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DEFORMATIONS OF MORPHISMS OF SHEAVES

DONATELLA IACONO AND ELENA MARTINENGO

Abstract. We analyse infinitesimal deformations of morphisms of locally free sheaves on a smooth projective variety X over an algebraically closed field of characteristic zero. In particular, we describe a differential graded Lie algebra controlling the deformation problem. As an application, we study infinitesimal deformations of the pairs given by a locally free sheaf and a subspace of its sections with a view towards Brill-Noether theory.

Contents

Introduction

Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves of $\mathcal{O}_X$-modules on X and $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. In this article, we are interested in the infinitesimal deformations of $\alpha : \mathcal{F} \to \mathcal{G}$, where both the sheaves $\mathcal{F}$ and $\mathcal{G}$ and the map α can be deformed. This is equivalent to deform the graph of α as a subsheaf of the direct sum $\mathcal{F} \oplus \mathcal{G}$, in such a way that the deformation of $\mathcal{F} \oplus \mathcal{G}$ is given by a deformation of $\mathcal{F}$ and a deformation of $\mathcal{G}$.

Not much is known about them. We tackle this problem using differential graded Lie algebras (dgLas). In [Ia06, Ia08], the first author investigated infinitesimal deformations of a holomorphic map gluing the dgLas that control the deformations of the domain, of the codomain and of the graph in the product. Here, we apply the same approach to deformations of a morphism of sheaves. Moreover, we extend it to any algebraically closed field $\mathbb{K}$ of characteristic zero, using semicosimplicial dgLas techniques as developed in [FMM12, FIM12].

The sheaves of dgLas associated to the infinitesimal deformations of a sheaf and of the graph inside the direct sum are classically known, each of them forms a semicosimplicial dgLa $\mathfrak{g}^\Delta$ and the Čech functor $H^1_{sc}(\mathfrak{g}^\Delta)$ controls the corresponding deformations. A deformation of the morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is obtained from the data of the three involved deformations via a totalisation process. Finally, applying the strong tool of Hinich's Theorem of descent of Deligne groupoids [Hin97], we get a specific differential graded Lie algebra that controls the infinitesimal deformations of $\alpha : \mathcal{F} \to \mathcal{G}$ via the Deligne

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functor in groupoids. This dgLa $\operatorname{Tot}(H(\mathcal{V})^\Delta)$ is the Thom-Whitney dgLa associated to a suitable semicosimplicial dgLa $H(\mathcal{V})^\Delta$ (see Section 2 for details).

Theorem (Theorem 2.7). *The functor* $\operatorname{Def}_{(\mathcal{F},\alpha,\mathcal{G})}$ *of infinitesimal deformations of the morphism* $\alpha : \mathcal{F} \to \mathcal{G}$ *is equivalent to the Deligne functor* $\operatorname{Del}_{\operatorname{Tot}(H(\mathcal{V})^\Delta)}$ *associated to the Thom-Whitney dgLa of the semicosimplicial dgLa* $H(\mathcal{V})^\Delta$*.*

This approach holds for any algebraically closed field $\mathbb{K}$ of characteristic zero. If we restrict our attention to the field of complex numbers, we can consider the dgLas associated to the Dolbeault resolutions of the relevant sheaves, instead of the Čech semicosimplicial dgLas and the Thom-Withney construction, as explained in Remark 2.11.

As an application of our study of the infinitesimal deformations of a morphism of sheaves, we analyse the infinitesimal deformations of a locally free sheaf $\mathcal{E}$ over a smooth projective variety X over $\mathbb{K}$ together with a linear subspace $U \subseteq H^0(X,\mathcal{E})$ of its global sections. Such a pair $(\mathcal{E}, U)$ is called a coherent system. The study of the moduli space and deformations of coherent systems is very classical and it is a wide and still very active research field. We refer the reader to the introduction of [IM23] for more details and references. The idea behind our approach is to view any infinitesimal deformation of the pair $(\mathcal{E}, U)$ as an infinitesimal deformation of the morphism $s : U \otimes \mathcal{O}_X \to \mathcal{E}$ of sheaves of $\mathcal{O}_X$-modules, induced by the inclusion $U \subseteq H^0(X,\mathcal{E})$. For this correspondence, we need the technical assumption that $H^1(X, \mathcal{O}_X) = 0$, so that the sheaf $U \otimes \mathcal{O}_X$ has only trivial infinitesimal deformations. To study the infinitesimal deformations of $s : U \otimes \mathcal{O}_X \to \mathcal{E}$, we can directly apply the previous construction to find a suitable semicosimplicial dgLa, whose Thom-Whitney dgLa controls infinitesimal deformations of the pair $(\mathcal{E}, U)$ via the Deligne functor.

Moreover, under the assumption that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, its cohomology groups fit into an exact sequence (see Equation 8), from which we obtain much information about the tangent and the obstructions spaces of the deformation problem of $(\mathcal{E}, U)$ and a link with the Petri map. The dgLa we described is quite involved, but its importance lies on the fact that it holds over any algebraically closed field $\mathbb{K}$ of characteristic zero.

In our article [IM23], we generalised many classical results concerning coherent systems of line bundles over a curve, to the case of a vector bundle of any rank on a smooth projective variety of any dimension over $\mathbb{C}$. Here, we recover these results over any algebraically closed field $\mathbb{K}$ of characteristic zero, under the additional hypothesis that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, see Corollaries 3.5, 3.6, 3.7 and 3.8. This additional hypothesis do not allow us to generalise the results obtained over $\mathbb{C}$, for curves of genus $g > 0$.

The Thom-Whitney dgLa constructed in this way is unfortunately quite complicated to be handled. For this reason, we even present a more explicit model of a semicosimplicial dgLa whose total complex controls deformations of the pair $(\mathcal{E}, U)$ (see Section 3.3).

The connection between the moduli space of coherent systems and Brill-Noether theory is obviously very close. The classical Brill-Noether theory concerns the subvarieties $W^k_d(C)$ of $\operatorname{Pic}^d(C)$ of line bundles on a curve C of degree d having at least $k+1$ independent sections and much is classically known about it. During the last years, several generalisations of this problem were investigated. The general case of vector bundles on varieties of higher dimension is still quite mysterious and many generalisations are studied just over the field of complex numbers. A parallel approach to this problem is the one based on dgL pairs, used to locally analyse the cohomology jump loci of some moduli spaces over any field $\mathbb{K}$ [BW15, BR19, B24]

We decide to follow a different approach. Note that the functor associated to the Brill-Noether loci, or to any deformation problem with cohomological constraints, is intrinsically more difficult to be studied, since it is not a deformation functor. Our idea is to get as much information as possible on infinitesimal deformations of a pair $(\mathcal{E}, U)$, that we tackle using dgLas even over any algebraically closed field $\mathbb{K}$ of characteristic zero. Then, using the natural link between coherent systems and Brill-Noether theory, we deduce some information on the last deformation problem. In our point of view this approach is very natural and very explicit and takes advantage from the very powerful tool of dgLas in deformation theory.

In particular, we are able to recover the results of [IM23] about the infinitesimal deformations of $\mathcal{E}$ with at least k independent sections (see Section 4) for any algebraically closed field $\mathbb{K}$ of characteristic zero, adding the hypothesis that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$. This additional hypothesis do not allow us to generalise the results obtained over $\mathbb{C}$, for curves of genus $g > 0$.

The paper is organised as follows. With the aim of providing an introduction to the subject, we include the notion of differential graded Lie algebras, semicosimplicial dgLas and the associated deformation and Deligne functors in Section 1.

In Section 2, we review the notion of infinitesimal deformations of a morphism of locally free sheaves and we describe a dgLa that controls these deformations, as the Thom-Whitney dgLa of a specific semicosimplicial dgLa.

Section 3 is devoted to the infinitesimal deformations of a locally free sheaf together with a subspace of sections, viewed as deformations of a morphism of sheaves. We actually provide a second dgLa which can be constructed explicitly.

Finally, in the last section, we apply our results to study infinitesimal deformations of a locally free sheaf together with at least a prescribed number of independent sections.

Throughout the paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero; **Sets** denotes the category of sets in a fixed universe and $\mathbf{Art}_{\mathbb{K}}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. For an element $A \in \mathbf{Art}_{\mathbb{K}}$, its maximal ideal is indicated by $\mathfrak{m}_A$.

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1. Preliminaries

1.1. **Differential graded Lie algebras, deformation and Deligne functors.** In this subsection, we introduce the basic definitions and properties of differential graded Lie algebras, together with their associated deformation and Deligne functors.

For full details, we refer the reader to [Man99, Man04, Man09, Man22].

Definition 1.1. A *differential graded Lie algebra*, briefly a *dgLa*, is the data $(L, d, [\ ,\ ])$, where $L = \bigoplus_{i\in\mathbb{Z}} L^i$ is a $\mathbb{Z}$-graded vector space over $\mathbb{K}$, $d : L^i \to L^{i+1}$ is a linear map, such that $d \circ d = 0$, and $[\ ,\ ] : L^i \times L^j \to L^{i+j}$ is a bilinear map, such that:

- $[\ ,\ ]$ is graded skewsymmetric, i.e., $[a, b] = -(-1)^{\deg a \deg b}[b, a]$,
- $[\ ,\ ]$ verifies the graded Jacoby identity, i.e., $[a, [b, c]] = [[a, b], c]+(-1)^{\deg a \deg b}[b, [a, c]]$,
- $[\ ,\ ]$ and d verify the graded Leibniz's rule, i.e., $d[a, b] = [da, b] + (-1)^{\deg a}[a, db]$,

for every a, b and c homogeneous elements.

Definition 1.2. Let $(L, d_L, [\ ,\ ]_L)$ and $(M, d_M, [\ ,\ ]_M)$ be two dgLas, a *morphism of dglas* $\varphi : L \to M$ is a degree zero linear morphism that commutes with the brackets and the differentials.

A *quasi-isomorphism* of dgLas is a morphism of dgLas that induces an isomorphism in cohomology.

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Definition 1.3. The *Thom-Whitney homotopy fibre product* of two morphisms of dgLas $h : L \to M$ and $g : N \to M$ is the differential graded Lie algebra defined as

$$L \times_M N := \{(l, n, m(t, dt)) \in L \times N \times M[t, dt] \mid m(0) = h(l),\ m(1) = g(n)\}.$$

Here, we denote by $M[t, dt]$ the dgLa $M \otimes K[t, dt]$, where t has degree zero, dt has degree 1 and $d^2 t = 0$ as well as $(dt)^2 = 0$. Moreover, $m(0)$ and $m(1)$ denotes the evaluation in $t = 0$ and $t = 1$, respectively.

Definition 1.4. Let L be a nilpotent differential graded Lie algebra, we define:

$$\mathrm{Def}(L) = \frac{\mathrm{MC}(L)}{\sim_{\text{gauge}}},$$

where:

$$\mathrm{MC}(L) = \left\{ x \in L^1 \mid dx + \frac{1}{2}[x, x] = 0 \right\}$$

is the set of the Maurer-Cartan elements, and the gauge action is the action of $\exp(L^0)$ on $\mathrm{MC}(L)$, given by:

$$e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a, -])^n}{(n+1)!}([a, x] - da).$$

If L is any dgLa, we define the *deformation functor associated to* L as the functor

$$\mathrm{Def}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Sets},$$

that associates to every $A \in \mathbf{Art}_{\mathbb{K}}$ the set

$$\mathrm{Def}_L(A) := \mathrm{Def}(L \otimes \mathfrak{m}_A).$$

We recall that the tangent space to the deformation functor Def_L is the first cohomology space $H^1(L)$ of the dgLa L. Moreover, a complete obstruction theory for the functor Def_L can be naturally defined and its obstruction space is the second cohomology space $H^2(L)$ of the dgLa L.

If the functor of deformations of a geometric object $\mathcal{X}$ is isomorphic to the deformation functor associated to a dgLa L, then we say that L *controls* the deformations of $\mathcal{X}$.

Any morphism $\varphi : L \to M$, induces a morphism $\varphi : \mathrm{Def}_L \to \mathrm{Def}_M$, that is an isomorphism, whenever φ is a quasi-isomorphism.

Definition 1.5. A *small category* is a category whose morphisms form a set.

A *groupoid* is a small category such that every morphism is an isomorphism. We denote the category of groupoids by **Grpds**.

For every groupoid G, the set of isomorphism classes of objects is denoted by $\pi_0(G)$.

Let L be a nilpotent dgLa, we define $C(L)$ as the groupoid whose set of objects is $\mathrm{MC}(L)$ and whose morphisms between two objects x and y are defined as the set

$$\mathrm{Mor}_{C(L)}(x, y) = \{e^a \in \exp(L^0) \mid e^a * x = y\}.$$

The *irrelevant stabilizer* of a Maurer-Cartan element $x \in \mathrm{MC}(L)$ is the normal subgroup

$$I(x) = \{e^{du+[x,u]} \mid u \in L^{-1}\} \subseteq \mathrm{Mor}_{C(L)}(x, x).$$

Definition 1.6. The *Deligne groupoid* of a nilpotent differential graded Lie algebra L is the groupoid $\mathrm{Del}(L)$ having as objects the Maurer-Cartan elements of L and as morphisms

$$\mathrm{Mor}_{\mathrm{Del}(L)}(x, y) = \frac{\mathrm{Mor}_{C(L)}(x, y)}{I(x)} = \frac{\mathrm{Mor}_{C(L)}(x, y)}{I(y)},$$

where the second equality is a natural isomorphism (see [Man22, Lemma 6.5.5]).

If L is any dgLa, we define the *Deligne functor*

$$\mathrm{Del}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds},$$

as the functor that associates to every $A \in \mathbf{Art}_{\mathbb{K}}$ the groupoid

$$\mathrm{Del}_L(A) := \mathrm{Del}(L \otimes \mathfrak{m}_A).$$

It is immediate from the definitions that there is an equivalence of functors:

$$\pi_0(\mathrm{Del}_L) \cong \mathrm{Def}_L\,.$$

Remark 1.7. Note that, if the nilpotent dgLa L is concentrated in non negative degrees, then the irrelevant stabilisers are trivial and the morphisms of the Deligne groupoid $\mathrm{Del}(L)$ coincide with the ones induced by the gauge action.

1.2. **Semicosimplical differential graded Lie algebras.** Here, we recall some preliminaries on the semicosimplicial dgLas, their total object and the deformation functors associated to them. We mainly follow [FMM12, FIM12], see also [Man22].

Definition 1.8. A *semicosimplicial differential graded Lie algebra* is a covariant functor $\mathbf{\Delta}_{\mathrm{mon}} \to \mathbf{DGLA}$, from the category $\mathbf{\Delta}_{\mathrm{mon}}$, whose objects are finite ordinal sets and whose morphisms are order-preserving injective maps between them, to the category of dgLas. Equivalently, a semicosimplicial dgLa $\mathfrak{g}^\Delta$ is a diagram

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathfrak{g}_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots,$$

where each $\mathfrak{g}_i$ is a dgLa, and for each $i > 0$, there are $i+1$ morphisms of dgLas

$$\partial_{k,i} \colon \mathfrak{g}_{i-1} \to \mathfrak{g}_i, \qquad k = 0, \ldots, i,$$

such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$.

A *morphism of semicosimplicial differential graded Lie algebras* $f : \mathfrak{g}^\Delta \to \mathfrak{h}^\Delta$, is given by a sequence $\{f_i : \mathfrak{g}_i \to \mathfrak{h}_i\}$ of morphisms of dgLas, commuting with the maps $\partial_{k,i}$.

In particular, every $\mathfrak{g}_i$ is a vector space and so we can consider the graded vector space $\bigoplus_{n\geq 0} \mathfrak{g}_n[-n]$ that has two differentials, i.e.,

$$d = \sum_n (-1)^n d_n, \qquad \text{where} \quad d_n \text{ is the differential of } \mathfrak{g}_n,$$

and

$$\partial = \sum_i \partial_i, \qquad \text{where} \quad \partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}.$$

Note that $d\partial + \partial d = 0$, thus the graded vector space $\bigoplus_{n\geq 0} \mathfrak{g}_n[-n]$, endowed with the differential $D = d + \partial$, is a complex, called the *total complex*, but it can not be endowed with a structure of dgLa.

However, there is a dgLa that is naturally associated to any semicosimplicial dgLa and that is quasi-isomorphic to the total complex. It is constructed as follows. For every $n \geq 0$, we denote by Ω_n the differential graded commutative algebra of polynomial differential forms on the standard n-simplex Δ^n:

$$\Omega_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(\sum_{i=0}^n t_i - 1, \sum_{i=0}^n dt_i)}.$$

Definition 1.9. The *Thom-Whitney dgLa* associated to the semicosimplicial dgLa $\mathfrak{g}^\Delta$ is

$$\mathrm{Tot}(\mathfrak{g}^\Delta) = \{(x_n)_n \in \prod_n \Omega_n \otimes \mathfrak{g}_n \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1} \quad \forall\, 0 \leq k \leq n\},$$

where, for $k = 0, \ldots, n$, $\delta^{k,n} \colon \Omega_n \to \Omega_{n-1}$ are the face maps and $\partial_{k,n} \colon \mathfrak{g}_{n-1} \to \mathfrak{g}_n$ are the maps of the semicosimplicial dgLa $\mathfrak{g}^\Delta$.

As already mentioned, $\mathrm{Tot}(\mathfrak{g}^\Delta)$ is quasi-isomorphic, as graded vector space, to the total complex $\left(\bigoplus_{n\geq 0} \mathfrak{g}_n[-n], d + \partial\right)$.

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Remark 1.10. Let $h : L \to M$ and $g : N \to M$ be two morphisms of dgLas and consider the semicosimplicial dgLa

$$L \oplus N \underset{g}{\overset{h}{\rightrightarrows}} M \rightrightarrows 0 \rightrightarrows \cdots .$$

In this case, the associated Thom-Whitney dgLa is nothing else that the Thom-Whitney homotopy fibre product of Definition 1.3.

Example 1.11. Given a sheaf $\mathcal{L}$ of dgLas on a topological space X and an open cover $\mathcal{V} = \{V_i\}_i$ of X, the Čech semicosimplicial dgLa $\mathcal{L}(\mathcal{V})$ is given by:

$$\prod_i \mathcal{L}(V_i) \rightrightarrows \prod_{i<j} \mathcal{L}(V_{ij}) \rightrightarrows \prod_{i<j<k} \mathcal{L}(V_{ijk}) \rightrightarrows \cdots ,$$

where, as usual, $V_{ij} = V_i \cap V_j$, $V_{ijk} = V_i \cap V_j \cap V_k$ and so on, denote the intersections, $\mathcal{L}(V_i) = \Gamma(V_i, \mathcal{L})$ stays for the sections and the morphisms $\partial_{k,i}$ are the restriction maps. Here, the total complex associated to the Čech semicosimplicial dgLie algebra $\mathcal{L}(\mathcal{V})$ is the Čech complex $\check{C}(\mathcal{V}, \mathcal{L})$ of the sheaf $\mathcal{L}$. In particular, by the quasi isomorphism between the total complex and the Thom-Whitney dgLa, we have $H^k(\operatorname{Tot}(\mathcal{L}(\mathcal{V}))) \cong \check{H}^k(\mathcal{V}, \mathcal{L})$, for all $k \in \mathbb{Z}$.

According to [FMM12, FIM12], we define the following deformation functor associated to a semicosimplicial dgLa.

Definition 1.12. Let $\mathfrak{g}^\Delta$ be a semicosimplicial dgLa. The functor

$$Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

is defined, for all $A \in \mathbf{Art}_{\mathbb{K}}$, by

$$Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A) = \left\{ (l, m) \in (\mathfrak{g}^1_0 \oplus \mathfrak{g}^0_1) \otimes \mathfrak{m}_A \;\middle|\; \begin{array}{l} dl + \frac{1}{2}[l, l] = 0, \\ \partial_{1,1} l = e^m * \partial_{0,1} l, \\ \partial_{0,2} m \bullet -\partial_{1,2} m \bullet \partial_{2,2} m = dn + [\partial_{2,2}\partial_{0,1} l, n] \\ \qquad\qquad \text{for some } n \in \mathfrak{g}^{-1}_2 \otimes \mathfrak{m}_A \end{array} \right\},$$

where the symbol $\bullet$ stands for the Baker Campbell Hausdorff product in a Lie algebra.

Two elements (l_0, m_0) and $(l_1, m_1) \in Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A)$ are equivalent under the relation $\sim$ if and only if there exist elements $a \in \mathfrak{g}^0_0 \otimes \mathfrak{m}_A$ and $b \in \mathfrak{g}^{-1}_1 \otimes \mathfrak{m}_A$ such that

$$\begin{cases} e^a * l_0 = l_1 \\ -m_0 \bullet -\partial_{1,1} a \bullet m_1 \bullet \partial_{0,1} a = db + [\partial_{0,1} l_0, b]. \end{cases} \tag{1}$$

Definition 1.13. Let $\mathfrak{g}^\Delta$ be a semicosimplicial dgLa, the functor

$$H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

is defined, for all $A \in \mathbf{Art}_{\mathbb{K}}$, by

$$H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A) = \frac{Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A)}{\sim}.$$

Note that any morphism of semicosimplicial dgLas $\mathfrak{g}^\Delta \to \mathfrak{h}^\Delta$ induces a natural transformation of functors $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \to H^1_{\mathrm{sc}}(\exp \mathfrak{h}^\Delta)$.

Remark 1.14. If $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, i.e., every dgLa $\mathfrak{g}_i$ is concentrated in degree zero, then the functor $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$ reduces to the one defined in [FMM12].

Moreover, if $\mathfrak{g}^\Delta$ is a semicosimplicial dgLa, such that every dgLa is concentrated in non negative degrees, we can easily view the above functor as a functor in groupoids. More explicitly, we have

$$\widetilde{H}^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds},$$

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where, for any $A \in \mathbf{Art}_{\mathbb{K}}$, the set of object is

$$Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})(A) = \left\{ (l,m) \in (\mathfrak{g}^1_0 \oplus \mathfrak{g}^0_1) \otimes \mathfrak{m}_A \;\middle|\; \begin{array}{l} dl + \frac{1}{2}[l,l] = 0, \\ \partial_{1,1} l = e^m * \partial_{0,1} l, \\ \partial_{0,2} m \bullet -\partial_{1,2} m \bullet \partial_{2,2} m = 0 \end{array} \right\}, \tag{2}$$

and the isomorphisms between two objects (l_0, m_0) and $(l_1, m_1) \in Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})(A)$ are given by the elements $a \in \mathfrak{g}^0_0 \otimes \mathfrak{m}_A$ as above, i.e., such that

$$\begin{cases} e^a * l_0 = l_1 \\ -m_0 \bullet -\partial_{1,1} a \bullet m_1 \bullet \partial_{0,1} a = 0. \end{cases} \tag{3}$$

We recall the following result [FIM12, Theorem 4.10], that relates these functors with the ones associated to the dgLas.

Theorem 1.15. *Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial dgLa, such that $H^j(\mathfrak{g}^i) = 0$ for all $i \geq 0$ and $j < 0$. Then, there is an equivalence of functors:*

$$\mathrm{Def}_{\mathrm{Tot}(\mathfrak{g}^{\Delta})} \cong H^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta}).$$

In particular, the functor $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})$ is a deformation functor, its tangent space is $H^1(\mathrm{Tot}(\mathfrak{g}^{\Delta}))$ and its obstructions are contained in $H^2(\mathrm{Tot}(\mathfrak{g}^{\Delta}))$.

In particular, if the functor of deformations of a geometric object $\mathcal{X}$ is isomorphic to the deformation functor $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^{\Delta})$ associated to a semicosimiplicial dgLa $\mathfrak{g}^{\Delta}$, then the Thom-Whitney dgLa $\mathrm{Tot}(\mathfrak{g}^{\Delta})$ controls the deformations of $\mathcal{X}$.

Example 1.16. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on a projective variety X. Denote by $\mathcal{E}nd(\mathcal{E})$ the sheaf of $\mathcal{O}_X$-modules endomorphisms of $\mathcal{E}$. It is a classical fact that $\mathcal{E}nd(\mathcal{E})$ encodes all the information of the infinitesimal deformations of $\mathcal{E}$. Over the field of complex numbers $\mathbb{C}$, it can be rephrased saying that the dgLa $A^{0,*}_X(\mathcal{E}nd(\mathcal{E}))$ controls the deformations of $\mathcal{E}$ via the Maurer-Cartan functor modulo the gauge equivalence [Fu03]. Over any algebraically closed field $\mathbb{K}$ of characteristic zero, one can consider the Čech semicosimplicial Lie algebra associated to the sheaf $\mathcal{E}nd(\mathcal{E})$ and with an open affine cover $\mathcal{V} = \{V_i\}_i$ of X:

$$\mathcal{E}nd(\mathcal{E})(\mathcal{V}): \ \prod_i \mathcal{E}nd(\mathcal{E})(V_i) \rightrightarrows \prod_{i<j} \mathcal{E}nd(\mathcal{E})(V_{ij}) \rightrightarrows \prod_{i<j<k} \mathcal{E}nd(\mathcal{E})(V_{ijk}) \rightrightarrows \cdots.$$

In [FMM12] the authors proved that the functor $H^1_{sc}(\exp \mathcal{E}nd(\mathcal{E})(\mathcal{V}))$ of Definition 1.13 is isomorphic to the functor of infinitesimal deformations of $\mathcal{E}$. Moreover, the cohomology of the total complex of the semicosimplicial dgLa above is $H^k(\mathrm{Tot}(\mathcal{E}nd(\mathcal{E})(\mathcal{V}))) \cong H^k(X, \mathcal{E}nd(\mathcal{E}))$ for all $k \in \mathbb{Z}$, according to the classical fact that the tangent space to the functor of the infinitesimal deformations of $\mathcal{E}$ is $H^1(X, \mathcal{E}nd(\mathcal{E}))$ and the obstructions are contained in $H^2(X, \mathcal{E}nd(\mathcal{E}))$.

1.3. **Semicosimplicial groupoid and descent theorem.** This subsection is dedicated to semicosimplicial objects in the category of groupoids, their total groupoid and the fundamental Hinich's Theorem on descent of Deligne groupoids. Here, we mainly follow [Man22].

Definition 1.17. A *semicosmplicial groupoid* is a covariant functor $\mathbf{\Delta}_{\mathrm{mon}} \to \mathbf{Grpds}$, from the category $\mathbf{\Delta}_{\mathrm{mon}}$, whose objects are finite ordinal sets and whose morphisms are order-preserving injective maps between them, to the category of groupoids. Equivalently, a semicosimplicial groupoid G^{Δ} is a diagram

$$G_0 \rightrightarrows G_1 \rightrightarrows G_2 \rightrightarrows \cdots,$$

where each G_i is a groupoid, and for each $i > 0$, there are $i+1$ morphisms of groupoids

$$\partial_{k,i} \colon G_{i-1} \to G_i, \qquad k = 0, \dots, i,$$

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such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$. Since every groupoid is a small category, equality of morphisms of groupoids is intended in the strict sense.

A *morphism of semicosimplicial groupoids* $f : G^\Delta \to H^\Delta$, is given by a sequence $\{f_i : G_i \to H_i\}$ of morphisms of groupoids, commuting with the maps $\partial_{k,i}$.

Example 1.18. Let $\mathfrak{g}^\Delta$ be a semicosimplicial dgLa

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \rightrightarrows\!\!\!\rightarrow \mathfrak{g}_2 \rightrightarrows\!\!\!\rightrightarrows \cdots,$$

such that every $\mathfrak{g}_i$ is a nilpotent dgLa. Applying the Deligne groupoid of Definition 1.6, one obtains the semicosimplicial groupoid

$$\mathrm{Del}_{\mathfrak{g}^\Delta} : \ \mathrm{Del}_{\mathfrak{g}_0} \rightrightarrows \mathrm{Del}_{\mathfrak{g}_1} \rightrightarrows\!\!\!\rightarrow \mathrm{Del}_{\mathfrak{g}_2} \rightrightarrows\!\!\!\rightrightarrows \cdots.$$

Analogously, for any $A \in \mathbf{Art}_{\mathbb{K}}$ and any semicosimplicial dgLa $\mathfrak{g}^\Delta$, one defines the semicosimplicial groupoid

$$\mathrm{Del}_{\mathfrak{g}^\Delta}(A) : \ \mathrm{Del}_{\mathfrak{g}_0}(A) \rightrightarrows \mathrm{Del}_{\mathfrak{g}_1}(A) \rightrightarrows\!\!\!\rightarrow \mathrm{Del}_{\mathfrak{g}_2}(A) \rightrightarrows\!\!\!\rightrightarrows \cdots.$$

Definition 1.19. Let G^Δ be a semicosimplicial groupoid. The *total groupoid* $\mathrm{Tot}(G^\Delta)$ of G^Δ is the groupoid defined as follows:

- the objects of $\mathrm{Tot}(G^\Delta)$ are the elements of the form (l, m), where l is an object in G_0 and $m : \partial_{0,1}l \to \partial_{1,1}l$ is a morphism in G_1 such that the following diagram commutes in G_2

$$\begin{array}{ccccc} & & \partial_{0,2}\partial_{0,1}l & & \\ & \swarrow\!\!= & & \searrow^{\partial_{0,1}m} & \\ \partial_{1,2}\partial_{0,1}l & & & & \partial_{0,2}\partial_{1,1}l \\ \downarrow{\scriptstyle \partial_{1,1}m} & & & & \| \\ \partial_{1,2}\partial_{1,1}l & & & & \partial_{2,2}\partial_{0,1}l \\ & =\!\!\searrow & & \swarrow_{\partial_{2,1}m} & \\ & & \partial_{1,2}\partial_{1,1}l, & & \end{array}$$

- the morphisms between two objects (l_0, m_0) and (l_1, m_1) are the morphisms a in G_0 such that the following diagram commutes

$$\begin{array}{ccc} \partial_{0,1}l_0 & \xrightarrow{m_0} & \partial_{1,1}l_0 \\ {\scriptstyle \partial_{0,1}a}\downarrow & & \downarrow{\scriptstyle \partial_{1,1}a} \\ \partial_{0,1}l_1 & \xrightarrow[m_1]{} & \partial_{1,1}l_1. \end{array}$$

Proposition 1.20. [Man22, Proposition 7.5.5] *Let $f : G^\Delta \to H^\Delta$ be a morphism of semicosimplicial groupoids, such that for every $n = 0, 1, 2$ the component $f_n : G_n \to H_n$ is an equivalence of groupoids, then $f : \mathrm{Tot}(G^\Delta) \to \mathrm{Tot}(H^\Delta)$ is also an equivalence of groupoids.*

We end up this section with the following theorem due to Hinich on descent of Deligne groupoids.

Theorem 1.21. [Hin97, Corollary 4.1.1] *Let $\mathfrak{g}^\Delta$ be a nilpotent semicosimplicial dgLa, such that every $\mathfrak{g}_n$ is trivial in negative degrees. Then, there exists a natural equivalence of groupoids*

$$\mathrm{Del}_{\mathrm{Tot}(\mathfrak{g}^\Delta)} \to \mathrm{Tot}(\mathrm{Del}_{\mathfrak{g}^\Delta}).$$

Remark 1.22. We remark that a generalisation of the previous Hinich's descent Theorem for the Deligne-Getzler ∞-groupoid was proved in [Ba17].

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2. Deformations of a morphism of locally free sheaves

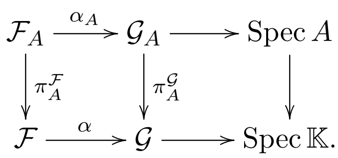
In this section, we would like to analyse the infinitesimal deformations of a morphism of locally free sheaves and explicitly describe a differential graded Lie algebra that controls the deformation problem.

2.1. **Geometric deformations.** Let X be a smooth projective variety over $\mathbb{K}$, $\mathcal{F}$ and $\mathcal{G}$ locally free sheaves of $\mathcal{O}_X$-modules over X and $\alpha\colon \mathcal{F} \to \mathcal{G}$ a morphism of them. First of all we recall some classical definitions.

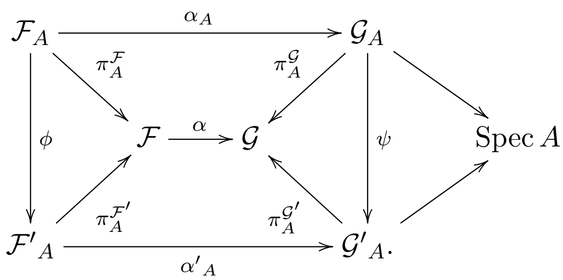
Definition 2.1. *An infinitesimal deformation of $\mathcal{F}$ over $A \in$* $\mathbf{Art}_{\mathbb{K}}$ is a locally free sheaf $\mathcal{F}_A$ of $\mathcal{O}_X \otimes A$-module on $X \times \operatorname{Spec} A$ together with a morphism $\pi_A : \mathcal{F}_A \to \mathcal{F}$, such that the obvious restriction of scalars $\pi_A : \mathcal{F}_A \otimes_A \mathbb{K} \to \mathcal{F}$ is an isomorphism.

Let $(\mathcal{F}_A, \pi_A)$ and $(\mathcal{F}'_A, \pi'_A)$ be two deformations of the sheaf $\mathcal{F}$ over A. They are *isomorphic*, if there exists an isomorphism $\phi\colon \mathcal{F}_A \to \mathcal{F}'_A$ of $\mathcal{O}_X \otimes A$-modules, that commutes with the restrictions to $\mathcal{F}$.

Definition 2.2. *An infinitesimal deformation of the morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ over $A \in$* $\mathbf{Art}_{\mathbb{K}}$ is a morphism $\alpha_A : \mathcal{F}_A \to \mathcal{G}_A$ of locally free sheaves of $\mathcal{O}_X \otimes A$-modules over $X \times \operatorname{Spec} A$, where $\mathcal{F}_A$ and $\mathcal{G}_A$ are deformations of $\mathcal{F}$ and $\mathcal{G}$ over A respectively, such that the following diagram is commutative:



Two deformations $\alpha_A : \mathcal{F}_A \to \mathcal{G}_A$ and $\alpha'_A : \mathcal{F}'_A \to \mathcal{G}'_A$ of $\alpha : \mathcal{F} \to \mathcal{G}$ over A are *isomorphic*, if there exist a pair (ϕ, ψ) of isomorphisms of sheaves $\phi : \mathcal{F}_A \to \mathcal{F}'_A$ and $\psi : \mathcal{G}_A \to \mathcal{G}'_A$, such that the following diagram commutes:



We recall that, the trivial deformation of a sheaf $\mathcal{F}$ over A is given by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times \operatorname{Spec} A} = \mathcal{F} \otimes_{\mathbb{K}} A$ and that the trivial deformation of $\alpha\colon \mathcal{F} \to \mathcal{G}$ over $A \in \mathbf{Art}_{\mathbb{K}}$ is given by the trivial extension $\alpha \otimes \mathrm{Id}_A \colon \mathcal{F} \otimes_{\mathbb{K}} A \to \mathcal{G} \otimes_{\mathbb{K}} A$.

Note that, since $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, any infinitesimal deformation of $\alpha\colon \mathcal{F} \to \mathcal{G}$ is locally trivial, i.e., it is locally in X isomorphic to the trivial deformation.

Definition 2.3. The *functor of infinitesimal deformations of the morphism $\alpha : \mathcal{F} \to \mathcal{G}$* is the functor

$$\operatorname{Def}_{(\mathcal{F},\alpha,\mathcal{G})} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds},$$

that associates, to any $A \in \mathbf{Art}_{\mathbb{K}}$, the groupoid $\operatorname{Def}_{(\mathcal{F},\alpha,\mathcal{G})}(A)$, whose objects are the deformations of the morphism α over A and whose morphisms are the isomorphisms of them. Note that

$$\pi_0(\operatorname{Def}_{(\mathcal{F},\alpha,\mathcal{G}))(A)}) = \{\text{isomorphism classes of deformations of the morphism } \alpha \text{ over } A\}$$

is the classical functor of deformations in **Sets**.

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Remark 2.4. Among all the infinitesimal deformations of $\alpha\colon \mathcal{F} \to \mathcal{G}$ over $A \in \mathbf{Art}_{\mathbb{K}}$, there are the infinitesimal deformations of the morphism α in which $\mathcal{F}$ and $\mathcal{G}$ deform trivially, i.e., the deformations $\alpha_A\colon \mathcal{F} \otimes_{\mathbb{K}} A \to \mathcal{G} \otimes_{\mathbb{K}} A$ such that just the map deforms. The groupoid of these deformations defines a subfunctor $\mathrm{Def}_\alpha : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds}$ of the functor $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})}$.

Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of locally free sheaves and γ its graph in $\mathcal{F} \oplus \mathcal{G}$. We define γ as the image of the morphism of sheaves $(\mathrm{Id}, \alpha) : \mathcal{F} \to \mathcal{F} \oplus \mathcal{G}$.

For future use, we would like to observe that a deformation of $\alpha : \mathcal{F} \to \mathcal{G}$ over $A \in \mathbf{Art}_{\mathbb{K}}$ as in Definition 2.2 can be also seen as the collection of the following data:

- two deformations $\mathcal{F}_A$ and $\mathcal{G}_A$ of $\mathcal{F}$ and $\mathcal{G}$ over A, respectively;
- a deformation $\gamma_A \subseteq (\mathcal{F} \oplus \mathcal{G})_A$ of the inclusion $\gamma \subseteq \mathcal{F} \oplus \mathcal{G}$ over A;
- an isomorphism f_A between the deformations $\mathcal{F}_A \oplus \mathcal{G}_A$ and $(\mathcal{F} \oplus \mathcal{G})_A$.

Two of such collections of data $(\mathcal{F}_A, \mathcal{G}_A, \gamma_A \subseteq (\mathcal{F} \oplus \mathcal{G})_A, f_A)$ and $(\mathcal{F}'_A, \mathcal{G}'_A, \gamma'_A \subseteq (\mathcal{F} \oplus \mathcal{G})'_A, f'_A)$ define isomorphic deformations of $\alpha : \mathcal{F} \to \mathcal{G}$, if there exist:

- two isomorphisms $\phi : \mathcal{F}_A \to \mathcal{F}'_A$ and $\psi : \mathcal{G}_A \to \mathcal{G}'_A$ of the deformations of $\mathcal{F}$ and $\mathcal{G}$ over A, respectively,
- an isomorphism $\chi : (\mathcal{F} \oplus \mathcal{G})_A \to (\mathcal{F} \oplus \mathcal{G})'_A$ of the deformations of $\mathcal{F} \oplus \mathcal{G}$ over A, with $\chi(\gamma_A) \subseteq \gamma'_A$,

such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_A \oplus \mathcal{G}_A & \xrightarrow{f_A} & (\mathcal{F} \oplus \mathcal{G})_A \\ \Big\downarrow{\scriptstyle (\phi,\psi)} & & \Big\downarrow{\scriptstyle \chi} \\ \mathcal{F}'_A \oplus \mathcal{G}'_A & \xrightarrow{f'_A} & (\mathcal{F} \oplus \mathcal{G})'_A. \end{array}$$

We point out that this approach is similar to the one used in [Ho73, Ho74, Ho76, Se06] to analyse deformations of holomorphic maps of complex manifolds. In particular, this was applied in [Ia06, Ia08] and in [Man22] to investigate these deformations via dgLas.

2.2. **The local case.** First of all, we analyse the infinitesimal deformations of a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ in the local case. We assume that X is a smooth affine variety over $\mathbb{K}$ and $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of the free sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Under this hypothesis, we are able to find out a dgLa that controls the deformations of α.

We denote by $\mathcal{E}nd(\mathcal{F})$, $\mathcal{E}nd(\mathcal{G})$ and $\mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})$ the free sheaves of the $\mathcal{O}_X$-modules endomorphisms of the sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \oplus \mathcal{G}$, respectively. Moreover, let $\mathcal{L}$ be the subsheaf of $\mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})$ that preserves the graph $\gamma \subseteq \mathcal{F} \oplus \mathcal{G}$ of the morphism α, i.e.,

$$\mathcal{L} := \{\varphi \in \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G}) \mid \varphi(\gamma) \subseteq \gamma\}.$$

Definition 2.5. In the above notation, we define the dgLa $H(X, \mathcal{F}, \alpha, \mathcal{G})$ as the Thom-Whitney homotopy fibre product of the diagram of Lie algebras

$$\begin{array}{ccc} & & \Gamma(X, \mathcal{L}) \\ & & \Big\downarrow{\scriptstyle h} \\ \Gamma(X, \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) & \xrightarrow{g} & \Gamma(X, \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})), \end{array}$$

where h is the inclusion and g is defined as the map $(\varphi, \psi) \mapsto \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}$. More explicitly, according to Definition 1.3, the elements of $H(X, \mathcal{F}, \alpha, \mathcal{G})$ are of the form $(x, y, z(t, dt))$, where

$$x \in \Gamma(X, \mathcal{L}), \quad y \in \Gamma(X, \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})),$$
$$z(t, dt) \in \Gamma(X, \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G}))[t, dt],$$

such that $z(0) = h(x), z(1) = g(y)$.

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Then, we can prove the following result.

Theorem 2.6. *In this setting, the functor* $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})}$ *of infinitesimal deformations of the morphism* $\alpha : \mathcal{F} \to \mathcal{G}$ *on* X *is equivalent to the Deligne functor* $\mathrm{Del}_{H(X,\mathcal{F},\alpha,\mathcal{G})}$ *associated to the dgLa* $H(X,\mathcal{F},\alpha,\mathcal{G})$.

Proof. By Definition 2.5, $H(X,\mathcal{F},\alpha,\mathcal{G})$ is the Thom-Whitney dgLa of the following semicosimplicial Lie algebra:

$$\mathfrak{h}^{\Delta} : \ \Gamma(X,\mathcal{L}) \oplus \Gamma(X,\mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) \underset{g}{\overset{h}{\rightrightarrows}} \Gamma(X,\mathcal{E}nd(\mathcal{F}\oplus\mathcal{G})) \rightrightarrows\!\!\!\to 0.$$

By Hinich's Theorem on descent of Deligne groupoids (Theorem 1.21), there is an equivalence of functors of groupoids

$$\mathrm{Del}_{H(X,\mathcal{F},\alpha,\mathcal{G})} \cong \mathrm{Tot}(\mathrm{Del}_{\mathfrak{h}^{\Delta}}),$$

i.e., for every $A \in \mathbf{Art}_{\mathbb{K}}$, there is an equivalence of the groupoids $\mathrm{Del}_{H(X,\mathcal{F},\alpha,\mathcal{G})}(A) \cong \mathrm{Tot}(\mathrm{Del}_{\mathfrak{h}^{\Delta}})(A)$.

Let us describe explicitly the objects and the morphisms of the groupoid $\mathrm{Tot}(\mathrm{Del}_{\mathfrak{h}^{\Delta}})(A)$, for any $A \in \mathbf{Art}_{\mathbb{K}}$. Its objects are elements of the form (x,y,z), where:

$$x \in \mathrm{Del}(\Gamma(X,\mathcal{L}) \otimes \mathfrak{m}_A), \quad y \in \mathrm{Del}(\Gamma(\mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) \otimes \mathfrak{m}_A),$$

and so $x = y = 0$, since we are dealing with Lie algebras, and

$$z = e^w \in \exp(\Gamma(X,\mathcal{E}nd(\mathcal{F}\oplus\mathcal{G})) \otimes \mathfrak{m}_A),$$

such that $e^w * 0 = 0 \in \Gamma(X,\mathcal{L}) \otimes \mathfrak{m}_A$. A morphism between two of such objects, e^w and e^t, is of the form e^a, where $a = (a_1,a_2,a_3) \in (\Gamma(X,\mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) \oplus \Gamma(X,\mathcal{L})) \otimes \mathfrak{m}_A$, such that $e^w e^{h(a)} = e^t e^{g(a)}$.

These data corresponds to the data of the objects and morphisms of the groupoid $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})}(A)$. Indeed, when X is an affine smooth variety, any deformation of a free sheaf on X is trivial and the only datum of a deformation of α is an isomorphism of the direct sum of the trivial deformations of $\mathcal{F}$ and $\mathcal{G}$ to the trivial deformation of $\mathcal{F}\oplus\mathcal{G}$, that preserves γ, this is e^w.

A morphism between two of such deformations, e^w and e^t, is given by an isomorphism of the trivial deformation of $\mathcal{F}$, an isomorphism of the trivial deformation of $\mathcal{G}$, an isomorphism of the trivial deformation of $\mathcal{F}\oplus\mathcal{G}$ that preserve γ. These data are (a_1,a_2,a_3), respectively, and they have to respect compatibilities with the isomorphisms above, that are expressed by the equation $e^w e^{h(a)} = e^t e^{g(a)}$. □

2.3. **The global case.** Let, now, X be a smooth projective variety over $\mathbb{K}$, $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism of locally free sheaves on X and $\mathcal{V} = \{V_i\}_i$ an open affine cover of X trivialising both $\mathcal{F}$ and $\mathcal{G}$.

Consider the following semicosimplicial dgLa

$$H(\mathcal{V})^{\Delta} : \ \prod_i H(V_i,\mathcal{F},\alpha,\mathcal{G}) \rightrightarrows \prod_{i<j} H(V_{ij},\mathcal{F},\alpha,\mathcal{G}) \rightrightarrows\!\!\!\to \prod_{i<j<k} H(V_{ijk},\mathcal{F},\alpha,\mathcal{G}) \cdots,$$

where $H(V_i,\mathcal{F},\alpha,\mathcal{G})$ is the Thom-Whitney homotopy fibre of Definition 2.5.

Then, we can prove the following result.

Theorem 2.7. *In this setting, the functor* $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})}$ *of infinitesimal deformations of the morphism* $\alpha : \mathcal{F} \to \mathcal{G}$ *on* X *is equivalent to the Deligne functor* $\mathrm{Del}_{\mathrm{Tot}(H(\mathcal{V})^{\Delta})}$ *associated to the Thom-Whitney dgLa of the semicosimplicial dgLa* $H(\mathcal{V})^{\Delta}$.

Proof. Applying Hinich's Theorem on descent of Deligne groupoids (Theorem 1.21), there exists an equivalence of groupoids:

$$\mathrm{Del}_{\mathrm{Tot}(H(\mathcal{V})^{\Delta})} \cong \mathrm{Tot}(\mathrm{Del}_{H(\mathcal{V})^{\Delta}}).$$

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By Proposition 1.20 and by the local case, analysed in Theorem 2.6, we have

$$\mathrm{Tot}(\mathrm{Del}_{H(\mathcal{V})^\Delta}) \cong \mathrm{Tot}(\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(\mathcal{V})^\Delta}).$$

Here $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(\mathcal{V})^\Delta}$ is the semicosimplicial functor in groupoids

$$\prod_i \mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(V_i)} \rightrightarrows \prod_{i<j} \mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(V_{ij})} \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \prod_{i<j<k} \mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(V_{ijk})} \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \cdots,$$

where $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(V)}$ is the functor of infinitesimal deformations of $\alpha_{|V} : \mathcal{F}_{|V} \to \mathcal{G}_{|V}$, for any $V \subset X$ affine open subset that trivialises both $\mathcal{F}$ and $\mathcal{G}$. Moreover, by a classical argument, global deformations are given by gluing local ones, then

$$\mathrm{Tot}(\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})(\mathcal{V})^\Delta}) \cong \mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})},$$

as desired. □

Corollary 2.8. *In the above notation, the Thom-Whitney dgLa associated to the semicosimplicial dgLa $H(\mathcal{V})^\Delta$ controls the infinitesimal deformations of the morphism of locally free sheaves $\alpha : \mathcal{F} \to \mathcal{G}$.*

Remark 2.9. We note that, for any coherent sheaf of dgLas H, the quasi isomorphism class of the Thom-Whitney dgLa $\mathrm{Tot}(H(\mathcal{V})^\Delta)$ does not depend on the choice of the affine open cover $\mathcal{V}$ of X [FIM12]. Then, the previous construction does not depend on the choice of the cover.

Remark 2.10. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of locally free sheaves of $\mathcal{O}_X$-modules on X and let $\mathcal{V} = \{V_i\}_i$ be an affine open cover of X, we can construct a bisemicosimplicial object associated to these data as follows. On every open set $V_i \in \mathcal{V}$, we have

$$\Gamma(V_i,\mathcal{L}) \oplus \Gamma(V_i, \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) \underset{g}{\overset{h}{\rightrightarrows}} \Gamma(V_i, \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})) \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} 0.$$

Since h and g are morphisms of sheaves, they commute with restrictions of every open subsets, inducing morphisms of the Čech semicosimplicial objects

$$\mathcal{L}(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{F})(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{G})(\mathcal{V}) \underset{g}{\overset{h}{\rightrightarrows}} \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})(\mathcal{V}) \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} 0.$$

In Theorem 2.7, we define, for every open set V_i, the dgLas $H(V_i,\mathcal{F},\alpha,\mathcal{G})$ as the Thom-Withney homotopy fiber products; they form a semicosimplicial dgLas $H(\mathcal{V})^\Delta$ and we consider the associated Thom-Withney dgLa $\mathrm{Tot}(H(\mathcal{V})^\Delta)$. This construction does not depend on the order [Ia10]. We could first consider the Thom-Withney dgLas of the Čech semicosimplicial sheaves of Lie algebras $\mathcal{L}(\mathcal{V})$, $\mathcal{E}nd(\mathcal{F})(\mathcal{V})$, $\mathcal{E}nd(\mathcal{G})(\mathcal{V})$ and $\mathcal{E}nd(\mathcal{F}\oplus\mathcal{G})(\mathcal{V})$ and obtain the semicosimplicial dgLa

(4)

$$\mathrm{Tot}(\mathcal{L}(\mathcal{V})) \oplus \mathrm{Tot}(\mathcal{E}nd(\mathcal{F})(\mathcal{V})) \oplus \mathrm{Tot}(\mathcal{E}nd(\mathcal{G})(\mathcal{V})) \underset{g}{\overset{h}{\rightrightarrows}} \mathrm{Tot}(\mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})(\mathcal{V})) \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} 0$$

and then apply Thom-Withney homotopy fibre product. This dgLa is quasi isomorphic to $\mathrm{Tot}(H(\mathcal{V})^\Delta)$.

Since the cohomology of the total complex of the semicosimplicial dgLa $H(\mathcal{V})^\Delta$ is the same as the one of the Thom-Withney homotopy fibre, according to [Ia08, Section 4], we get the following exact sequence:

$$\ldots \to H^i(\mathrm{Tot}(H(\mathcal{V})^\Delta) \to H^i(X, \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G}) \oplus \mathcal{L}) \to$$
$$\to H^i(X, \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})) \to H^{i+1}(\mathrm{Tot}(H(\mathcal{V})^\Delta) \to \ldots$$

Moreover, since the map $h : \mathcal{L} \to \mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})$ is injective, thanks to [Ia08, Lemma 3.1], the following sequence is also exact:

(5)

$$\cdots H^i(\mathrm{Tot}(H(\mathcal{V})^\Delta) \to H^i(X, \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{G})) \to H^i(X, \operatorname{coker} h) \to H^{i+1}(X, \mathrm{Tot}(H(\mathcal{V})^\Delta) \cdots.$$

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Note that, since $\mathrm{Tot}(H(\mathcal{V})^{\Delta})$ controls the infinitesimal deformations of $\alpha : \mathcal{F} \to \mathcal{G}$, its first cohomology space is isomorphic to the tangent space of the functor $\mathrm{Def}_{(\mathcal{F},\alpha,\mathcal{G})}$, and its second cohomology space is an obstruction space.

Remark 2.11. If we work over the field of complex numbers, we can consider the Dolbeault resolutions of the endomorphisms of sheaves, and avoid the Čech semicosimplicial dgLas. More precisely, we can consider the two morphisms of dgLas

$$A^{0,*}_X(\mathcal{L}) \oplus A^{0,*}_X(\mathcal{E}nd(\mathcal{F})) \oplus A^{0,*}_X(\mathcal{E}nd(\mathcal{G})) \underset{g}{\overset{h}{\rightrightarrows}} A^{0,*}_X(\mathcal{E}nd(\mathcal{F} \oplus \mathcal{G})),$$

analogous to the previous h and g. Then, the associated Thom-Whitney homotopy fibre product of Definition 1.3 is a dgLa that controls infinitesimal deformations of $\alpha : \mathcal{F} \to \mathcal{G}$. Note that this construction is the analogous of the one in Equation (4) of Remark 2.10.

3. Deformations of a locally free sheaf and a subspace of sections

In this section, we would like to apply the previous results to the infinitesimal deformations of a locally free sheaf with a subspace of its sections, finding a dgLa that controls them.

3.1. **Geometric deformations.** Let X be a smooth projective variety over $\mathbb{K}$, $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and $U \subseteq H^0(X, \mathcal{E})$ be a subspace of global sections of $\mathcal{E}$. We study the infinitesimal deformations of $\mathcal{E}$ which preserve the subspace U. We start with some definitions.

Definition 3.1. Let $A \in \mathbf{Art}_{\mathbb{K}}$. An *infinitesimal deformation* of the pair $(\mathcal{E}, U)$ over A is the data $(\mathcal{E}_A, \pi_A, i_A)$ of:

- a deformation $(\mathcal{E}_A, \pi_A)$ of $\mathcal{E}$ over A,
- a morphism $i_A : U \otimes A \to H^0(X \times \mathrm{Spec}\, A, \mathcal{E}_A)$,

such that the following diagram commutes

$$\begin{array}{ccc} U \otimes A & \xrightarrow{i_A} & H^0(X \times \mathrm{Spec}\, A, \mathcal{E}_A) \\ \downarrow \pi & & \downarrow \pi_A \\ U & \overset{i}{\hookrightarrow} & H^0(X, \mathcal{E}). \end{array} \tag{6}$$

Two deformations $(\mathcal{E}_A, \pi_A, i_A)$, $(\mathcal{E}'_A, \pi'_A, i'_A)$ are *isomorphic* if there exist an isomorphism $\phi : \mathcal{E}_A \to \mathcal{E}'_A$ of sheaves of $\mathcal{O}_X \otimes A$-modules, such that $\pi'_A \circ \phi = \pi_A$ and an isomorphism $\psi : U \otimes A \to U \otimes A$, that makes the diagram commutative:

$$\begin{array}{ccc} U \otimes A & \xrightarrow{i_A} & H^0(X \times \mathrm{Spec}\, A, \mathcal{E}_A) \\ \downarrow \psi & & \downarrow \phi \\ U \otimes A & \xrightarrow{i'_A} & H^0(X \times \mathrm{Spec}\, A, \mathcal{E}'_A). \end{array}$$

Note that, this implies that ϕ induces an isomorphism $\phi : i_A(U \otimes A) \to i'_A(U \otimes A)$. In the following, we will often shorten the notation of such deformations with $(\mathcal{E}_A, i_A)$.

Definition 3.2. The *functor of infinitesimal deformations of* $(\mathcal{E}, U)$ is

$$\mathrm{Def}_{(\mathcal{E},U)} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds},$$

that associates to any $A \in \mathbf{Art}_{\mathbb{K}}$ the groupoid $\mathrm{Def}_{(\mathcal{E},U)}(A)$, whose objects are the infinitesimal deformations of the pair $(\mathcal{E}, U)$ over A and whose morphisms are the isomorphisms among them. Note that

$$\pi_0(\mathrm{Def}_{(\mathcal{E},U))}(A)) = \{\text{isomorphism classes of deformations of the pair } (\mathcal{E}, U) \text{ over } A\}$$

is the classical functor of deformations in **Sets**.

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Remark 3.3. Let X be a smooth projective variety over $\mathbb{K}$ with $H^1(X, \mathcal{O}_X) = 0$. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and $U \subseteq H^0(X, \mathcal{E})$ a subspace of global sections of it. Let $s : U \otimes \mathcal{O}_X \to \mathcal{E}$ be the morphism of sheaves of $\mathcal{O}_X$-modules associated to $i : U \hookrightarrow H^0(\mathcal{E})$ via the obvious correspondence between global sections and morphisms from $\mathcal{O}_X$ to $\mathcal{E}$.

According to Definition 2.3, we denote by

$$\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpds},$$

the functor of infinitesimal deformations of the morphism s, that associates, to any $A \in \mathbf{Art}_{\mathbb{K}}$, the groupoid $\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}(A)$, whose objects are the infinitesimal deformations of the morphism s over A and whose morphisms are the isomorphisms between them.

We claim that, for any $A \in \mathbf{Art}_{\mathbb{K}}$ there is a 1-1 correspondence between the objects of the groupoids

$$\mathrm{Def}_{(\mathcal{E},U)}(A) \to \mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}(A).$$

Indeed, given a deformation $(\mathcal{E}_A, i_A)$ of $(\mathcal{E}, U)$ over A as in Definition 3.1, there is just one morphism of sheaves of $\mathcal{O}_X \otimes A$-modules $s_A : U \otimes \mathcal{O}_X \otimes A \to \mathcal{E}_A$ associated to i_A, it makes the following diagram commutative

$$\begin{array}{ccc} U\otimes\mathcal{O}_X\otimes A & \xrightarrow{s_A} & \mathcal{E}_A \\ \downarrow \pi & & \downarrow \pi_A \\ U\otimes\mathcal{O}_X & \xrightarrow{s} & \mathcal{E}, \end{array} \tag{7}$$

and it defines an object of $\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}(A)$. On the other hand, since $H^1(X, \mathcal{O}_X) = 0$, every infinitesimal deformation of the domain $U \otimes \mathcal{O}_X$ is trivial. A deformation of s over A is just the data of a deformation $\mathcal{E}_A$ of $\mathcal{E}$ and of a map $s_A : U \otimes \mathcal{O}_X \otimes A \to \mathcal{E}_A$, such that the diagram (7) is commutative. Taking global sections, one gets the required deformed map $i_A : U \otimes A \to H^0(X \times \operatorname{Spec} A, \mathcal{E}_A)$ and the commutative diagram (6).

Analogously, we can prove that isomorphisms of deformations correspond each others. This assures that, under the hypothesis $H^1(X, \mathcal{O}_X) = 0$, the functors $\mathrm{Def}_{(\mathcal{E},U)}$ and $\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}$ are are equivalent.

In [Mar08, Mar09], the second author introduced the problem of deformations of the pair $(\mathcal{E}, U)$ and found a dgLa that controls it on a variety X over the field of complex numbers. In [IM23], we studied these deformations in more details and we generalised to vector bundles over a smooth complex projective variety some classical results known for line bundles over curves, concerning the description of the tangent space, the smoothness of the functor $\mathrm{Def}_{(\mathcal{E},U)}$ and the liftings of a section. Here, we would like to study the infinitesimal deformations of the pair $(\mathcal{E}, U)$ over an algebraically closed field $\mathbb{K}$ of characteristic zero using the analysis of the previous sections.

3.2. **The dgLa that controls deformations of** $(\mathcal{E}, U)$**.** Let X be a smooth projective variety over $\mathbb{K}$, we assume here $H^1(X, \mathcal{O}_X) = 0$. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X and $U \subseteq H^0(X, \mathcal{E})$ be a subspace of global sections of it.

Let $s : U \otimes \mathcal{O}_X \to \mathcal{E}$ be the morphism of sheaves of $\mathcal{O}_X$-modules associated to $i : U \hookrightarrow H^0(\mathcal{E})$. As observed in Remark 3.3, the infinitesimal deformations of the pair $(\mathcal{E}, U)$ are equivalent to the infinitesimal deformations of the morphism s. We indicate with γ the graph of the morphism s, that is a subsheaf of $(U \otimes \mathcal{O}_X) \oplus \mathcal{E}$.

Let $\mathcal{V} = \{V_i\}_i$ be an affine open cover of X, we denote by $\mathcal{E}nd(\mathcal{E}), \mathcal{E}nd(U \otimes \mathcal{O}_X)$ and $\mathcal{E}nd\,((U \otimes \mathcal{O}_X) \oplus \mathcal{E})$ the sheaves of O_X-modules endomorphisms of $\mathcal{E}, U \otimes \mathcal{O}_X$ and $(U \otimes \mathcal{O}_X) \oplus \mathcal{E}$, respectively. Let $H(\mathcal{V})^\Delta$ be the Cech semicosimplicial dgLa associated to

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the Thom-Whitney homotopy fibres of the diagram of sheaves of Lie algebras

$$\begin{array}{ccc} & & \mathcal{L} \\ & & \downarrow h \\ \mathcal{E}nd(U\otimes\mathcal{O}_X)\oplus\mathcal{E}nd(\mathcal{E}) & \xrightarrow{g} & \mathcal{E}nd\left((U\otimes\mathcal{O}_X)\oplus\mathcal{E}\right), \end{array}$$

where $h:\mathcal{L}=\{\varphi\in\mathcal{E}nd\left((U\otimes\mathcal{O}_X)\oplus\mathcal{E}\right)\mid\varphi(\gamma)\subseteq\gamma\}\to\mathcal{E}nd\left((U\otimes\mathcal{O}_X)\oplus\mathcal{E}\right)$ is the inclusion and g is defined as the map $(\varphi,\psi)\mapsto\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}$. Applying the results of the previous section, we get the following result

Theorem 3.4. *In the above notation and under the hypothesis $H^1(X,\mathcal{O}_X)=0$, the functor $\mathrm{Def}_{(\mathcal{E},U)}$ of deformations of $(\mathcal{E},U)$ is equivalent to the Deligne functor associated to the Thom-Whitney dgLa of the semicosimplicial dgLa $H(\mathcal{V})^\Delta$. Moreover, under the additional hypothesis that $H^2(X,\mathcal{O}_X)=0$, the cohomology of the dgLa $\mathrm{Tot}(H(\mathcal{V})^\Delta)$ fits into the exact sequence*

$$\begin{aligned} &0\to H^0(\mathrm{Tot}(H(\mathcal{V})^\Delta))\to H^0(X,\mathcal{E}nd(\mathcal{E}))\to\mathrm{Hom}(U,H^0(\mathcal{E})/U)\to\\ &\to H^1(\mathrm{Tot}(H(\mathcal{V})^\Delta))\to H^1(X,\mathcal{E}nd(\mathcal{E}))\xrightarrow{\alpha}\mathrm{Hom}(U,H^1(X,\mathcal{E}))\to\\ &\to H^2(\mathrm{Tot}(H(\mathcal{V})^\Delta))\to H^2(X,\mathcal{E}nd(\mathcal{E}))\to\mathrm{Hom}(U,H^2(X,\mathcal{E})). \end{aligned}\tag{8}$$

Proof. The equivalence

$$\mathrm{Def}_{(\mathcal{E},U)}\cong\mathrm{Del}_{\mathrm{Tot}(H(\mathcal{V})^\Delta)}$$

is a direct consequence of Theorem 2.7 and Remark 3.3. According to the sequence (5), the cohomology of the dgLa $\mathrm{Tot}(H(\mathcal{V})^\Delta)$ fits in the exact sequence

(9)

$$\cdots H^i(\mathrm{Tot}(H(\mathcal{V})^\Delta)\to H^i(X,\mathcal{E}nd(U\otimes\mathcal{O}_X)\oplus\mathcal{E}nd(\mathcal{E}))\to H^i(X,\operatorname{coker}h)\to H^{i+1}(\mathrm{Tot}(H(\mathcal{V})^\Delta)\cdots.$$

Observe that, since X is projective,

$$H^0(X,\mathcal{E}nd(U\otimes\mathcal{O}_X)\oplus\mathcal{E}nd(\mathcal{E}))=\mathrm{Hom}(U,U)\oplus H^0(X,\mathcal{E}nd(\mathcal{E}));$$

while for $i=1,2$, under the hypothesis $H^i(X,\mathcal{O}_X)=0$, we get

$$H^i(X,\mathcal{E}nd(U\otimes\mathcal{O}_X)\oplus\mathcal{E}nd(\mathcal{E}))=\left(H^i(X,\mathcal{O}_X)\otimes U^\vee\otimes U\right)\oplus H^i(X,\mathcal{E}nd(\mathcal{E})=H^i(X,\mathcal{E}nd(\mathcal{E})).$$

Moreover, we can describe explicitly the $\operatorname{coker}h$ via the following isomorphisms

$$\operatorname{coker}h\cong\mathcal{H}om\left(\gamma,\frac{(U\otimes\mathcal{O}_X)\oplus\mathcal{E}}{\gamma}\right)\cong\mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E}).$$

The first one is given by restricting an endomorphism of $(U\otimes\mathcal{O}_X)\oplus\mathcal{E}$ to γ and projecting it to the quotient. The second one is obtained just observing that $U\otimes\mathcal{O}_X\cong\gamma$ and $\dfrac{(U\otimes\mathcal{O}_X)\oplus\mathcal{E}}{\gamma}\cong\mathcal{E}$. Thus, for $i=0$, we have

$$H^0(\operatorname{coker}h)=H^0(\mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E}))=\mathrm{Hom}(U,H^0(X,\mathcal{E}))=\mathrm{Hom}(U,U)\oplus\mathrm{Hom}\left(U,\frac{H^0(X,\mathcal{E})}{U}\right),$$

where the last equality follows applying $\mathrm{Hom}(U,-)$ to the exact sequence of vector spaces

$$0\to U\to H^0(X,\mathcal{E})\to H^0(X,\mathcal{E})/U\to 0.$$

While, for $i=1,2$, we have

$$H^i(\operatorname{coker}h)=H^i(\mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E}))=\mathrm{Hom}(U,H^i(X,\mathcal{E})).$$

Thus, the first terms of the exact sequence (9) become

(10)

$$0\to H^0(\mathrm{Tot}(H(\mathcal{V})^\Delta))\to\mathrm{Hom}(U,U)\oplus H^0(X,\mathcal{E}nd(\mathcal{E}))\xrightarrow{f}\mathrm{Hom}(U,U)\oplus\mathrm{Hom}\left(U,\frac{H^0(X,\mathcal{E})}{U}\right)\to$$

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$$\to H^1(\mathrm{Tot}(H(\mathcal{V})^{\Delta})) \to H^1(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^1(X,\mathcal{E})) \to H^2(\mathrm{Tot}(H(\mathcal{V})^{\Delta})) \to$$
$$\to H^2(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^2(X,\mathcal{E})).$$

The last step is to make explicit the morphism f of (10). It is immediate to see that f is the morphism

$$\mathrm{Hom}(U,U) \oplus H^0(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{f} \mathrm{Hom}(U,U) \oplus \mathrm{Hom}\left(U, \frac{H^0(X,\mathcal{E})}{U}\right),$$

whose first component is the identity on $\mathrm{Hom}(U,U)$. Thus the exact sequence (10) induces the sequence (8). $\square$

The exact sequence (8) of Theorem 3.4 is a generalisation over any algebraically closed field of characteristic zero of the exact sequence (7) in [IM23], from which we get most of our results, that now can be immediately generalised to any algebraically closed field $\mathbb{K}$ of characteristic zero.

Let us denote by $r_U : \mathrm{Def}_{(\mathcal{E},U)} \to \mathrm{Def}_{\mathcal{E}}$, the forgetful functor, i.e., the functor that ignores all information about U. Then, under the hypothesis $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, we have the following results, that holds for over any algebraically closed field $\mathbb{K}$ of characteristic zero.

Corollary 3.5. [IM23, Corollary 3.11] *If the map* $\alpha : H^1(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^1(X,\mathcal{E}))$ *that appears in the exact sequence* (8) *is surjective or, that is equivalent, if* $\mathrm{Hom}(U, H^1(X,\mathcal{E})) = 0$*, then the forgetful morphism* $r_U : \mathrm{Def}_{(\mathcal{E},U)} \to \mathrm{Def}_{\mathcal{E}}$ *is smooth.*

Corollary 3.6. [IM23, Corollary 3.13] *In the notation above, we have*

$$\dim t_{\mathrm{Def}_{(\mathcal{E},U)}} \geq \dim t_{\mathrm{Def}_{\mathcal{E}}} - m \cdot \dim H^1(X,\mathcal{E}),$$

where m is the dimension of $U \subseteq H^0(X,\mathcal{E})$.

Corollary 3.7. [IM23, Corollary 3.11] *A section $s \in H^0(X,\mathcal{E})$ can be extended to a section of a first order deformation of $\mathcal{E}$ associated to an element $a \in H^1(X, \mathcal{E}nd(\mathcal{E}))$ if and only if $a \cup s = 0 \in H^1(X,\mathcal{E})$.*

Corollary 3.8. [IM23, Corollary 3.15] *The tangent space to the deformations of the pair $(\mathcal{E}, H^0(\mathcal{E}))$ can be identified with*

$$t_{\mathrm{Def}_{(\mathcal{E},H^0(\mathcal{E}))}} = \{a \in H^1(X, \mathcal{E}nd(\mathcal{E})) \mid a \cup s = 0, \ \forall\ s \in H^0(X,\mathcal{E})\}.$$

Remark 3.9. We underline that the results of [IM23] hold over the field of complex numbers $\mathbb{C}$ without the additional hypothesis that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$.

3.3. **An explicit model for the semicosimplicial dgla that controls deformations of $(\mathcal{E}, U)$.** In this section, we describe another semicosimplicial dgLa whose total complex controls deformations of the pairs $(\mathcal{E}, U)$ over a field $\mathbb{K}$. The construction is more explicit and it avoids the Theorem on descent of Deligne groupoids (Theorem 1.21).

In the previous notation, given the morphism of sheaves $s : U \otimes \mathcal{O}_X \to \mathcal{E}$, we consider the complex of sheaves

$$\mathcal{Q} : U \otimes \mathcal{O}_X \xrightarrow{s} \mathcal{E},$$

where the sheaves $U \otimes \mathcal{O}_X$ and $\mathcal{E}$ are settled in degree zero and one, respectively. As above, s is the morphism associated to $i : U \hookrightarrow H^0(\mathcal{E})$ and it is a differential, $s^2 = 0$, since there is no degree 2. A similar construction is used in [Man22, pag. 234-235] to analyse the embedded deformations. Consider the following complex of sheaves concentrated in degree 0 and 1:

$$\mathcal{H}om^{\geq 0}(\mathcal{Q},\mathcal{Q}) = \begin{cases} \mathcal{H}om^0(\mathcal{Q},\mathcal{Q}) &= \mathcal{E}nd(U \otimes \mathcal{O}_X) \oplus \mathcal{E}nd(\mathcal{E}) \\ \mathcal{H}om^1(\mathcal{Q},\mathcal{Q}) &= \mathcal{H}om(U \otimes \mathcal{O}_X, \mathcal{E}) \\ \mathcal{H}om^k(\mathcal{Q},\mathcal{Q}) &= 0 \quad \forall\ k \neq 0, 1, \end{cases}$$

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This is a sheaf of dgLas. The differential is given by s, the bracket is the usual one in $\mathcal{E}nd(U\otimes\mathcal{O}_X)$ and $\mathcal{E}nd(\mathcal{E})$, it is the trivial bracket between an element in $\mathcal{E}nd(U\otimes\mathcal{O}_X)$ and an element in $\mathcal{E}nd(\mathcal{E})$ and it is given by the composition between elements in degree 0 and 1.

Next, fix an open affine cover $\mathcal{V}=\{V_i\}_i$ of X trivialising $\mathcal{E}$ and let $\mathfrak{g}^\Delta$ be the Čech semicosimplicial dgLa $\mathfrak{g}^\Delta$ associated to the sheaf $\mathcal{H}om^{\geq 0}(\mathcal{Q},\mathcal{Q})$ and to the open cover $\mathcal{V}$. The aim of this section is to prove the following result.

Theorem 3.10. *The Čech semicosimplicial dgLa $\mathfrak{g}^\Delta$ associated to the sheaf $\mathcal{H}om^{\geq 0}(\mathcal{Q},\mathcal{Q})$ controls the infinitesimal deformations of the morphism $s: U\otimes\mathcal{O}_X\to\mathcal{E}$, i.e., the functor $\widetilde{H}^1_{sc}(\exp\mathfrak{g}^\Delta)$ and the functor $\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}$ are isomorphic.*

Proof. For any $A\in\mathbf{Art}_{\mathbb{K}}$, we define a 1-1 correspondence between the objects of the groupoids

$$\widetilde{H}^1_{sc}(\exp\mathfrak{g}^\Delta)(A)\to\mathrm{Def}_{(U\otimes\mathcal{O}_X,\mathcal{E},s)}(A).$$

An element in $Z^1_{sc}(\exp\mathfrak{g}^\Delta)(A)$ is a pair $(l,m)\in\left(\mathfrak{g}^1_0\oplus\mathfrak{g}^0_1\right)\otimes\mathfrak{m}_A$, where the element $l=\{l_i\}_i\in\prod_i\mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E})(V_i)\otimes\mathfrak{m}_A$ and the element $m=\{\nu_{ij},\mu_{ij}\}_{ij}\in\prod_{i<j}(\mathcal{E}nd(U\otimes\mathcal{O}_X)(V_{ij})\oplus\mathcal{E}nd(\mathcal{E})(V_{ij}))\otimes\mathfrak{m}_A$, and it has to satisfy the equations in (2).

Using these data, we can define a deformation of the morphism $s: U\otimes\mathcal{O}_X\to\mathcal{E}$ over A as follows.

- The deformation $\mathcal{E}_A$ of the locally free sheaf $\mathcal{E}$ over A is obtained by gluing the local trivial deformations $\{\mathcal{E}|_{V_i}\otimes A\}_i$ by the isomorphisms $\{e^{\mu_{ij}}:\mathcal{E}_{|V_{ij}}\otimes A\to\mathcal{E}_{|V_{ij}}\otimes A\}_{ij}$. Note that the third equation of (2) gives rise in our case to the equation $\partial_{0,2}\mu\bullet-\partial_{1,2}\mu\bullet\partial_{2,2}\mu=0$, that assures that the given gluing functions are compatible on the triple intersections.
- Similarly, the data $\{e^{\nu_{ij}}: U\otimes\mathcal{O}_X|_{V_{ij}}\otimes A\to U\otimes\mathcal{O}_X|_{V_{ij}}\otimes A\}_{ij}$ are gluing isomorphisms of the local trivial deformations of the sheaf $U\otimes\mathcal{O}_X$ over A in the open sets V_{ij}. The third equation in (2) gives rise to the equation $\partial_{0,2}\nu\bullet-\partial_{1,2}\nu\bullet\partial_{2,2}\nu=0$ and it assures that the gluing isomorphisms are compatible on the triple intersections.
- The morphism $s_A: U\otimes\mathcal{O}_X\otimes A\to\mathcal{E}_A$ is locally defined as $\{s|_{V_i}+l_i: U\otimes\mathcal{O}_X|_{V_i}\otimes A\to\mathcal{E}|_{V_i}\otimes A\}_i$. Note that the second equation of (2) is $\delta_{11}l=e^m*\delta_{01}l$ and it can be rewritten as $s+\delta_{11}l=e^\mu(s+\delta_{01}l)e^{-\nu}$ (see for example [IM23, Formula 8]). It says that the local maps $\{s|_{V_i}+l_i\}_i$ glue to a global one.

Note that the first equation $dl+\frac{1}{2}[l,l]=0$ is trivial in our case, because our dgLas are zero in degree 2.

Thus every element in $Z^1_{sc}(\exp\mathfrak{g}^\Delta)(A)$ exactly defines an infinitesimal deformation of the morphism $s: U\otimes\mathcal{O}_X\to\mathcal{E}$ on A. Let now $(l_0,m_0),(l_1,m_1)\in Z^1_{sc}(\exp\mathfrak{g}^\Delta)(A)$ be isomorphic elements. We prove that they define two isomorphic deformations in $\mathrm{Def}_{(U\otimes\mathcal{O}_X,s,\mathcal{E})}(A)$.

- By the first equation in (3), there exists $a\in\mathfrak{g}^0_0\otimes\mathfrak{m}_A$ such that $e^a*l_0=l_1$. Note that $a=(\beta,\alpha)=\{\beta_i,\alpha_i\}_i\in\prod_i(\mathcal{E}nd(U\otimes\mathcal{O}_X)(V_i)\oplus\mathcal{E}nd(\mathcal{E})(V_i))\otimes\mathfrak{m}_A$ and as usual, the equation can be rewritten as $s+l_1=e^\alpha(s+l_0)e^{-\beta}$. Then, for every i, $e^{\alpha_i}:\mathcal{E}_{|V_i}\otimes A\to\mathcal{E}_{|V_i}\otimes A$ and $e^{\beta_i}: U\otimes\mathcal{O}_X|_{V_i}\otimes A\to U\otimes\mathcal{O}_X|_{V_i}\otimes A$ define local isomorphisms that make the following diagram commutative:

$$\begin{array}{ccc} U\otimes\mathcal{O}_X|_{V_i}\otimes A & \xrightarrow{s+(l_0)_i} & \mathcal{E}|_{V_i}\otimes A \\ \Big\downarrow{\scriptstyle e^{\beta_i}} & & \Big\downarrow{\scriptstyle e^{\alpha_i}} \\ U\otimes\mathcal{O}_X|_{V_i}\otimes A & \xrightarrow{s+(l_1)_i} & \mathcal{E}|_{V_i}\otimes A. \end{array}$$

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- The second equation in (3) gives rise to the following two equations $-\mu_0 \bullet -\partial_{1,1}\alpha \bullet \mu_1 \bullet \partial_{0,1}\alpha = 0$ and $-\nu_0 \bullet -\partial_{1,1}\beta \bullet \nu_1 \bullet \partial_{0,1}\beta = 0$ that assure that the isomorphisms e^{α_i} and e^{β_i} glue to a global one.

Thus, as wanted, the two deformations defined by $(l_0, m_0), (l_1, m_1)$ are isomorphic.

In this way, we obtain every infinitesimal deformation of the morphism $s : U \otimes \mathcal{O}_X \to \mathcal{E}$. Indeed, on a fix open affine set V_i, the infinitesimal deformations of $\mathcal{E}$ and $U \otimes \mathcal{O}_X$ are trivial, while a deformation of $s : U \otimes \mathcal{O}_X \to \mathcal{E}$ over V_i, is given by $s|_{V_i} + l_i : U \otimes \mathcal{O}_X|_{V_i} \otimes A \to \mathcal{E}|_{V_i} \otimes A$, for $l_i \in \mathcal{H}om(U \otimes \mathcal{O}_X, \mathcal{E})(V_i) \otimes \mathfrak{m}_A$. Then, we have to glue these data along double intersections to give a deformation of the morphism s on X. Therefore, we need isomorphisms $\tilde{\mu}_{ij} : \mathcal{E}_{|V_{ij}} \otimes A \to \mathcal{E}_{|V_{ij}} \otimes A$ for all i, j, that satisfy the cocycle condition on triple intersections: $\tilde{\mu}_{jk} \circ \tilde{\mu}_{ik}^{-1} \circ \tilde{\mu}_{ij} = \mathrm{Id}$, to glue the trivial deformations of $\mathcal{E}$ to a global one. Moreover, we need isomorphisms $\tilde{\nu}_{ij} : U \otimes \mathcal{O}_X|_{V_{ij}} \otimes A \to U \otimes \mathcal{O}_X|_{V_{ij}} \otimes A$ such that the following diagram commutes

$$\begin{array}{ccc} U \otimes \mathcal{O}_X|_{V_i} \otimes A & \xrightarrow{(s|_{V_i} + l_i)_{|V_{ij}}} & \mathcal{E}|_{V_i} \otimes A \\ \downarrow{\tilde{\nu}_{ij}} & & \downarrow{\tilde{\mu}_{ij}} \\ U \otimes \mathcal{O}_X|_{V_j} \otimes A & \xrightarrow[(s|_{V_j} + l_j)_{|V_{ij}}]{} & \mathcal{E}|_{V_j} \otimes A, \end{array}$$

i.e., $(s|_{V_j} + l_j)_{|V_{ij}} \circ \tilde{\nu}_{ij} = \tilde{\mu}_{ij} \circ (s|_{V_i} + l_i)_{|V_{ij}}$. Then, the local trivial deformations will glue to a global non trivial deformation of s.

Since we are in characteristic zero, we can take the logarithm to obtain $(\tilde{\mu}_{ij}, \tilde{\nu}_{ij}) = (e^{\mu_{ij}}, e^{\nu_{ij}})$ with $\{(\nu_{ij}, \mu_{ij})\}_{ij} \in \prod_{i<j} (\mathcal{E}nd(U \otimes \mathcal{O}_X)(V_{ij}) \oplus \mathcal{E}nd(\mathcal{E})(V_{ij})) \otimes \mathfrak{m}_A$.

Therefore, any deformation of the morphism s over A is given by an element $(l, m) \in \left(\mathfrak{g}_0^1 \oplus \mathfrak{g}_1^0\right) \otimes \mathfrak{m}_A$, where $l = \{l_i\}_i \in \prod_i \mathcal{H}om(U \otimes \mathcal{O}_X, \mathcal{E})(V_i) \otimes \mathfrak{m}_A$ and the element $m = \{\nu_{ij}, \mu_{ij}\}_{ij} \in \prod_{i<j} (\mathcal{E}nd(U \otimes \mathcal{O}_X)(V_{ij}) \oplus \mathcal{E}nd(\mathcal{E})(V_{ij})) \otimes \mathfrak{m}_A$, that has to satisfy the conditions in (2).

Analogously, we can prove that isomorphisms of deformations correspond to the existence of an element $a \in \mathfrak{g}_0^0 \otimes \mathfrak{m}_A$ that satisfy the condition in Equation (3). □

As a direct consequence of Remark 3.3, Theorems 3.10 and 1.15, we get the following result.

Corollary 3.11. *Under the hypothesis $H^1(X, \mathcal{O}_X) = 0$, there are natural isomorphisms of functors*

$$\mathrm{Def}_{(\mathcal{E},U)} \cong \mathrm{Def}_{(U \otimes \mathcal{O}_X, s, \mathcal{E})} \cong H^1_{sc}(\exp \mathfrak{g}^\Delta) \cong \mathrm{Def}_{\mathrm{Tot}(\mathfrak{g}^\Delta)}.$$

This corollary provides another explicit description of a dgLa controlling the infinitesimal deformations of the pair $(\mathcal{E}, U)$.

Fix an open affine cover $\{V_i\}_i$ of X. As above, let $\mathfrak{g}^\Delta$ and $\mathfrak{h}^\Delta$ be the Čech semicosimplicial dgLas associated to the sheaf $\mathcal{H}om^{\geq 0}(\mathcal{Q}, \mathcal{Q})$ and to the sheaf $\mathcal{E}nd(\mathcal{E})$, respectively. There is an obvious surjective morphism of semicosimplicial dgLas $\mathfrak{g}^\Delta \to \mathfrak{h}^\Delta$ and, denoting with $\mathfrak{m}^\Delta$ the kernel of it, we get the following exact sequence of semicosimplicial dgLas:

$$0 \to \mathfrak{m}^\Delta \to \mathfrak{g}^\Delta \to \mathfrak{h}^\Delta \to 0. \tag{11}$$

Lemma 3.12. *Under the hypothesis that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, the cohomology of the total complex of the semicosimplicial dgLa $\mathfrak{m}^\Delta$ is given by:*

$$H^0\left(\mathrm{Tot}(\mathfrak{m}^\Delta)\right) = 0, H^1\left(\mathrm{Tot}(\mathfrak{m}^\Delta)\right) = \mathrm{Hom}\left(U, H^0(\mathcal{E})/U\right) \text{ and } H^2\left(\mathrm{Tot}(\mathfrak{m}^\Delta)\right) = \mathrm{Hom}(U, H^1(\mathcal{E})).$$

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Proof. The double complex associated to the semicosimplicial dgLa $\mathfrak{m}^\Delta$ is
(12)

$$\begin{array}{ccccccc}
\prod_i \mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E})(V_i) & \xrightarrow{\check\delta} & \prod_{i<j} \mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E})(V_{ij}) & \xrightarrow{\check\delta} & \prod_{i<j<k} \mathcal{H}om(U\otimes\mathcal{O}_X,\mathcal{E})(V_{ijk}) & \xrightarrow{\check\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\prod_i \mathcal{E}nd(U\otimes\mathcal{O}_X)(V_i) & \xrightarrow{\check\delta} & \prod_{i<j} \mathcal{E}nd(U\otimes\mathcal{O}_X)(V_{ij}) & \xrightarrow{\check\delta} & \prod_{i<j<k} \mathcal{E}nd(U\otimes\mathcal{O}_X)(V_{ijk}) & \xrightarrow{\check\delta} & \dots
\end{array}$$

where the horizontal differential $\check\delta$ is the Čech one, while the vertical d is the differential of the dgLa involved.

Let $\{E_k^{p,q}\}$ be the spectral sequence associated to the double complex (12). Calculating the Čech cohomology of it, we get the first page:
(13)

$$\begin{array}{ccccc}
\mathrm{Hom}(U, H^0(X,\mathcal{E})) & \mathrm{Hom}(U, \check H^1(X,\mathcal{E})) & \mathrm{Hom}(U, \check H^2(X,\mathcal{E})) & \dots & \mathrm{Hom}(U, \check H^k(X,\mathcal{E})) \\
\uparrow d & \uparrow d & \uparrow d & & \uparrow d \\
\mathrm{Hom}(U,U) & 0 & 0 & \dots & H^k(X,\mathcal{O}_X)\otimes \mathrm{Hom}(U,U)
\end{array}$$

where we use that $\check H^0(X, \mathcal{E}nd(U\otimes\mathcal{O}_X)) = \mathrm{Hom}(U,U)$ and $\check H^k(X, \mathcal{E}nd(U\otimes\mathcal{O}_X)) = 0$ for $k = 1, 2$ and $\check H^k(X, \mathcal{H}om(U\otimes\mathcal{O}_X, \mathcal{E})) = \mathrm{Hom}(U, \check H^k(X,\mathcal{E}))$ for all $k \geq 0$. Since the differential from the second page is always zero, the spectral sequence abuts to $E_2^{p,q} = E_\infty^{p,q}$. This gives the expected cohomology of the total complex $\mathrm{Tot}(\mathfrak{m}^\Delta)$. □

From the exact sequence (11) and thanks to Lemma 3.12, we recover the exact sequence (8):

$$\begin{aligned}
&0 \to H^0(\mathrm{Tot}(\mathfrak{g}^\Delta)) \to H^0(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^0(X,\mathcal{E})/U) \\
&\to H^1(\mathrm{Tot}(\mathfrak{g}^\Delta)) \to H^1(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{\alpha} \mathrm{Hom}(U, H^1(X,\mathcal{E})) \xrightarrow{\beta} \\
&\to H^2(\mathrm{Tot}(\mathfrak{g}^\Delta)) \xrightarrow{\gamma} H^2(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^2(X,\mathcal{E})).
\end{aligned} \tag{14}$$

Thus, the two dgLas $\mathrm{Tot}(H(\mathcal{V})^\Delta)$ and $\mathrm{Tot}(\mathfrak{g}^\Delta)$, constructed to control the infinitesimal deformations of the pair $(\mathcal{E}, U)$, have actually the same tangent and obstructions space.

Remark 3.13. Note that, if we strengthen the hypothesis of Theorem 3.4 and of Lemma 3.12, assumining that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, both the exact sequences (8) and (14) continue in higher degrees giving rise to the exact sequences:

$$\dots \to H^i(\mathrm{Tot}(\mathfrak{g}^\Delta)) \to H^i(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^i(X,\mathcal{E})) \to H^{i+1}(\mathrm{Tot}(\mathfrak{g}^\Delta)) \to \dots$$

and

$$\dots \to H^i(\mathrm{Tot}(H(\mathcal{V})^\Delta) \to H^i(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^i(X,\mathcal{E})) \to H^{i+1}(\mathrm{Tot}(H(\mathcal{V})^\Delta)) \to \dots$$

respectively. Thus, in this case the two dgLas $\mathrm{Tot}(H(\mathcal{V})^\Delta)$ and $\mathrm{Tot}(\mathfrak{g}^\Delta)$ are quasi-isomorphic.

4. Deformations of a locally free sheaf preserving some sections

In this section, we consider a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on a smooth projective variety X over $\mathbb{K}$, such that $\dim H^0(X, \mathcal{E}) \geq k$. We analyse the infinitesimal deformations of $\mathcal{E}$ such that at least k independent sections of $\mathcal{E}$ lift to the deformed locally free sheaf. Our approach is the same as in [IM23] improved with the techniques developed in the previous sections that allow to generalise the results to any algebraically closed field $\mathbb{K}$ of characteristic zero.

We start giving the following definition.

Definition 4.1. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on a smooth projective variety X over $\mathbb{K}$, such that $h^0(X,\mathcal{E}) \geq k$. Let $Gr(k, H^0(X,\mathcal{E}))$ be the Grassmannian of all subspaces of $H^0(X,\mathcal{E})$ of dimension k. We define the functor $\mathrm{Def}^k_{\mathcal{E}} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Sets}$, that associates to every $A \in \mathbf{Art}_{\mathbb{K}}$ the set

$$\mathrm{Def}^k_{\mathcal{E}}(A) = \bigcup_{U \in Gr(k,H^0(X,\mathcal{E}))} r_U(\mathrm{Def}_{(\mathcal{E},U)}(A)),$$

where $r_U : \mathrm{Def}_{(\mathcal{E},U)} \to \mathrm{Def}_{\mathcal{E}}$ is the forgetful maps of functors. We call it the functor of infinitesimal deformations of $\mathcal{E}$ with at least k sections.

As observed in loc. cit., the functor $\mathrm{Def}^k_{\mathcal{E}}$ is a functor of Artin rings, but unfortunately, it is not a deformation functor. This makes the analysis of it more difficult: its first order deformations do not necessary form a vector space, and its obstructions do not have a clear meaning in term of the corresponding moduli space.

Here we get the following result, that is a version of [IM23, Theorem 5.3] over any algebraically closed field $\mathbb{K}$ of characteristic zero.

Theorem 4.2. *Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, such that $H^i(X,\mathcal{O}_X) = 0$ for $i = 1, 2$. If $h^0(X,\mathcal{E}) = k$, then the tangent space to the deformation functor $\mathrm{Def}^k_{\mathcal{E}}$ is*

$$t_{\mathrm{Def}^k_{\mathcal{E}}} = \mathrm{Def}^k_{\mathcal{E}}(\mathbb{K}[\epsilon]) = \{a \in H^1(X, \mathcal{E}nd(\mathcal{E})) \ \mid \ a \cup s = 0, \ \forall \ s \in H^0(X,\mathcal{E})\}.$$

If instead $h^0(X,\mathcal{E}) \geq k+1$, then the first order deformations of $\mathcal{E}$ with at least k sections are described by the cone

$$\mathrm{Def}^k_{\mathcal{E}}(\mathbb{K}[\epsilon]) = \{\nu \in H^1(X, \mathcal{E}nd(\mathcal{E})) \mid \exists U \in Gr(k, H^0(X,\mathcal{E})) \text{ such that } \nu \cup s = 0, \forall s \in U\}$$

and the vector space generated by it, that we call the tangent space to $\mathrm{Def}^k_{\mathcal{E}}$, is

$$t_{\mathrm{Def}^k_{\mathcal{E}}} = H^1(X, \mathcal{E}nd(\mathcal{E})).$$

Proof. In the case $h^0(X,\mathcal{E}) = k$, the functor $\mathrm{Def}^k_{\mathcal{E}}$ is in one-to-on e correspondence with the functor $\mathrm{Def}_{(\mathcal{E},H^0(\mathcal{E}))}$ and the tangent space is described in Corollary 3.8 as

$$t_{\mathrm{Def}^k_{\mathcal{E}}} \cong t_{\mathrm{Def}_{(\mathcal{E},H^0(\mathcal{E}))}} = \{a \in H^1(X, \mathcal{E}nd(\mathcal{E})) \ \mid \ a \cup s = 0, \ \forall \ s \in H^0(X,\mathcal{E})\}.$$

If $h^0(X,\mathcal{E}) \geq k+1$, by definition,

$$\mathrm{Def}^k_{\mathcal{E}}(\mathbb{K}[\epsilon]) = \bigcup_{U \in Gr(k,H^0(X,\mathcal{E}))} r_U(\mathrm{Def}_{(\mathcal{E},U)}(\mathbb{K}[\epsilon])).$$

For each $U \in Gr(k, H^0(X,\mathcal{E}))$, we calculate the image via r_U of the tangent space to the infinitesimal deformations of the pair $(\mathcal{E}, U)$ using the exact sequence (8)

$$\cdots \to H^1(\mathrm{Tot}(H(\mathcal{V})^\Delta)) \stackrel{r_U}{\to} H^1(X, \mathcal{E}nd(\mathcal{E})) \stackrel{\alpha}{\to} \mathrm{Hom}(U, H^1(X,\mathcal{E})) \cdots .$$

Thus

$$r_U(\mathrm{Def}_{(\mathcal{E},U)}(\mathbb{K}[\epsilon])) = \ker \alpha = \{\nu \in H^1(X, \mathcal{E}nd(\mathcal{E})) \mid \nu \cup s = 0, \forall s \in U\}$$

and the first statement is proved.

The second statement follows from a classical linear algebra argument, that can be found for example in [ACGH85, Proposition 4.2]. □

Remark 4.3. We notice that our explicit description of the tangent space is also a particular case of the one of the Zariski tangent space to the cohomology jump functors done in [B24, Proposition 2.7] using dgl pairs.

Finally, we prove a new version of [IM23, Propositions 5.5-5.6] that concerns the smoothness of the functor $\mathrm{Def}^k_{\mathcal{E}}$.

Proposition 4.4. *Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, such that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$. If there exists an $U \in Gr(k, H^0(X, \mathcal{E}))$ such that $\mathrm{Hom}(U, H^1(X, \mathcal{E})) = 0$ or, in an equivalent way, such that the map $\alpha : H^1(X, \mathcal{E}nd(\mathcal{E})) \to \mathrm{Hom}(U, H^1(X, \mathcal{E}))$ in sequence (8) is surjective, then*

$$\mathrm{Def}_{\mathcal{E}} \text{ is smooth } \Leftrightarrow \mathrm{Def}_{(\mathcal{E},U)} \text{ is smooth } \Leftrightarrow \mathrm{Def}^k_{\mathcal{E}} \text{ is smooth.}$$

Proof. From Corollary 3.5, the two equivalent hypotheses imply that the forgetful morphism r_U is smooth. Then, the first equivalence is obvious. As regard the second equivalence, since the obstruction is complete, each $\mathcal{E}_A \in \mathrm{Def}^k_{\mathcal{E}}(A)$ comes from a pair $(\mathcal{E}_A, i_A) \in \mathrm{Def}_{(\mathcal{E},U)}(A)$, for every $A \in \mathbf{Art}_{\mathbb{K}}$. The above argument obviously implies the equivalence between the smoothness of $\mathrm{Def}_{(\mathcal{E},U)}$ and $\mathrm{Def}^k_{\mathcal{E}}$. □

Proposition 4.5. *Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, such that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$. If there exists an $U \in Gr(k, H^0(X, \mathcal{E}))$ such that $H^2(\mathrm{Tot}(H(\mathcal{V})^{\Delta})) = 0$, then both the functors $\mathrm{Def}_{(\mathcal{E},U)}$ and $\mathrm{Def}^k_{\mathcal{E}}$ are smooth.*

Proof. Since $H^2(\mathrm{Tot}(H(\mathcal{V})^{\Delta})) = 0$, the functor $\mathrm{Def}_{(\mathcal{E},U)}$ is smooth and the relative obstruction to r_U is zero, thus r_U is smooth too. These two properties assure that $\mathrm{Def}^k_{\mathcal{E}}$ is smooth too. □

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