Ergod. Th. & Dynam. Sys. (1987), 7, 263-265
Printed in Great Britain

Loose Bernoullicity is preserved under exponentiation by integrable functions

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(*Received* 23 *November* 1985)

Abstract. It is known that if Ω is a Lebesgue space, $T:\Omega\to\Omega$ is a loosely Bernoulli transformation, and L is a fixed non-zero integer, then the transformation $S=T^L$ will again be loosely Bernoulli on each ergodic component. In this note, the above stated result is extended to include the case where L is an arbitrary integrable integer-valued function on Ω.

Let $(\Omega, \mathcal{F}, \mu)$ be a Lebesgue space and $T:\Omega\to\Omega$ an invertible, ergodic, measure-preserving transformation. For L an integrable integer-valued function on Ω, we consider the transformation $U(\omega)=T^{L(\omega)}(\omega)$. In general, U will not be invertible and will not preserve μ. Moreover, not every point $\omega\in\Omega$ will belong to a bilateral U-orbit, i.e. a set $S=\{\omega_i: i\in\mathbb{Z}\}$ where $U(\omega_i)=\omega_{i+1}$.

On the other hand, it was shown in [**1**] that with the above hypothesis there exists a set Ω_1 of full measure (of points satisfying a certain finiteness condition) and the set $A\subseteq\Omega_1$ of points which also belong to a bilateral U-orbit has strictly positive measure. Moreover, the transformation $U=T^L$ restricted to A is invertible and preserves μ_A, but may not be ergodic. In this article we note that the behavior of U on almost all ergodic components can be explicitly described.

More precisely, we claim that there exists a U-invariant set $B\subseteq A$ with $\mu_A(B)=1$ such that there are finitely many (possibly zero) sets C with $\mu(C)>0$ and U restricted to C aperiodic ergodic; and all other ergodic components of U in B are finite rotations. To see this, note that if there exists $C\subseteq A$ with $\mu(C)>0$ and $U(C)=C$, then by ergodicity μ-a.e. T-orbit contains points in C and hence a complete U-orbit lying in C. However, by [**1**, Theorem 2(b)] there can be at most finitely many aperiodic U-orbits (cardinality of orbit not finite) on a T-orbit. The number m of aperiodic U-orbits on a T-orbit is an upper bound for the number of such sets C. Thus the set B consists of a union of all periodic U-orbits and a finite number of such C, where U restricted to C is aperiodic.

We now describe the behavior of U on the aperiodic components of B. For any U-invariant $C\subseteq B$ with $\mu(C)>0$ for which U restricted to C is aperiodic ergodic, let $L':C\to\mathbb{Z}$ be the integer-valued function such that $U|_C=(T_C)^{L'}$, where $U|_C$ is the restriction and T_C is the induced transformation. Clearly $|L'|\le|L|$. Moreover, it follows from aperiodicity that the values $\sum_{i=1}^n L'(U^i\omega)$, $n=1,2,\ldots$, are disjoint

https://doi.org/10.1017/S0143385700003990 Published online by Cambridge University Press

for μ_C-a.e. ω, hence

$$\limsup_{n\to\infty} \frac{1}{n}\left|\sum_{i=1}^{n} L'(U^i\omega)\right| \geq \tfrac{1}{2}.$$

By the ergodic theorem, we must therefore have $|\int L'\,d\mu_C| \neq 0$. Assume $\int L'\,d\mu_C > 0$, the other case being similar.

By ergodicity we can choose a set D with $\mu_C(D) > \frac{1}{2}$ and a positive integer N such that for $n \geq N$ we have

$$\frac{1}{n}\sum_{i=1}^{n} L'(U^i\omega) > \tfrac{1}{2}\int L'\,d\mu_C > 0 \qquad \text{for } \omega \in D.$$

Now choose $E \subseteq D$, $\mu_C(E) > 0$, so that for $\omega \in E$ we have $\inf\{i > 0\colon U^i\omega \in E\} \geq N$. Then the function $\bar{L}\colon E \to \mathbb{Z}$ such that $(U|_C)_E = U_E = (T_E)^{\bar{L}}$ satisfies $\bar{L} > 0$ and $\int \bar{L}\,d\mu_E = \int L'\,d\mu_C$.

We now observe that T_E is a factor of a tower transformation over U_E. Let $\hat{E}$ be the subset of $E \times \{0, 1, 2, \ldots\}$ below the graph of $\bar{L}$. Let $\hat{\mu}$ be defined as

$$\hat{\mu}(F) = \sum_{i=0}^{\infty} \mu_E(F_j)\left(\int \bar{L}\,d\mu_E\right)^{-1}$$

where F_j denotes the section of F at j. Let $\hat{U}\colon \hat{E} \to \hat{E}$ be defined by $\hat{U}(\omega, i) = (\omega, i+1)$ if $0 \leq i < \bar{L}(\omega) - 1$, while $\hat{U}(\omega, \bar{L}(\omega)-1) = (U_E(\omega), 0)$. Then it is well known that $\hat{U}$ is an ergodic transformation on $\hat{E}$ which preserves $\hat{\mu}$, and

$$\hat{\mu}(E \times \{0\}) = \left(\int \bar{L}\,d\mu_E\right)^{-1} = \left(\int L'\,d\mu_C\right)^{-1}.$$

Define $\Phi\colon \hat{E} \to E$ to be the map taking (ω, i) to $(T_E)^i\omega$. Note that $T_E \circ \Phi = \Phi \circ \hat{U}$, so $\hat{U}$ is a skew product over T_E. As mentioned earlier, the number m of aperiodic U_E-suborbits on a T_E-orbit is finite and, by ergodicity, constant almost everywhere. Since $\bar{L} > 0$ and U_E is ergodic, it is easy to see that $m = \int \bar{L}\,d\mu_E$. (Hence $\int L'\,d\mu_C$ must be a positive integer. See also [**2**; Proposition 10].) Moreover $\hat{U}$ is an m-point extension of T_E, i.e. $\hat{U}$ is a skew product of T_E with the symmetric group on the integers $\{1, 2, \ldots, m\}$. Since one can write each m-point extension of T_E as the transformation induced on the set $E \times \{1, 2, \ldots, m\}$ by an m-point extension of T (where the skewing on $\Omega - E$ is the identity), we see that $U|_C$ is Kakutani equivalent to a finite extension of T. In particular, let $(\hat{T}, \hat{\Omega})$ denote the m-point extension of T. We have that $U|_C$ induces U_E, where E has relative measure $\mu(E)/\mu(C)$ in C. Moreover, $\hat{T}$ induces $\hat{U}$, which in turn induces U_E, where E has relative measure $\mu(E)/m$ in $\hat{\Omega}$. Then if $m > 1$ or $\mu(E) < 1$, we have that $\hat{T}$ induces $U|_C$ from [**3**, lemma 1.3]. If $m = 1$ and $\mu(E) = 1$, then $U|_C \simeq T$, so trivially $U|_C$ is induced by an m-point extension of T. We summarize these results in the following theorem.

THEOREM. *Let $(\Omega, \mathcal{F}, \mu)$ be a Lebesgue space and $T\colon \Omega \to \Omega$ an invertible ergodic measure-preserving transformation. Let L be an integrable integer-valued function. Then for $U(\omega) = T^{L(\omega)}(\omega)$, there is a maximal U-invariant set $A \subseteq \Omega$, with $\mu(A) > 0$, on which U is invertible and preserves μ_A. Moreover, there is a set $B \subseteq A$ with $\mu_A(B) = 1$ such that there are at most finitely many sets $C \subseteq B$ with $\mu(C) > 0$ and $U|_C$ aperiodic* *ergodic. On each of these, $U|_C$ is induced by a finite extension of T. For all other $C \subseteq B$ with $U|_C$ ergodic, $U|_C$ is a finite rotation.*

COROLLARY. *If T is loosely Bernoulli and L is an arbitrary integrable function, T^L is loosely Bernoulli on each ergodic component.*

Proof. By [**3**, Lemma 6.6], loose Bernoullicity is preserved under inducing. By [**4**] and [**3**; Corollary 7.9], it is also preserved under finite extensions.

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