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DISTANCE-TRANSITIVE GRAPHS OF VALENCY FIVE

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1. Introduction

If u and v are vertices of the (finite, connected) graph Γ, let $d(u,v)$ denote the length of the shortest path joining u to v in Γ. The graph Γ is said to be *distance-transitive* if whenever $d(u,v)=d(u',v')$, there exists an automorphism g of Γ such that $u^g=u'$ and $v^g=v'$. Distance-transitive graphs of valency 3 and 4 were originally classified [**2**, **11**, **12**, **13**] by using a computer to generate all "feasible intersection arrays" (cf. [**1**, Chapter 20]). In both cases a classification has since been given by hand [**4**, **5**]. We continue this latter tradition and prove the following theorem—which was recently proved independently by Ivanov *et al.* using a computer [**10**].

Theorem. *A distance-transitive graph of valency five is one of the following:* K_6, $(2.K_6)_2$, $(2.K_6)_4$, H_{11}, Q_5, $\square_5$, $K_{5,5}$, $5.K_{5,5}$, O_5, $2.O_5$, $2.\square_5$, Σ_6, $P_3(4)$, $P_4(4)$.

K_6 is the complete graph, and $K_{5,5}$ the complete bipartite graph, of valency five. Q_5 is the 5-dimensional cube, and $\square_5$ is the antipodal quotient of Q_5. O_5 is the odd graph of valency five. H_{11} is the incidence graph of points and blocks in the unique symmetric 2-(11, 5, 2) design. Σ_6 has as vertices the thirty-six subgroups of order 20 in S_6, two such being adjacent when they intersect in a subgroup of order 4 ([**1**, p. 153]). $P_3(4)$ and $P_4(4)$ are the incidence graphs respectively of the projective plane of order 4 and the classical generalised quadrangle of order (4, 4) associated with $PSp(4,4)$. If Δ is a graph of diameter d, then $r.\Delta$ denotes an r-fold antipodal covering of Δ with diameter $2d$, whereas $(r.\Delta)_\gamma$ denotes an r-fold antipodal covering of Δ with diameter $2d+1$ in which the parameter $c_{d+1}=\gamma$. The antipodal covers occurring in the theorem are all unique. (For the definitions and basic properties of the parameters a_i, b_i, c_i associated with the distance-transitive graph Γ see [**1**, Chapter 20]. For information about antipodal coverings see [**6**].)

2. The two outstanding cases

Given a group G acting distance-transitively on the graph Γ we introduce an extra parameter "s". An *s-arc* in Γ is a sequence $(u(0), u(1), \ldots, u(s))$ of $s+1$ vertices, each adjacent to the next and such that $u(i-1)\neq u(i+1)$ $(0<i<s)$. The group G is *s-arc-transitive* on Γ if G acts transitively on the set of *s-arcs* of Γ, but not on the set of $(s+1)$-arcs.

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Lemma 1. ([**3**, Lemma 2.4 and Proposition 3.5]). *Let Γ be a distance-transitive graph of valency 5 and girth g. Let $\{u,v\}$ be an edge of Γ. Then $G=\operatorname{Aut}\Gamma$ is s-arc-transitive on Γ for some $s\geqq 1$ and one of the following holds:*

(i) $g=3$ *and* $\Gamma=K_6$, *or* $\Gamma=(2.K_6)_2$—*the icosahedron*;

(ii) $g=4$, $\Gamma=(2.K_6)_4$, *and* $s=1$ *or* 2;

(iii) $g=4$, $s=2$, *and* $\Gamma=H_{11}$, $\Gamma=Q_5$, $\Gamma=\square_5$, *or* $\Gamma=K_{5,5}$;

(iv) $g=4$, $s=3$, *and* $\Gamma=K_{5,5}$;

(v) $g=5$, $s=2$, *and either* (a) $G_1(u)=1$, $G(u)=F_{5.4}$, A_5, *or* S_5, *or* (b) $G_1(uv)=1\neq G_1(u)=Z_2$, $G(u)=F_{5.4}\leftthreetimes Z_4=F_{5.4}\times Z_2$;

(vi) $g=6$, $\Gamma=5.K_{5,5}$, *and* $s=2$ *or* 3;

(vii) $g=6$, $s=3$, *and* $\Gamma=O_5$, *or* $\Gamma=2.O_5$;

(viii) $g=6$, $s=4$, *and* $\Gamma=P_3(4)$;

(ix) $g=7$, $s=3$, *and either* (a) $G_1(uv)=1\neq G_1(u)=Z_4$, $G(u)=F_{5.4}\times Z_4$, *or* (b) $G_1(uv)=1\neq G_1(u)=A_4$ *or* S_4, $G(u)=A_5\times A_4$, $S_5\leftthreetimes S_4$, *or* $S_5\times S_4$;

(x) $g\geqq 8$, $s\geqq 4$.

Here $G(u)$ denotes the stabiliser of the vertex u in G, $G_1(u)$ denotes the pointwise stabiliser of u and each of its neighbours, and $G_1(uv)=G_1(u)\cap G_1(v)$. $F_{5.4}$ is the Frobenius group of order 20.

Of the three outstanding cases (v), (ix), and (x) in Lemma 1, the last may appear the most intractable. The program proposed in [**3**] was based on the assumption that this case was likely to be the first to be resolved in complete generality (that is, for all valencies). This has recently been borne out by a result of Weiss [**14**], of which the following is a corollary.

Lemma 2. ([**14**]). *Let Γ be a distance-transitive graph of valency 5 and girth $\geqq 8$. If Γ is s-arc transitive for some $s\geqq 4$, then $\Gamma=P_4(4)$.*

The rest of this paper is devoted to the two remaining cases: (v) $g=5$, $s=2$, and (ix) $g=7$, $s=3$.

3. Case (v)

Case (v). $g=5$, $s=2$ *and either* (a) $G_1(u)=1$, $G(u)=F_{5.4}$, A_5, *or* S_5, *or* (b) $G_1(uv)=1\neq G_1(u)=Z_2$, $G(u)=F_{5.4}\times Z_2$.

We consider the possible values of the parameter a_2 in turn.

If $a_2=4$, then Γ would be a Moore graph of valency 5 and diameter 2, contrary to [**8**]. Suppose $a_2=3$. Let (u,v,w) be a 2-arc in Γ. Then $G_1(u)=1$ (since $G_1(v)\lhd G(vw)$ acts $\frac{1}{2}$-transitively on $\Gamma(w)-\{v\}$ and has a fixed point, namely $\Gamma(w)\cap\Gamma_3(u)$; so $G_1(v)=G_1(w)\lhd\langle G(v),G(w)\rangle=G$). Also $G(u)\neq F_{5.4}$ (otherwise $\langle\Gamma_2(u)\rangle$ would be a trivalent Cayley graph for $F_{5.4}$ with girth $\geqq 5$, which is impossible). Thus $G(u)=A_5$ or S_5. Give v the label 1, and label the other vertices of $\Gamma(u)$ with 2, 3, 4, 5. Then $G(uw)=G(uvw)=A_3$ or S_3, and so fixes exactly two points in $\Gamma(u)$, namely $v=1$ and one other—say 2. So we may label each vertex $w \in \Gamma_2(u)$ with the ordered pair $(1,2)$ of points in $\Gamma(u)$ which are fixed by $G(uw)$, the first coordinate being precisely $\Gamma(u) \cap \Gamma(w)$. $G(uw)$ acts transitively on the three vertices of $\Gamma(w) \cap \Gamma_2(u)$, hence none of these vertices is joined to 1 or 2 (otherwise we would get a 3-gon or a 4-gon), none of them has 2 as second coordinate (or they all would, whence $\langle \Gamma_2(u) \rangle$ consists of five copies of K_4, so Γ would contain a 3-gon), and none of them has 1 as second coordinate (or they all would and $(1,2)$, $(3,1)$, $(1,5)$, $(4,1)$ would form a 4-gon in $\langle \Gamma_2(u) \rangle$). We may therefore assume that the three remaining vertices of $\Gamma(u)$ have been labelled in such a way that $(1,2)$ is joined to $(3,4)$, $(4,5)$, $(5,3)$. It follows that $G(uw) \neq S_3$, so $G(u) = A_5$. But then $(3,4)$ is joined to $(1,2)$, $(2,5)$, $(5,1)$, so $\langle \Gamma_2(u) \rangle = (2.O_3)_1$—the dodecahedron. Thus the vertex 2 is in $\Gamma_3((1,2))$, and $\Gamma(2) \cap \Gamma_2((1,2)) \supseteqq \{u, (2,3), (2,4), (2,5)\}$ so $c_3 \geqq 4$. Now $c_3 \neq 5$ (otherwise $G(u) = A_5$ could not act transitively on the four points of $\Gamma_3(u)$). Hence $c_3 = 4$. But then $a_3 = 0$ (as $a_3 = 1$ with $k_3 = 5$ is impossible), so $b_3 = 1$, $c_4 = 5$, and $\Gamma = 2.\square_5$.

Suppose next that $a_2 = 2$. Let (u, v, w) be a 2-arc. Then $G(u,w)$ leaves $\Gamma(w) \cap \Gamma_2(u)$ invariant, so $G(u) \neq A_5$ or S_5. Hence $G(u) = F_{5.4}$ or $F_{5.4} \times Z_2$, and $\langle \Gamma_2(u) \rangle = 4C_5$ or $2C_{10}$ (since neither $F_{5.4}$ nor $F_{5.4} \times Z_2$ can act vertex transitively on C_{20}). Let $A = \Gamma(v) \cap \Gamma_2(u)$, $B = \Gamma(u) - \{v\}$, $C = \Gamma(B) \cap \Gamma(A)$ (where $\Gamma(B)$ consists of all vertices adjacent to some vertex in B, and similarly for $\Gamma(A)$), $D = \Gamma(A) \cap \Gamma_3(u)$, $E = (\Gamma(B) \cap \Gamma_2(u)) - \Gamma(A)$. Now $|\Gamma(e) \cap B| = 1$ for each $e \in E$ (since $c_2 = 1$). Also E is not a union of connected components of $\langle \Gamma_2(u) \rangle$ (since $5 \nmid |E|$), so we must have $\Gamma(E) \cap C \neq \varnothing$. Thus since $E \subseteqq \Gamma_3(v)$ and $B \cup C \subseteqq \Gamma_2(v)$ we see that $c_3 \geqq 2$. Moreover $|\Gamma(c) \cap E| \leqq 1$ for each $c \in C$, so $\Gamma(E) \cap E \neq \varnothing$ whence $a_3 \geqq 1$ (since $E \subseteqq \Gamma_3(v)$). Hence either $c_3 = 2$ or $c_3 = 4$. Suppose $c_3 = 4$. Then $a_3 = 1$. Hence Γ has intersection array $\{5,4,2;1,1,4\}$ and is primitive, so the group $G = \operatorname{Aut}\Gamma$ (of order 36.20 or 36.40) has a unique minimal normal subgroup M. Thus $M = A_6$, $\Gamma = \Sigma_6$, and $\operatorname{Aut}\Gamma = \operatorname{Aut} S_6$.

Now suppose that $c_3 = 2$—still with $a_2 = 2$, $a_3 \geqq 1$. Then $a_3 \neq 3$ (otherwise Γ would be primitive on 46 points, contradicting [**15**, Theorem 31.1]). Thus $a_3 = 2$ or $a_3 = 1$. We saw above that $\Gamma(E) \cap E$ and $\Gamma(E) \cap C$ are non-empty. Moreover interchanging u and v interchanges A and B, D and E, and leaves C invariant, so $\Gamma(D) \cap D$ and $\Gamma(D) \cap C$ are non-empty. Thus if $G(u) = F_{5.4} \times Z_2$, then we have the partial intersection diagram of Fig. 1 (since $G(uv)$ is then transitive on C, D, E). If $G(u) = F_{5.4}$, then $G(uv)$ has two orbits of length four on each of C, D, E: if $\langle \Gamma_2(u) \rangle = 4C_5$, then the vertices of A belong to distinct components and it is easy to see that the same partial intersection diagram applies; if $\langle \Gamma_2(u) \rangle = 2C_{10}$, then A must consist of two "opposite pairs" of vertices—one from each of the two 10-gons—and it is not hard to see that this forces the same partial intersection diagram. If $a_3 = 1$ and $b_3 = 2$, then for each 3-arc (u, v, w, x) with $u \in \Gamma_3(x)$, $b_3 = 2$ implies that $\Gamma(u) \cap \Gamma_4(x) = \{v_1, v_2\}$. Thus if $\Gamma_2(x) \cap \Gamma(u) = \{v, v'\}$ and $\Gamma(v') \cap \Gamma(x) = \{w'\}$, then we can choose $z' \in \Gamma(w') \cap \Gamma_2(u) \cap \Gamma(v)$ (since $w' \in C$). Moreover $\{v_1, v_2\} = \Gamma(u) \cap \Gamma_3(w)$ implies that $v' \in \Gamma(u) \cap \Gamma_2(w)$ so we can choose $z \in \Gamma(w) \cap \Gamma_2(u) \cap \Gamma(v')$. But then the 3-arc $(uvwz)$ in $\langle \Gamma_2(w') \rangle$ has $\Gamma(u) \cap \Gamma(w') = \{v'\} = \Gamma(z) \cap \Gamma(w')$, so the circuit containing $(uvwz)$ in $\langle \Gamma_2(w') \rangle$ cannot be "rotated" by an element of order 5 in $G(w')$. Thus $a_3 = 2$, $b_3 = 1$ and $c_4 \geqq c_3 = 2$. Now $c_4 \neq 5$ (otherwise $O_5(G(u))$ fixes each vertex $y \in \Gamma_4(u)$, so $O_5(G(u)) = O_5(G(y)) \lhd \langle G(u), G(y) \rangle = G$). Also $c_4 \neq 3$ (since $3 \nmid 20$). If $c_4 = 4$, then $a_4 \neq 1$ (since $k_4 = 5$ is odd); so $a_4 = 0$, $b_4 = 1$, $c_5 = 5$, and Γ would be a 2-fold covering of a Moore graph of valency 5, contrary to [**8**]. Suppose $c_4 = c_3 = 2$. Then $a_4 \neq 3$ (or else $\langle \Gamma_4(u) \rangle$ is trivalent on ten vertices with girth $\geqq 5$, and so is isomorphic

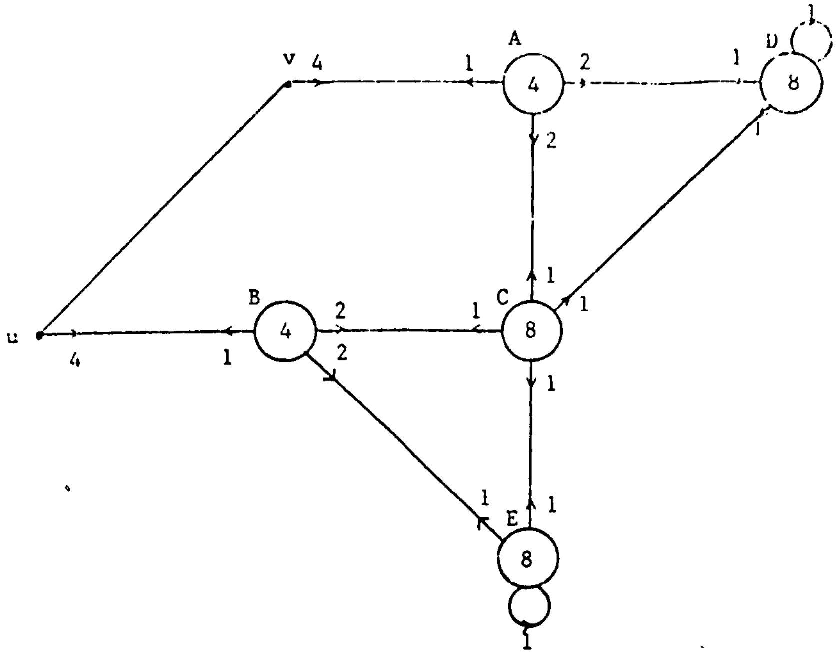

FIGURE 1

to the Petersen graph; but then for each $y \in \Gamma_4(u)$ we would have $C_6 = \langle \Gamma_2(y) \cap \Gamma_4(u) \rangle \subseteqq \langle \Gamma_2(y) \rangle = 4C_5$ or $2C_{10}$). Hence $a_4 = 2$, $b_4 = 1$, so $c_5 = 5$ or 2. Moreover $c_5 \neq 5$ (or Γ would be primitive on 58 points, contradicting [**15**, Theorem 31.1]). Thus $c_5 = 2$. Since $k_5 = 5$ is odd we cannot have $a_5 = 3$. Thus $a_5 = 2$, $b_5 = 1$, $c_6 = 5$. But then $\langle \Gamma_5(u) \cup \Gamma_6(u) \rangle$ contains a triangle. This completes the case $a_2 = 2$.

Suppose $a_2 = 1$. Then $b_2 = 3$, and $a_3 \geqq 1$ (since $\langle \Gamma_3(v) \cap \Gamma_2(u) \rangle = 6K_2$). Hence $c_3 \leqq 4$. Moreover $c_3 \neq 4$ (since if $c_3 = 4$ then $a_3 = 1$ and Γ would be primitive on 41 vertices, contradicting [**9**, Satz 21.3]). Suppose $c_3 = 3$. Then $a_3 = 1$ and $b_3 = 1$ (otherwise $a_3 = 2$ and Γ would be primitive on 46 vertices, contrary to [**15**, Theorem 31.1]). Hence $c_4 \neq 5$ (otherwise $k_4 = 4$ and either $G(u) \geqq A_5$ could not act transitively on $\Gamma_4(u)$, or $G(u)$ is soluble and $O_5(G(u))$ fixes each $y \in \Gamma_4(u)$, so $O_5(G(u)) = O_5(G(y)) \lhd \langle G(u), G(y) \rangle = G$). Thus $c_4 = 4$, $k_4 = 5$ is odd, so $a_4 \neq 1$; hence $a_4 = 0$, $b_4 = 1$, $c_5 = 5$, and Γ would be a 2-fold cover of a Moore graph of valency 5, contradicting [**8**]. Suppose $c_3 = 2$. Then $k_3 = 30$, so $G(u) = A_5$ or S_5. If we label the vertices of $\Gamma(u)$ with the symbols $1, 2, 3, 4, 5$, each vertex w of $\Gamma_2(u)$ receives a natural label (i, j) where $\Gamma(w) \cap \Gamma(u) = \{i\}$ and $G(uw)$ fixes $i, j \in \Gamma(u)$. Since $a_2 = 1$ the vertex (i, j) must be joined to the vertex (j, i). Let $v = 1$, $w = (1, 2)$, $x \in \Gamma(w) \cap \Gamma_3(u)$. Then $G(ux)$ interchanges the two vertices $(1, 2)$, $(i, j) \in \Gamma(x) \cap \Gamma_2(u)$. Now $i \neq 1$ (since Γ contains no 4-gons), and $i \neq 2$ (otherwise either $j = 1$ so Γ contains a 3-gon, or $j \neq 1$ and the 2-arc $((1, 2), (2, 1), 2)$ lies in more than one 5-gon). Similarly $j \neq 1$. Thus either (i) $\{1, 2\} \cap \{i, j\} = \varnothing$, or (ii) $j = 2$.

(*i*) Suppose $\{1, 2\} \cap \{i, j\} = \varnothing$. Then $k_3 = 30$, so $G(u)$ must have an orbit of length 30 on the sixty pairs $\{(i, j), (k, l)\}$ with $\{i, j\} \cap \{k, l\} = \varnothing$. Hence $G(u) = A_5$. We may assume that $\Gamma(x) \cap \Gamma_2(u) = \{(1, 2), (3, 4)\}$ and give x the label $\{(1, 2), (3, 4)\}$. $G(ux) = \langle (13)(24) \rangle$ has precisely two fixed points in $\Gamma_3(u)$, namely $x=\{(1,2),(3,4)\}$ and $x'=\{(2,1),(4,3)\}$. Thus $a_3=2$ (otherwise x would have to be joined to x' and we would get the 4-gon $(x,x',(2,1),(1,2))$). Let $A=\Gamma(v)\cap\Gamma_2(u)$, $D=\Gamma_2(v)\cap\Gamma_3(u)=\Gamma(A)\cap\Gamma_3(u)$. Since $a_2=1$ in $\Gamma_2(v)$ we must have $\langle D\rangle=6K_2$. Thus the two vertices in $\Gamma(x)\cap\Gamma_3(u)$ are of the form $\{(1,i),(j,k)\}$ and (applying $G(ux)=\langle(13)(24)\rangle$) $\{(3,i'),(j',k')\}$. Using this and the fact that $\Gamma(x)\cap\Gamma_3(u)$ must be a self-paired suborbit of $G(u)^{\Gamma_3(u)}$ we get just five possibilities for $\Gamma(x)\cap\Gamma_3(u)=\{y,y'\}$ say: (a) $y=\{(3,5),(1,4)\}$, $y'=\{(1,5),(3,2)\}$; (b) $y=\{(5,4),(3,2)\}$, $y'=\{(5,2),(1,4)\}$; (c) $y=\{(1,4),(5,2)\}$, $y'=\{(3,2),(5,4)\}$; (d) $y=\{(2,1),(3,5)\}$, $y'=\{(4,3),(1,5)\}$; (e) $y=\{(4,5),(3,1)\}$, $y'=\{(2,5),(1,3)\}$. In cases (a)–(c) we get $y'\in\Gamma(y)$, so Γ has girth 3; in case (d) we get a 4-gon $((1,2),\ x,y,\ (2,1))$; in case (e) the 2-arc (y,x,y') lies in two 5-gons—one in $\langle\Gamma_3(u)\rangle$, the other being $(y,x,y',\ (1,3),\ (3,1))$. Thus case (i) does not occur.

(*ii*) Suppose now that $j=2$: that is, if $x\in\Gamma(1,2)\cap\Gamma_3(u)$, then we may assume that $\Gamma(x)\cap\Gamma_2(u)=\{(1,2),(3,2)\}$. Then each vertex x in $\Gamma_3(u)$ receives a natural label $x=(2,(13))$ whose second coordinate is an unordered pair, or transposition, not involving the first coordinate. The 2-arc $(1,(1,2),\ (2,(13)))$ lies in a unique 5-gon, which we may take to be $(1,(1,2),(2,(13)),(5,(14)),(1,5))$. Thus $G(u)=A_5$ (otherwise $G(ux)=\langle(13),(45)\rangle$ and the $G(ux)$-orbit containing $(5,(14))$ would have size four, contradicting $a_3\leqq 3$). Hence $G(ux)=\langle(13)(45)\rangle$, so $(2,(13))$ is also joined to $(4,(35))$, whence $a_3\geqq 2$. The $G(u)$ images of the edge $\{(2,(13)),\ (5,(14))\}$ form six 5-gons in $\langle\Gamma_3(u)\rangle$. But then we get two 5-gons $((1,(52)),\ (3,(24)),\ (5,(14)),\ x',y')$ containing a single 2-arc—one with $x'=(2,(13))$, $y'=(4,(35))$, and the other with $x'=(1,5)$, $y'=(5,1)$—contradicting $a_2=1$. Thus case (ii) does not occur. Hence $c_3\neq 2$.

Suppose $c_3=1$. Then $k_3=60$, so $G(u)=A_5$ or S_5. As before $\langle\Gamma_2(v)\cap\Gamma_3(u)\rangle=6K_2$, so $a_3\geqq 1$, $b_3\leqq 3$. Each edge (w,w') in $\langle\Gamma_2(u)\rangle$ lies in exactly three 2-arcs (w,w',x') with $x'\in\Gamma_3(u)$. And each such 2-arc lies in a unique 5-gon (w,w',x',x'',x). Since $G(u)$ acts transitively on such 2-arcs, it must act transitively on the 30 corresponding 5-gons. Hence the possible vertices x'' opposite the edge (w,w') must form a single $G(u)$-orbit whose length divides 30. Thus $x''\in\Gamma_4(u)$, k_4 divides 30, and $c_4=|\Gamma(x'')\cap\Gamma_3(u)|\geqq|\{x',x\}|=2$. So either (a) $k_4=30$, and $b_3=2$, $c_4=4$ or $b_3=1$, $c_4=2$, or (b) $k_4=15$ and $b_3=1$, $c_4=4$. Each 5-gon (u,v,w,w',v') in Γ has stabiliser A_3 or S_3, and so is fixed by a unique subgroup $\langle h\rangle$ of order three. There is an element $g\in G$ of order five which "rotates" the 5-gon and which normalises, and hence centralises, the subgroup $\langle h\rangle$. In both case (a) and case (b) we have $k_0=1$ and $k_1\equiv k_2\equiv k_3\equiv k_4\equiv 0 \pmod 5$. Moreover $k_i\not\equiv 0 \pmod 5$ only if $c_i=5$, whence $i=d$. Thus since $G(u)=A_5$ or S_5 acts transitively on each $\Gamma_i(u)$ we have $k_i\geqq 5$ for each $i\geqq 5$. (The apparent possibilities $k_d=1$ or 2 would force Γ to be antipodal, and this is not possible if $k_4|30$.) It follows that $\sum_{i=0}^{d}k_i=|G:G(u)|\not\equiv 0 \pmod 5$. Thus the element g must fix some vertex z. Since $G(z)=A_5$ or S_5, the element h cannot fix z, so h acts semiregularly on the fixed points of g: in particular, g fixes at least three points. But if g fixes the vertex $z'\in\Gamma_j(z)$ for some first $j\geqq 1$, then $\langle g\rangle$ must act transitively on $\Gamma(z')$, so $c_j=5$, $j=d$. Since $k_d\neq 2,3,4$, the only possibility in case (a) or (b) above is that $k_4=30$, $b_4=1$, $c_5=5$, $k_5=6$. But then g fixes only $1+0+0+0+0+1$ points, which is a contradiction. This completes the case $a_2=1$, $c_3=1$, and hence completes case (v).

4. Case (ix)

Case (ix). *girth* 7, $s=3$, $G_1(uv)=1$ *and either* (a) $G_1(u)=Z_4$, $G(u)=F_{5.4}\times Z_4$, *or* (b) $G_1(u)=A_4$ *or* S_4, $G(u)=A_5\times A_4$, $S_5 \curlywedge S_4$, *or* $S_5\times S_4$.

Suppose first that $G(u)\trianglerighteq A_5\times A_4$. If $(0,1,2,3)$ is a 3-arc in Γ, the stabiliser $H=G(0123)\geqq A_3\times A_3$ fixes a unique circuit $C=(0,1,2,3,\ldots,t-1)$ containing the 3-arc $(0,1,2,3)$. Moreover for each i, $O_3(G_1(i)\cap H)=A_3$ fixes $\Gamma(i+2k)(\operatorname{mod} t)$ pointwise and acts non-trivially on each $\Gamma(i+2k-1)(\operatorname{mod} t)$; (compare [**5**, Lemma 3] or [**7**, Lemmas 5.2–5.4]). Hence in particular t must be even, so $t\geqq 8$, $a_3=3$, $b_3=1$. Let $\Gamma(3)\cap\Gamma_3(0)=\{x_1,x_2,x_3\}$, and let $\Gamma_2(x_i)\cap\Gamma(0)=\{y_i\}$ $(i=1,2,3)$. Then $O_3(G_1(0)\cap H)$ must permute the x_i cyclically while fixing the y_i pointwise, which is impossible. Thus we may assume that $G(u)=F_{5.4}\times Z_4$.

Γ has girth 7 so $a_3\geqq 1$, $b_3\leqq 3$. If $a_3=4$, then Γ would be a Moore graph of diameter 3—contradicting [**8**]. If $a_3=3$, then $\langle\Gamma_3(u)\rangle$ has girth $\geqq 7$ and so must consist of two components of size 40 (otherwise $\langle\Gamma_3(u)\rangle$ would be a Cayley graph for $F_{5.4}\times Z_4$, whereas this group cannot be generated either by three involutions, or by a single element of order $\geqq 7$ together with an involution). But each component would then be a Cayley graph of girth $\geqq 7$ for some subgroup H of index 2 in $G(u)$. However $H=F_{5.4}\times Z_2$ and $H=F_{5.4}\curlywedge Z_4$ cannot be generated either by three involutions, or by an element of order $\geqq 7$ and an involution; and though $H=D_{10}\times Z_4$ can be so generated (namely by an element g of order 20 and an involution t), the corresponding Cayley graph has girth 6 (since t inverts g^4 and centralises g^5, so $gtg=g^{-1}tg^{-1}$). Suppose $a_3=2$. Then $b_3=2$, so $c_4=2$ or 4 (since k_4 must divide $|G(u)|=80$). If $c_4=4$ then either (a) $a_4=1$ and Γ is primitive on 2.73 points, contradicting [**15**, Theorem 31.1], or (b) $a_4=0$, $b_4=1$, in which case we must have $c_5=4$, $a_5=1$ (since $c_5=5$ implies $5\nmid k_5$, whence for $z\in\Gamma_5(u)$, $O_5(G(u))=O_5(G(z))\lhd\langle G(u),G(z)\rangle=G$). But then if $v\in\Gamma(u)$, $\langle\Gamma_5(u)\rangle=5K_2$ and $|\Gamma_4(v)\cap\Gamma_5(u)|=8$, so $\langle\Gamma_4(v)\rangle$ must contain an edge, contradicting $a_4=0$. Thus we may assume that $c_4=2$. Clearly $a_4\neq 3$ (for the same reasons as $a_3\neq 3$). If $a_4=2$, then $\langle\Gamma_4(u)\rangle$ is a union of circuits of length $\geqq 7$, so $\langle\Gamma_4(u)\rangle=4C_{20}$ or $8C_{10}$ (since $F_{5.4}\times Z_4$ cannot act regularly on C_{80}, $2C_{40}$, $5C_{16}$, or $10C_8$). Let $y\in\Gamma_4(u)$, $y_1,y_2\in\Gamma(y)\cap\Gamma_4(u)$, $\{z\}=\Gamma(y)\cap\Gamma_5(u)$. Then $b_4=1$, so $\Gamma(u)\cap\Gamma_5(y)=\{v^*\}$ say. Now $b_5\leqq b_4=1$. If $c_5=2$ then $\Gamma(y)\cap\Gamma_4(v^*)=\Gamma(y)\cap\Gamma_3(u)$, and either $d>5$, $b_5=1$, $a_5=2$, $\{z\}=\Gamma(y)\cap\Gamma_6(v^*)$, so $\Gamma(y)\cap\Gamma_4(u)\subseteqq\Gamma(y)\cap\Gamma_5(v^*)$, or $d=5$, $a_5=3$, and again $\Gamma(y)\cap\Gamma_4(u)\subseteqq\Gamma(y)\cap\Gamma_5(v^*)$. Thus the component of $\langle\Gamma_4(u)\rangle$ containing y lies in $\Gamma_5(v^*)$. Since $G(u)$ acts transitively on $\Gamma(u)$, the number of components of $\langle\Gamma_4(u)\rangle$ should be divisible by $|\Gamma(u)|=5$, a contradiction. Hence $c_5=4$ (since $c_5=5$ would imply that $O_5(G(u))=O_5(G(z))\lhd\langle G(u),G(z)\rangle=G$ for each $z\in\Gamma_5(u)$). Moreover $a_5=0$ (otherwise Γ would be primitive on 2.103 points, contradicting [**15**, Theorem 31.1]). If $c_6=5$, then Γ would be a 5-fold cover of $P_3(4)$, contrary to [**6**]. Thus $c_6=4$, $a_6=0$ (otherwise Γ would be primitive on 211 points contradicting [**9**, Satz 21.3]), $b_6=1$, $c_7=5$ and Γ is a 2-fold cover of a Moore graph of diameter 3, contrary to [**8**]. Thus $a_4\neq 2$.

Let $v\in\Gamma(u)$, $A=\Gamma(u)\cap\Gamma_2(v)$, $B=\Gamma(v)\cap\Gamma_2(u)$, $C=\Gamma_3(v)\cap\Gamma_2(u)$, $D=\Gamma_3(u)\cap\Gamma_2(v)$, $E=\Gamma_4(v)\cap\Gamma_3(u)$, $F=\Gamma_4(u)\cap\Gamma_3(v)$, $H=\Gamma_3(u)\cap\Gamma_3(v)$. Then A,B,C,D are all $G(uv)$-orbits. Since $G(uv)$ acts semi-regularly on $\Gamma_3(u)$, both $E=E_1\cup E_2$ and $H=H_1\cup H_2$ break up into two $G(uv)$-orbits. Interchanging u and v we see that $F=F_1\cup F_2$ also consists of two $G(uv)$-orbits. Since $a_3=2$ and $G(u)=F_{5.4}\times Z_4$ acts regularly on $\Gamma_3(u)$, we have $\langle\Gamma_3(u)\rangle=$

$4C_{20}$ or $8C_{10}$. Since $D \subseteqq \Gamma_2(v)$ and $a_2=0$, we have $\Gamma(D) \cap \Gamma_3(u) \subseteqq H$. Moreover any union of components in $\langle \Gamma_3(u) \rangle$ must have size divisible by 10, so $\Gamma(D) \cap H_i \neq \emptyset$ $(i=1,2)$. If $\Gamma(H) \cap H \neq \emptyset$, then since $D \cup H$ cannot be a union of components we must have one of the $G(uv)$-orbits on H, say $\langle H_1 \rangle = 8K_2$, $\Gamma(H_2) \cap E = E_2$ (say), $\Gamma(E_2) \cap E = E_1$, $\langle E_1 \rangle = 8K_2$. But then $|\Gamma(x) \cap E| = 2$ for each $x \in E$, which is impossible since $E \subseteqq \Gamma_4(v)$ and $a_4 \leqq 1$. Thus $\Gamma(H) \cap H = \emptyset$, $\Gamma(H) \cap E_i \neq \emptyset$ $(i=1,2)$, and we have the partial intersection diagram in Fig. 2. In particular $a_4 \geqq 1$, so $a_4 = 1$.

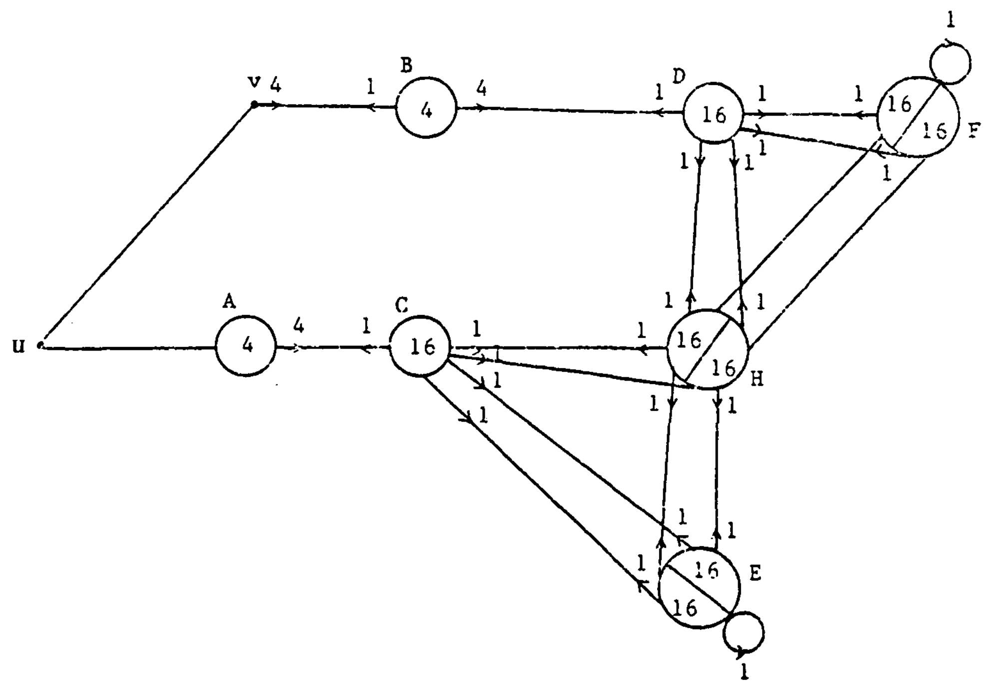

FIGURE 2

Suppose $\Gamma(E_i) \cap E_i \neq \emptyset$ $(i=1,2)$. Then $\Gamma(F_i) \cap F_i \neq \emptyset$ $(i=1,2)$. Let (y_1, y_2) be an edge in $\langle F_1 \rangle$. Then some element $g \in G(uv) = G_1(u) \times G_1(v)$ maps y_1 to y_2, and so inverts the edge (y_1, y_2). Hence $g^2 = 1$. Let $\Gamma_3(y_1) \cap \Gamma(u) = \{v, v'\}$. Then $y_1 \in \Gamma_3(v')$, so $|\Gamma(y_1) \cap \Gamma_3(v')| = 2$ (since $a_3 = 2$). Now $b_4 = b_3 = 2$ (since $a_4 = 1$), so $\Gamma(y_1) \cap \Gamma_5(u) = \Gamma(y_1) \cap \Gamma_4(v')$. Thus $y_2 \in \Gamma_3(v')$ and $\{v, v'\} = \Gamma_3(y_2) \cap \Gamma(u)$. Thus g fixes v'. If t_u, t_v denote the involutions in $G_1(u), G_1(v)$ respectively, then $g = t_u$ (since t_v and $t_u t_v$ have only one fixed point in $\Gamma(u)$). But then $\langle g \rangle \lhd G(u)$, so the $\langle g \rangle$-orbits in $\Gamma_4(u)$ form a block system for the action of $G(u)$ on $\Gamma_4(u)$. Hence g inverts every edge in $\Gamma_4(u)$. If $\{w_i\} = \Gamma(y_i) \cap \Gamma_2(v')$, and $\{y_i'\} = (\Gamma(w_i) \cap \Gamma_4(u)) - \{y_i\}$ $(i=1,2)$, then g must interchange y_1' and y_2', so (y_1', y_2') must be an edge. But then $(y_1, y_2, w_2, y_2', y_1', w_1)$ would be a 6-gon in Γ—a contradiction. Hence $\Gamma(F_i) \cap F_i = \emptyset = \Gamma(E_i) \cap E_i$ $(i=1,2)$, and each circuit in $\langle \Gamma_3(u) \rangle$ runs through the five $G(uv)$-orbits D, H_1, E_1, E_2, H_2 in turn. Let (y_1, y_2) be an edge in $\langle F \rangle$ with $y_i \in F_i$ $(i=1,2)$. As before we have $\Gamma_3(y_i) \cap \Gamma(u) = \{v, v'\}$, and let $\{w_i\} = \Gamma(y_i) \cap \Gamma_2(v')$ $(i=1,2)$. $G_1(u)$ has four orbits F_{i1}, F_{i2}, F_{i3}, F_{i4} of length 4 on each F_i $(i=1,2)$, which we may assume to be labelled such that $y_1 \in F_{11}$ and $y_2 \in F_{21}$, four orbits H_{i1}, H_{i2}, H_{i3}, H_{i4} of length 4 on each H_i, labelled such that $w_1 \in H_{11}$ and $w_2 \in H_{21}$, and four orbits C_1, C_2, C_3, C_4 of length 4 on C, labelled such that $C_1 = \Gamma(v')$. Moreover $(\Gamma_2(y_i) \cap \Gamma_2(u)) - \Gamma(v) \subseteq \Gamma(v') = C_1$, so $\Gamma_2(y_i^*) \cap \Gamma_2(u) \subseteq C_1$ for each $y_i^* \in F_{i1}$ $(i=1,2)$. It follows that each circuit in $\langle \Gamma_3(v) \rangle$ runs through five $G_1(u)$-orbits—say F_{11}, F_{21}, H_{21}, C_1, H_{11}—in turn. Thus the $G_1(u)$-orbits in $\Gamma_3(v)$ form a block system for the action of $G(v)$, with quotient $4C_5$. If $x \in \Gamma_3(v)$, let $C(x)$ be the cycle in $\langle \Gamma_3(v) \rangle$ containing the vertex x. Since $G(v)$ acts regularly on $\Gamma_3(v)$ the element $g' \in G(v)$ which maps y_1 (say) to the vertex opposite y_1 on $C(y_1)$ must belong to $G_1(u)$, and so induces a half turn on $C(y_1)$. But then $g' = t_u$ (since $g'^2 = 1$). Hence t_u commutes with $O_5(G(v))$, whereas t_u should invert $O_5(G(v))$.

Thus $a_3 = 1$ and $\langle \Gamma_3(u) \rangle = 40K_2$. Since 3 does not divide $|G(u)|$, $c_4 = 3$ and $k_4 = 80$. If $a_4 = 2$, then G is primitive with order $186.80 = 2^5.3.5.31$. There are 186 Sylow 5-subgroups (the Sylow 5-normalisers being precisely the vertex stabilisers). By Sylow's theorem there must be 32 Sylow 31-normalisers, each containing 31 Sylow 5-subgroups. Thus each Sylow 5-subgroup occurs in 32.31/186 Sylow 31-normalisers—a contradiction. Hence $a_4 \leqq 1$. Let $v \in \Gamma(u)$, $A = \Gamma(u) \cap \Gamma_2(v)$, $B = \Gamma(v) \cap \Gamma_2(u)$, $C = \Gamma_3(v) \cap \Gamma_2(u)$, $D = \Gamma_3(u) \cap \Gamma_2(v)$, $E = \Gamma_4(v) \cap \Gamma_3(u)$, $F = \Gamma_4(u) \cap \Gamma_3(v)$, and $H = \Gamma_3(u) \cap \Gamma_3(v)$. Then A, B, C, D, H are all $G(uv)$-orbits. Since $a_3 = 1$ we have for each $y \in D$, $\Gamma(y) \cap H \neq \varnothing$, and so for each $y \in E$ we must have $\Gamma(y) \cap E \neq \varnothing$. An element interchanging u and v would interchange E and F and leave H invariant. Hence for each $y \in F$ we have $\Gamma(y) \cap F \neq \varnothing$ and $\Gamma(y) \cap H = \varnothing$. It follows that $a_4 = 1$ and we get the partial intersection diagram in Fig. 3. Thus $b_4 = 1$ and $c_5 \geqq 4$ (since 3 does not divide 80). If $c_5 = 5$, then G is primitive of degree $202 = 2.101$ contradicting [**15**, Theorem 31.1]. Thus $c_5 = 4$. If $a_5 = 1$, then G is primitive of degree $206 = 2.103$, again contradicting [**15**, Theorem 31.1]. So $a_5 = 0$, $b_5 = 1$. Suppose $c_6 = 5$. Then Γ is not antipodal (since $a_4 \neq a_2$), so G is primitive. But then for each $z \in \Gamma_6(u)$, $O_5(G(u)) \lhd \langle G(u), G(z) \rangle = G$, a contradiction. Thus $c_6 = 4$, $a_6 = 0$, $b_6 = 1$, $c_7 = 5$ and Γ is a 2-fold antipodal covering of a Moore graph of diameter 3, contradicting [**8**]. This completes case (ix) and hence proves the theorem.

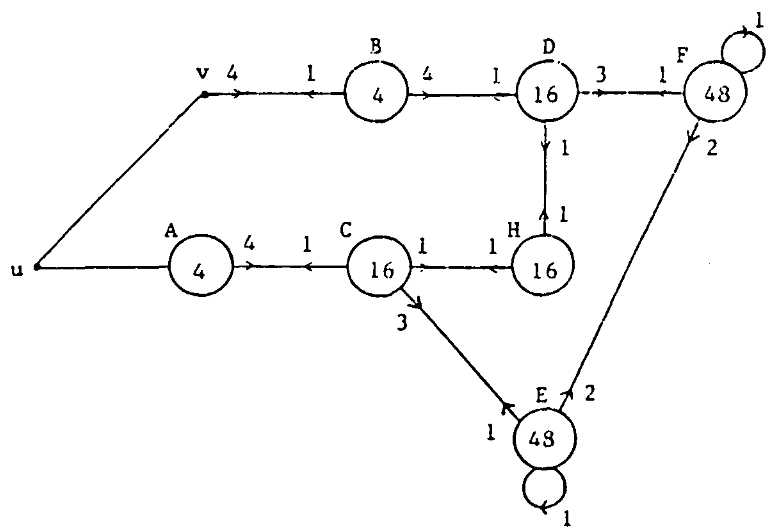

FIGURE 3

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REFERENCES

1. N. L. Biggs, *Algebraic Graph Theory* (Cambridge University Press, 1974).

2. N. L. Biggs and D. H. Smith, On trivalent graphs, *Bull. London Math. Soc.* **3** (1971), 155–158.

3. A. Gardiner, Classifying distance-transitive graphs, in *Combinatorial Mathematics IX*, Eds. E. J. Billington, S. Oates-Williams and A. P. Street (Springer Lecture Notes in Mathematics 952, 1982), 67–88.

4. A. Gardiner, On trivalent graphs, *J. London Math. Soc.* (2) **10** (1975), 507–512.

5. A. Gardiner, An elementary classification of distance-transitive graphs of valency four, *Ars Combin.*, to appear.

6. A. Gardiner, Antipodal covering graphs, *J. Combinatorial Theory B* **16** (1974), 255–273.

7. A. Gardiner, Arc transitivity in graphs III, *Quart. J. Math. Oxford* (2) **27** (1976), 313–323.

8. A. J. Hoffman and R. R. Singleton, On Moore graphs of diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

9. B. Huppert, *Endliche Gruppen* (Springer, 1967).

10. A. A. Ivanov, A. V. Ivanov and I. A. Faradžev, Distance-transitive graphs of valency 5, 6 and 7, *Zh. Vychisl. Mat. i Mat. Fiz.* **24** (1984), 1704–1718 (in Russian).

11. D. H. Smith, On tetravalent graphs, *J. London Math. Soc.* (2) **6** (1973), 659–662.

12. D. H. Smith, Distance-transitive graphs of valency four, *J. London Math. Soc.* (2) **8** (1974), 377–384.

13. D. H. Smith, On bipartite tetravalent graphs, *Discrete Math.* **10** (1974), 167–172.

14. R. Weiss, Distance-transitive graphs and generalised polygons, *Arch. Math.*, to appear.

15. H. Wielandt, *Permutation Groups* (Academic Press, 1964).

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