rings considered here are associative, contain unity $1 \neq 0$ and all modules are unital. For definitions and basic properties of quasi injective modules and quasi projective modules, we refer to [**9**] and [**20**] respectively. A prime ring R which is left noetherian, left hereditary and right noetherian, right hereditary is called an (hnp) ring. An (hnp) ring with no idempotent proper ideal is called a Dedekind prime ring. For basic properties of these rings we refer to [**2**] and [**3**]. Since an (hpn) ring R satisfies Goldie's conditions on left as well as on right, it has a classical quotient ring Q which is simple artinian; further, any one sided ideal of R is essential in R if and only if

Received March 28, 1973 and in revised form, September 6, 1973. This research was mostly done during the author's stay at Bedford College, London under the Younger Scientist Exchange Programme between the U.K. and India.

it contains a regular element [**7**]. A ring R which satisfies the minimum condition on both sides is said to be a generalized uniserial ring if for every primitive idempotent e of R the right (left) ideal eR (Re) has unique composition series; such rings are called serial rings by Eisenbud and Griffith [**1**]. A generalized uniserial ring which is a direct sum of primary rings is called a uniserial ring [**5**]. A ring R is called a principle ideal ring (PIR) if each of its left ideals is principal and each of its right ideals is principal [**8**]. Artinian PIR are precisely uniserial rings; this follows from [**8**, Chapter 4, Theorems 37 and 40] and the fact that every completely primary uniserial ring is a PIR and every uniserial ring is a finite direct sum of matrix rings over such rings. In a right R-module M, an element x is said to be a torsion element if $xa = 0$ for some regular element a of R; a module whose every element is a torsion element, is called a torsion module. For any module M, $E(M)$ will denote its injective hull and for any ring R, $J(R)$ will denote its Jacobson radical. A ring R is said to be a local ring if $R/J(R)$ is a division ring.

3. Generalized uniserial rings and quasi-injective modules. A module X is said to be uniserial if it has a unique composition series of finite length [**1**]. The following generalization of the Nakayama's Theorem was established by Eisenbud and Griffith [**1**, Theorem 17].

THEOREM 1. *Let R be a generalized uniserial ring. Then every R-module is a direct sum of uniserial modules.*

Let X be a uniserial, right R-module, where R is a generalized uniserial ring. Let

(1) $X = X_0 > X_1 > X_2 > \ldots > X_m = (0)$

be the composition series of X. If there exists a positive integer n such that the ith and jth composition factors of (1) are isomorphic if and only if $i \equiv j \pmod{n}$, we say that X is of periodicity n. Trivially, if all the composition factors of (1) are pairwise, non-isomorphic, then for any $n \geqq m$, we can say that X is of periodicity n. Now we establish the following:

THEOREM 2. *Let R be an indecomposable generalized uniserial ring. Then the right socle of R is homogeneous if and only if it has a Kupisch series $e_1R, e_2R, \ldots, e_nR$ such that $d(e_{i+1}R) = d(e_iR) + 1$ for $i < n$ (here for any module X, $d(X)$ denotes its length); further, if this condition holds, then every indecomposable right R-module is of periodicity n.*

Proof. By Kupisch [**10**] and Murase [**15**, Theorem 9] we can find orthogonal primitive idempotent $e_i (i = 1, 2, \ldots, n)$ of R, such that for any primitive idempotent e of R ,eR (Re) is isomorphic to one and only one of e_iR (Re_i) and further for $N = J(R)$,

(2) $d(e_iR) \geqq 2 \quad \text{for } i \geqq 2$

$$e_iR/e_iN \cong e_{i+1}N/e_{i+1}N^2 \quad \text{for } i < n,$$

and

(3) $e_nR/e_nN \cong e_1N/e_1N^2$ if $e_1N \neq (0)$

$$d(e_{i+1}R) \leqq d(e_iR) + 1 \text{ for } i < n$$

and

(4) $d(e_1R) \leqq d(e_nR) + 1.$

A series $e_1R, e_2R, \ldots, e_nR$ satisfying the above conditions is called a Kupisch series of R, of length n. Let $\rho_i = d(e_iR)$, then the composition series of e_iR is

$$e_iR > e_iN > e_iN^2 > \ldots > e_iN^{\rho_i - 1} > (0)$$

[**15**, p. 3] where N is the radical of R. If ρ is the index of nilpotency of N, then $\rho = \max(\rho_i)$.

Let R have a homogeneous right socle. Consider the case when $e_1N = (0)$. Then by [**15**, Theorem 15], $\rho = n$; (3) and (4) yield that $d(e_iR) = i$ for any i; so that $\rho_{i+1} = \rho_i + 1$ for every $i < n$.

Consider the case when $e_1N \neq (0)$. In this case it can be easily seen that any sequence got by a cyclic rotation of $e_1R, \ldots, e_nR$ is again a Kupisch series of R. Since the right socle of R is homogeneous, there exists $k \leqq n$ such that every minimal right ideal of R is isomorphic to e_kR/e_kN. By a cyclic rotation of $e_1R, e_2R, \ldots, e_nR$, we can take $k = n$; so that every minimal right ideal of R is isomorphic to e_nR/e_nN. In particular the minimal right ideal $e_nN^{\rho_n - 1} \cong e_nR/e_nN$; so that we can find smallest positive integer α, such that $e_nR/e_nN \cong e_nN^\alpha/e_nN^{\alpha+1}$. By the periodicity theorem of Eisenbud and Griffith [**1**, Theorem (2.3)], for any i and $j \leqq \rho_n - 1$, $e_nN^i/e_nN^{i+1} \cong e_nN^j/e_nN^{j+1}$ if and only if $i \equiv j \pmod{\alpha}$. By (3) and [**15**, Theorem 5], given any $i < n$, and $\beta \geqq 0$,

$$e_iN^\beta/e_iN^{\beta+1} \cong e_{i+1}N^{\beta+1}/e_{i+1}N^{\beta+2}$$

whenever $e_{i+1}N^{\beta+1} \neq (0)$ and $e_nN^\beta/e_nN^{\beta+1} \cong e_1N^{\beta+1}/e_1N^{\beta+2}$, whenever $e_1N^{\beta+1} \neq (0)$. Thus if $\alpha < n$, then $e_nN^\alpha/e_nN^{\alpha+1} \cong e_{n-\alpha}R/e_{n-\alpha}N$; hence $e_nR \cong e_{n-\alpha}R$, which is a contradiction. If $\alpha > n$, then

$$e_nN^\alpha/e_nN^{\alpha+1} \cong e_1N^{\alpha-(n-1)}/e_1N^{\alpha-n+2} \cong e_nN^{\alpha-n}/e_nN^{\alpha-n+1};$$

the minimality of α and the fact that $\alpha - n < \alpha$ yields a contradiction. Hence $\alpha = n$. Thus the periodicity of e_nR is n. Since $e_nN^n/e_nN^{n+1} \cong e_nR/e_nN \cong e_1N^{\rho_1 - 1} \cong e_nN^{\rho_1 - 2}/e_nN^{\rho_1 - 1}$, we get $\rho_1 - 2 \equiv 0 \pmod{n}$, so that $\rho_1 = k_1n + 2$ for some integer k_1. In general $\rho_i = k_in + i + 1$ for $1 \leqq i \leqq n$. For $i < n$, since $\rho_{i+1} \leqq \rho_i + 1$, we get $k_{i+1} \leqq k_i$. Since $\rho_1 \leqq \rho_n + 1$, we get $k_1n + 2 \leqq k_nn + n + 2$, i.e., $k_1 \leqq k_n + 1$. Hence

(5) $k_1 \geqq k_2 \geqq \ldots \geqq k_n \geqq k_1 - 1.$

Since the first and the last term differ only by one, there exists some $j \leqq n$ such that $k_i = k_1$ for $i \leqq j$ and $k_i = k_1 - 1$ for $i > j$. If $j = n$, then obviously

https://doi.org/10.4153/CJM-1974-110-5 Published online by Cambridge University Press

$\rho_{i+1} = \rho_i + 1$ for every $i < n$. Let $j < n$. By putting $f_1 = e_{j+1}, f_2 = e_{j+2} \ldots f_{n-j} = e_n, f_{n-j+1} = e_1, \ldots, f_n = e_j$ and by using (5) it follows immediately that

$$f_1R, \ldots, f_2R, \ldots, f_nR$$

is a Kupisch series of R such that $d(f_{i+1}R) = d(f_iR) + 1$ for every $i < n$.

Conversely, let R have a Kupisch series $e_1R, e_2R, \ldots, e_nR$ satisfying the given conditions. In general if $\rho_{i+1} = \rho_i + 1$ then $e_{i+1}N \cong e_iR$; and in that case the minimal right subideal of e_iR is isomorphic to that of $e_{i+1}R$. Consequently under our hypothesis all e_iR have isomorphic minimal right subideals. Hence the right socle of R is homogeneous.

Let X be a uniserial right R-module. If X is of periodicity m then every submodule and every factor module of X is also of periodicity m. It is clear from the above, that every $e_iR(i \leqq n)$ is isomorphic to a submodule of e_nR. Since e_nR is of periodicity n, we get that every e_iR is of periodicity n. As X/XN is an irreducible right R-module, $X/XN \cong e_iR/e_iN$ for some i; then by [**15**, p. 3] there exists $x \in X$ such that

$$x.\, e_iR > xe_iN > .xe_iN^2 > \ldots > x.\, e_iN^s = (0)$$

for some $s \leqq \rho_i$, is a composition series of X. Hence X is also of periodicity n.

A ring R is said to be right (left) bounded if every right (left) ideal of R containing a regular element contains a nonzero (two sided) ideal of R. The following theorem which we state without proof was proved by Eisenbud and Robson [**3**, Theorem 4.10].

THEOREM 3. *Let R be a hereditary noetherian prime ring. Then R is a right primitive or right bounded and is both if and only if R is simple artinian.*

Henceforth R will denote an (hnp) ring which is not right primitive, unless otherwise stated. By Theorem 3, R is right bounded. Let Q be the classical quotient ring of R, which we know is simple artinian. Since by Matlis [**11**] every injective right R-module is a direct sum of indecomposable injective right R-modules, to determine the structure of an injective right R-module it is enough to determine the structure of an indecomposable injective right R-module.

LEMMA 1. *Any indecomposable injective torsion free right R-module E is isomorphic to a minimal right ideal of Q.*

Proof. Since E is divisible and torsion free, E is a right Q-module. As Q is simple artinian the lemma follows.

Hence it only remains to determine the structure of an indecomposable injective right R-module which is not torsion free.

LEMMA 2(i). *If in a right R-module M, an element x is a torsion element, the xR is a torsion submodule with non-zero annihilator.*

(ii) *Any finitely generated, torsion, right R-module has nonzero annihilators.*

(iii) *If an indecomposable, injective right R-module E is not torsion free, then it is a torsion module.*

Proof. (i) Let $A = \operatorname{ann}_R(x)$. Since x is a torsion element, the right ideal A contains a regular element. Thus A is an essential right ideal of R. As R is right bounded, there exists a nonzero ideal B of R contained in A. Then $xRB = (0)$. This yields that xR is a torsion module with non-zero annihilator.

(ii) is an immediate consequence of (i).

(iii) Since E is not torsion free it has an element $x \neq 0$, which is a torsion element. Since xR is essential in E, every essential right ideal of R contains a regular element and by (i) xR is a torsion module, we get that E is a torsion module.

We now establish a theorem which generalizes the corresponding well known result for Dedekind's domains.

THEOREM 4. *Let R be an (hnp)-ring which is not right primitive. Let E be an indecomposable injective right R-module, which is not torsion free. Then E has an infinite properly ascending chain of sbumodules*

(6) $0 = x_0R < x_1R < x_2R < \ldots < x_nR < \ldots$

whose union is E such that

(i) *each* $x_{i+1}R/x_iR$ *is a simple R-module;*

(ii) *the members of the chain are the only submodules of E different from E; and*

(iii) *either all* $x_{i+1}R/x_iR$ *are pairwise non-isomorphic or there exists a positive integer n such that for any i, j,* $x_{i+1}R/x_iR \cong x_{j+1}R/x_jR$ *if and only if* $i \equiv j$ (mod n).

Proof. By Lemma 2, E is a torsion module. Consider $x \neq 0$ and $y \neq 0$ in E. Let $A = \operatorname{ann}(xR + yR)$. By Lemma 2(ii), $A \neq (0)$. By Eisenbud and Griffith [**1**, Corollary (3.2)], R/A is a generalized uniserial ring. As $xR + yR$ is a uniform right R/A-module, by Theorem 1, $xR + yR$ is a uniserial module; so that either $xR \subset yR$ or $yR \subset xR$ and xR is of finite length. This shows that the family of all submodules of E is totally ordered. Let $B = \operatorname{ann}_R(xR)$. xR is a faithful right R/B-module. As R/B is artinian, R/B is embeddable in $(xR)^{(m)}$, a direct sum of m copies of xR for some integer m. Let $S = R/B$ and e be any primitive idempotent of S. There exists m R-homomorphisms σ_i of eS into $(xR)^{(m)}$, with zero as intersection of their kernels. Since the family of R-submodules of eS is totally ordered (since S is a generalized uniserial ring), we get that at least one of the σ_i is a monomorphism. Hence eS is embeddable in xR. Since xR is a uniserial module it also follows that S is an indecomposable ring and has homogeneous socle, so that we can find a Kupisch series e_1S, $e_2S, \ldots, e_nS$ of S such that $d(e_{1+i}S) = d(e_iS) + 1$ for every $i < n$. Then $xR \cong e_nS$.

Since every nonzero ideal of R contains a regular element, a nonzero divisible right R-module must be faithful. Consequently E is faithful. As E is a torsion module, Lemma 2(ii) yields that E is of infinite length. Hence using the fact that the family of submodules of E is totally ordered and that every element of E generates a submodule of finite length, we get that there exists an infinite properly ascending chain of submodules of E

$$0 = x_0R < x_1R < x_2R < \ldots < x_kR < \ldots$$

whose union is E and every $x_{i+1}R/x_iR$ is a simple R-module. Either all the factors modules $x_{i+1}/R/x_iR$ are non-isomorphic or there exists smallest non-negative integers l, m with $l < m$ such that $x_{l+1}R/x_lR \cong x_{m+1}R/x_mR$. Take $x_{m+1} = x$. In the notations of the previous paragraphs the periodicity of xR is determined by the periodicity of e_nS; so that the periodicity of xR is n. Then it is clear that for any i, j, $x_{i+1}R/x_iR \cong x_{j+1}R/x_jR$ if and only if $i \cong j \pmod{n}$. This completes the proof.

COROLLARY 1. *Let R be a Dedekind prime ring which is not right primitive. Let E be an indecomposable, injective right R-module, which is not torsion free. Then, in E there exists an infinite ascending chains of cylic submodules*

$$(0) = x_0R < x_1R < x_2R < \ldots < x_kR < \ldots$$

such that its union is E and all $x_{i+1}R/x_iR$ are simple and isomorphic.

Proof. For any proper ideal A of R, R/A is a PIR with d.c.c., i.e., R/A is a uniserial ring; further if R/A is indecomposable clearly its Kupisch series is of length one. Hence the result follows.

THEOREM 5. *Let R be an (hnp)-ring which is not right primitive and Q be its classical quotient ring. Then every indecomposable injective right R-module is a homomorphic image of eQ, where e is a primitive idempotent of Q. Further every indecomposable injective torsion right R-module is a direct summand of Q/R.*

Proof. Consider any indecomposable injective right R-module E. If E is torsion free, then the result follows from Lemma 1. Let E be not torsion free. Then E is a torsion module and it has a unique simple submodule yR. If $P =$ ann (yR), we know that P is a prime ideal and R/P is artinian. Now for $P^* = \{q \in Q | qP \subset R\}$, $PP^* = 0_l(P) = \{q \in Q : qP \subset P\}$ [**2**, Lemma (1.2)]. Since $0_l(P) \supset R$, we get $P^* > R$, and hence there exists $x \in P^*$ such that $x \notin R$. Then $\bar{x} = x + R$ is a nonzero element of the right R-module Q/R such that $\bar{x}R$ is a faithful right R/P-module. We can choose $\bar{x}$ to be such that $\bar{x}R$ is simple. Then $\bar{x}R \cong yR$. Consequently as Q/R is injective, E is embeddable in Q/R. This immediately concludes the proof.

As an application of Theorem 4, we prove the following:

THEOREM 6. *Let R be an (hnp)-ring which is not right primitive and A be a proper ideal of R, such that R/A is an indecomposable ring. Then there exists a proper ideal B of R contained in A such that R/B is an indecomposable generalized*

https://doi.org/10.4153/CJM-1974-110-5 Published online by Cambridge University Press

uniserial ring with homogeneous socle and further R/B and R/A have Kupisch series of same lengths.

Proof. Let $S = R/A$ and $e_1S, e_2S \ldots e_nS$ be a Kupisch series of S; further let $J(S)$ be the radical of S. Each e_iS is a uniform torsion right R-module, so that if E_i is the right R-injective hull of e_iS, then it is an indecomposable injective, torsion, right R-module. For any $i < n$, using (3) we get $e_{i+1}S\, e_iS = e_{i+1}J(S)$. So we have nonzero homomorphism $\sigma_i : e_iS \rightarrow e_{i+1}S$ with image $e_{i+1}J(S)$. This homomorphism can be extended to a homomorphism $\eta_i : E_i \rightarrow E_{i+1}$. As homomorphic image of an injective R-module is injective, η_i is an epimorphism. So that for $1 < i \leqq n$ we have epimorphism $\lambda_i : E_1 \rightarrow E_i$ with $\lambda_i = \eta_{i-1} \ldots \eta_1$. Put λ_1 = identity map on E_1. Let $T_i = \lambda_i^{-1}(e_iS)$ and $K_i = \text{Ker } \lambda_i$. Then $K_i \subset K_{i+1}$ and $T_{i+1}/T_i \cong e_{i+1}S/e_{i+1}N(S)$. Put $T_0 = e_1J(S)$. Since all $e_iS/e_iJ(S)$ $(1 \leqq i \leqq n)$ are non-isomorphic, it follows that T_n/T_0 is a uniserial module of length n and of periodicity n. Now $T_n = xR$ for some $x(\neq 0) \in E_1$. Let $B = \text{ann}_R(xR)$. Since every e_iS is a homomorphic image of some submodule of xR, $SB = (0)$ so that from $S = R/A$, we get $B \subset A$. As seen during the proof of Theorem 4, R/B is an indecomposable generizaled uniserial ring with homogeneous right socle. We can find a Kupisch series $f_1S', f_2S', \ldots, f_mS'$ of $S' = R/B$ such that $d(f_{i+1}S') = d(f_iS') + 1$ and $xR \cong f_mS'$ has periodicity m. If $e_1J(S) = (0)$. Then as $n = d(xR) = d(f_mS')$ and all the composition factors modules of xR are non-isomorphic; we get $n = m$. Suppose that $e_1J(S) \neq (0)$ then as $e_nS/e_nJ(S) \cong e_1J(S)/e_1J(S)^2$ we get $T_n/T_{n-1} \cong e_1J(S)/e_1J(S)^2$ and that $T_n/e_1J(S)^2$ is a homomorphic image of xR such that it has periodicity n and length $n + 1$. Then as xR is of periodicity m, we get $n = m$. Hence we find that R/B has a Kupisch series of length n. This proves the theorem.

Definition 1. Let E be an indecomposable injective torsion right R-module, where R is an (hnp)-ring which is not right primitive. The unique infinite ascending chain of submodules of R

$$(0) = x_0R < x_1R < x_2R < \ldots < x_kR < \ldots$$

such that each $x_{i+1}R/x_iR$ is simple, is called the composition series of E, and each of $x_{i+1}R/x_iR$ is called ith composition factor module of E. Further if there exists a positive integer n such that ith and jth composition factor modules are isomorphic if and only if $i \equiv j \pmod{n}$, then n is called the periodicity of E; if no such n exists, then E is said to be of periodicity zero or infinity.

Let $\mathscr{E}$ be the class of all indecomposable injective torsion right R-modules, where R is an (hnp)-ring which is not right primitive. It is clear that if $E \in \mathscr{E}$ is of periodicity $n > 0$, there exists n and only n non-isomorphic member of $\mathscr{E}$ which are homomorphic images of E. If $F \in \mathscr{E}$ is one such, then there exists a homomorphism of E onto F with kernel of lengths $\leqq (n - 1)$ and kernel with this property is a uniquely determined submodule of E. If $E \in \mathscr{E}$ is of periodicity zero and F is a homomorphic image of E, then there exists a unique submodule K of E, such that $E/K \cong F$. For any $E, F \in \mathscr{E}$, define $M(E, F)$ as follows:

$M(E, F) = E$, if F is not a homomorphic image of E;

$=$ the submodule K of E such that $E/K \cong F$, in case F is a homomorphic image of E; if further E is of periodicity $n > 0$ we take $d(K) \leqq n - 1$.

For any $E, F \in \mathscr{E}$, we define E equivalent to F if and only if there exists submodules E' of E and F' of F such that $E' \neq E$ and $F' \neq F$ and $E/E' \cong F/F'$. It can be easily seen that this relation is an equivalence relation. Further under this equivalence relation any two equivalent members of $\mathscr{E}$ are of same periodicity and if any one of them is of finite periodicity, then they are homomorphic images of each other.

We now determine the structure of a quasi-injective right R-module.

THEOREM 7. *Let R be an (hnp)-ring which is not right primitive. Then a right R-module N is quasi-injective if and only if it satisfies the following.*

I. *If N is not a torsion module, then N is injective,*

II. *If N is a torsion module, then*

$$N = \oplus \sum_{i \in \Lambda} N_i,$$

where N_i are uniform right R-modules with the following properties: Let $E_i = E(N_i)$.

(i) *For any $i, j \in \Lambda$, $d(N)_i \leqq d(N_j) + d(M(E_i, E_j))$.*

Proof. We shall use the result that any module is quasi-injective if and only if it is invariant under every endomorphism of its injective hull [**9**, Theorem (1.1)].

Firstly, let us consider an indecomposable torsion free quasi-injective right R-module T. Since any quasi injective module over a noetherian ring is a direct sum of uniform modules by Miyashito [**14**], T is uniform. Since $E(T)$ is torsion free, for some primitive idempotent e of the classical quotient ring Q of R, $E(T) = eQ$. Since T is invarient under every R-endomorphism of eQ, $eQeT \subset T$. However $QeT = Q$; we get $T = eQ$. So that T is injective.

Let N be any quasi injective right R-module. $N = \oplus \sum_{i \in \Lambda} N_i$, for some uniform submodules N_i of N [**14**]. Suppose that N is not a torsion module, then one of these N_i, say N_i' must be torsion free. By the above paragraph $N_i' \cong eQ$ for some primitive idempotent e of Q. Let $E_i = E(N_i)$. Since N is invarient under every endomorphism of $E(N)$ and by Theorem 5, every E_i is a homomorphic image of eQ, we get $N_i = E_i$. Hence N is injective. So let N be a torsion module. Now

$$E(N) = \oplus \sum_{i \in \Lambda} E_i.$$

Consider any $i, j \in \Lambda$, it is clear from the definition of $M(E_i, E_j)$ that there exists a homomorphism $\eta : E_i \to E_j$ such that $\ker \eta = M(E_i, E_j)$. Since N is invariant under every endomorphism of $E(N)$, we get $\eta(N_i) \subset N_j$. Then using the fact that the family of submodules of E_i is totally ordered, it follows that $d(N_i) \leqq d(N_j) + d(M(E_i, E_j))$. Since the family of submodules of E_i is totally ordered and the kernel of every homomorphism of E_i into E_j contains $M(E_i, E_j)$, it follows that if the above condition is satisfied, then $\sigma(N_i) \subset N_j$ for any $\sigma : E_i \to E_j$ and then as every endomorphism of $E(N)$ is determined by homomorphisms between various E_i's the converse follows:

COROLLARY 3. *If N is a quasi-injective right R-module, then $N = M \oplus T$, where M is injective and T is a direct sum of uniserial R-modules. Further if N is not a torsion module, then $T = (0)$.*

Proof. If N is not a torsion module, by the above theorem N is injective. Let N be a torsion module. Now $N = \oplus \sum_{i \in \Lambda} N_i$ where N_i are uniform, Theorem 4 yields that if N_i is of infinite length then it must be injective, otherwise N_i is uniserial. Hence the corollary follows.

4. Quasi-projective modules. Rangaswamy and Vanaja [**18**] proved that a Dedekind domain D (commutative) is a complete discrete valuation ring of rank one if and only if its quotient field K is a quasi-projective D-module. In this section we generalize the above result to (hnp) ring which are not right primitive.

A Dedekind prime ring R which is complete with respect to the J-adic topology, where $J = J(R)$ is said to be a complete Dedekind prime ring. We prove the following:

THEOREM 8. *Let R be an (hnp) ring which is not right primitive and let Q be its classical quotient ring. Then Q is quasi projective right R-module if and only if $R = D_n$, where n is a positive integer, and D is a complete local Dedekind domain* (not necessarily commutative); *further in that case R is a Dedekind prime ring having $J(R)$ as its maximal ideal and Q is quasi projective as a left R-module.*

We firstly establish some other results.

THEOREM 9. *Let E be an indecomposable, injective, torsion right R-module, where R is an (hnp) ring, which is not right primitive. Then $D = \mathrm{Hom}_R(E, E)$ is a local Dedekind domain which is complete.*

Proof. Let $(0) = x_0R < x_1R < x_2R < \ldots < x_nR < \ldots$ be the composition series of E. If E is of periodicity zero, then each of its nonzero endomorphisms is an automorphism; so that D is a division ring. Let E be of periodicity $n > 0$. Since every nonzero endomorphism of E is an epimorphism, D is a domain. We consider any two nonzero elements σ and η of D. Now either $\ker \sigma \subset \ker \eta$ or $\ker \eta \subset \ker \sigma$. To be definite let $\ker \sigma \subset \ker \eta$ we define $\lambda \in D$ as follows:

As $\sigma(E) = E$, given $u \in E$, there exists $y \in E$ such that $\sigma(y) = u$. Define $\lambda(u) = \eta(y)$. Then λ is well-defined and $\eta = \lambda\sigma$. This proves that the family of left ideals of D is totally ordered, D is a left (PID). Further since the minimal submodules of $\sigma(E)$ and E are same, this gives that if $\ker \sigma \neq 0$, then $\ker \sigma = x_{kn}R$ for some k. Then for $J = J(D)$.

$$J^m = \{\sigma \in D \mid x_{mn} \in \ker \sigma\}.$$

We now prove that D is J-complete. Consider any sequence $\{\sigma_m\}$ in D such that $\sigma_k - \sigma_l \in J^l$ for every $k \geqq l \geqq 1$. This gives σ_l and σ_k agree upon $x_{ln}R$ whenever $k \geqq l$. Hence we can find $\sigma \in D$ such that $\sigma(x_{kn}) = \sigma_k(x_{kn})$. Then $\sigma - \sigma_k \in J^k$ for every k. Hence D is J-complete. Then by Michler [**12**, Satz (4.4)] D is also a principal right ideal ring. Then every one sided ideal of D is a power of its maximal ideal J. Hence the result follows.

THEOREM 10. *Let D be a local, complete, Dedekind prime ring and Q be its classical quotient ring. Then Q is quasi-projective as a right D-module and also as a left D-module.*

Proof. Since D is local by [**2**, Lemma 1.4], D is uniform as a right D-module. Hence D is free from zero divisors. Now by Theorem 3, D is right bounded. Consider any proper right ideal A of D. Then A contains a nonzero two sided ideal B. Since D/B is a local uniserial ring, A/B is a two sided ideal of D. Consequently A is a two sided ideal of D. Similarly every left ideal of D is two sided. Since in a local uniserial ring every ideal is a power of the maximal ideal, we get that every proper ideal in D is a power of its maximal ideal $J(D)$ so if we take $a \in J(D) - J(D)^2$, then $J(D) = aD = Da$ and for any $n \geqq 1$, $a^nD = Da^n$. Hence if $\alpha \in D$ is a unit then $\alpha a = a\beta$, $a\alpha = \gamma a$ for some units β and γ in D. Consider

$$\begin{array}{ccccc} & & Q & & \\ & & \big\downarrow{\scriptstyle \eta} & & \\ Q & \xrightarrow{\ \pi\ } & Q/K & \longrightarrow & 0 \end{array}$$

where η is a right D-homomorphism and π is natural homomorphism. Now $K = a^tD$ for some integer t. For each $n > 0$ $\eta(a^{-n}) = \overline{\alpha_n a^{k_n}} = \alpha_n a^{k_n} + K$ for some unit α_n in D and integer k_n. For $n > m > 0$,

$$\overline{\alpha_m a^{k_m}} = \eta(a^{-m}) = \eta(a^{-n})a^{n-m} = \overline{\alpha_n a^{k_n} a^{n-m}}.$$

This yields

$$\alpha_n a^{k_n+n} - \alpha_m a^{k_m+m} \in a^{m+t}D.$$

Since t is fixed, for large enough m, $m + t > 0$. Consequently eventually either all $k_n + n$ are positive or eventually all are negative and equal. In the former case $\alpha_n a^{k_n+n}$ is eventually in D and hence there exists $b \in D$ such that eventually $b - \alpha_n a^{k_n+n} \in a^{n+t}D$. Thus if we define $\sigma : Q \to Q$ by $\sigma(x) = bx$

for $x \in Q$, we have $\pi\sigma = \eta$. In the later case there exists m_0 such that $m_0 + t > 0$, $h_n + n = k_{m_0} + m_0$ for all $n \geqq m_0$. If we put $c = k_{m_0} + m_0$, we get $\alpha_n - \alpha_m \in a^{m+t-c}D$ for $n \geqq m \geqq m_0$. Hence there exists $\alpha \in D$ such that eventually $\alpha - \alpha_n \in a^{n+t-c}D$. If we define $\sigma : Q \rightarrow Q$ by $\sigma(x) = \alpha a^{-c}x$ we get $\pi\sigma = \eta$. Hence Q is quasi projective as a right D-module. Similarly Q is a quasi projective left D-module.

Proof of Theorem 8. Since R is not right primitive we have $R \neq Q$. Let Q be quasi projective as a right R-module. Consider any $\sigma \in \text{Hom}_R(Q/R, Q/R)$. Let $\pi : Q \rightarrow Q/R$ be the natural R-homomorphism. Since Q as a right R-module, is quasi-projective there exist right R-homomorphism $\sigma' : Q \rightarrow Q$ such that $\pi\sigma' = \sigma\pi$. Since $\ker \pi = R$ we get $\sigma'(R) \subset R$. Thus, if $\sigma'(1) = t$, it follows that $t \in R$ and for any $x \in Q$, $\sigma(x + R) = tx + R$. For any $t \in R$ let σ_t denote the left multiplication of Q/R by t. It follows that $t \rightarrow \sigma_t$; $t \in R$, is a ring homomorphism of R onto $\text{Hom}_R(Q/R, Q/R)$. As R does not have any nonzero right ideal which is a divisible R-module, it follows that the above mapping is an isomorphism. The same mapping is also a right R-isomorphism. Hence $\text{Hom}_R(Q/R, Q/R) \cong R$ both as a ring and as a right R-module. By using Theorem 5 and the fact that Q/R is a torsion, injective right R-module, we get Q/R is a direct sum of indecomposable injective torsion right R-modules and every indecomposable injective torsion right R-module is a direct summand of Q/R. Since $\text{Hom}_R(Q/R, Q/R) \cong R_R$ and R does not contain an infinite set of orthogonal idempotents, we get Q/R is a finite direct sum of indecomposable injective torsion right R-modules. Thus there are finitely many non-isomorphic indecomposable injective, torsion right R-modules. Consequently any indecomposable injective torsion right R-module is of finite periodicity. Since R does not have any non-trivial central idempotent, we get that all these injective modules are equivalent. So we can write

$$(7) \quad Q/R = (E_1 + E_2 + \ldots + E_{t_1}) + (E_{t_1+1} + \ldots + E_{t_1+t_2}) + \ldots + (\ldots + E_n)$$

where all E_i's are indecomposable and equivalent, but any two of the E_i's are isomorphic if and only if they occur within the same bracket. By Faith and Utumi [**4**, Theorem (3.1)] any $a \in R$ is in $J(R)$ if and only if $\{q \in Q/R | aq = 0\}$ is an essential right R-submodule of Q/R. Let us identify R with $\text{Hom}_R(Q/R, Q/R)$. In a natural way we can regard $\text{Hom}_R(E_i, E_j) \subset \text{Hom}_R(Q/R, Q/R)$. Then any $\sigma \in \text{Hom}_R(Q/R, Q/R)$ is expressible uniquely as $\sigma = \sum \sigma_{ji}$, with $\sigma_{ji} \in \text{Hom}_R(E_i, E_j)$. Using [**4**, Theorem (3.1)] we get $\sigma \in J(R)$ if and only if $\sigma_{ji} \in J(R)$ for all i, j; further for E_i and E_j occurring in different brackets in (7) we have $\text{Hom}_R(E_i, E_j) \subset J(R)$. It can be easily seen that given a maximal ideal M of R, for some fixed bracket on the right hand side of (7), M consists of all those $\sigma = \sum \sigma_{ji}$ such that for all E_i, E_j occurring within that bracket, $\sigma_{ji} \in J(R)$ i.e., σ_{ji} is not a monomorphism. Further notice the following: Let E, E', E'' be any three indecomposable injective torsion right R-modules and $\sigma : E \to E'$, $\eta : E' \to E''$ be nonzero R-homomorphisms. If $d(\ker \sigma) = t$, then for every $k \geqq 0$, the $(k + t)$th term in the composition series of E is mapped onto the kth term of the composition series of E'. From this it follows that if $\ker \sigma \neq (0)$ and $\ker \eta \neq (0)$, then $\ker (\eta\sigma)$ properly contains $\ker \sigma$. Using this fact and the above given form of the maximal ideals of R, it follows that no maximal ideal of R is an idempotent. Hence by [**3**, Propositions (2.2) and (4.5)] no proper ideal of R is an idempotent. Hence R is a Dedekind prime ring. Hence by Corollary 1, any indecomposable injective, torsion right R-module is of periodicity one, and thus all the E_i in (7) are isomorphic. Hence for $D = \mathrm{Hom}_R(E_1, E_1)$ we get $R \cong D_n$, where D by Theorem 9 is a local, complete Dedekind domain (not necessarily commutative).

Conversely, let $R = D_n$ where D satisfies the given conditions. Let R be the classical quotient ring of D. Then K_n is the classical quotient ring of R. Now by Theorem 10, K is quasi-projective as a right D-module (also as a left D-module). Since by Golan [**6**, Theorem (1.1)] quasi-projective modules are preserved under category equivalence, by using the Morita duality Theorem, it follows that for any primitive idempotent e of K_n, eK_n is quasi-projective as a right D_n-module (i.e. as a right R-module). Since K_n is a direct sum of n isomorphic minimal right ideals and by de Robert [**19**], a direct sum of finitely many copies of a quasi-projective module is quasi-projective, it follows that K_n is a quasi-projective right R-module. Similarly K_n is quasi-projective as a left R-module. The other part of the proof is immediate.

Remark. Let R be any (hnp)-ring with enough invertible ideals. By Eisenbud and Robson every finitely generated torsion right R-module is a direct sum of cyclic modules each of which is either unfaithful or completely faithful [**2**, Theorem (3.11)]. Let E be an indecomposable injective torsion right module. If E does not have any nonzero completely faithful submodule, by using the above mentioned result of Eisenbud and Robson, the same structure, as in Theorem 4, can be established for E. Theorem 6 also holds for any (hnp)-ring with enough invertible ideals.

Acknowledgement. The author expresses his sincere thanks to Prof. P. M. Cohn of Bedford College, London, for providing him with all the facilities for doing this research during his stay in that College. Thanks are also due to the referee for his various suggestions.

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https://doi.org/10.4153/CJM-1974-110-5 Published online by Cambridge University Press