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On Central Elements in the Universal Enveloping Algebras of the Orthogonal Lie Algebras

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Abstract. We present an analogy of the famous formula that the square of the Pfaffian is equal to the determinant for an alternating matrix for the case where the entries are the generators of the orthogonal Lie algebras. This identity clarifies the relation between the two sets of central elements in the enveloping algebra of the orthogonal Lie algebras. We employ systematically the exterior calculus for the proof.

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1. Introduction

For the universal enveloping algebra of the orthogonal Lie algebra realized as the alternating matrices $\mathfrak{o}_n = \{X \in \mathfrak{gl}_n\,;\, {}^tX + X = 0\}$, we have two types of central elements which are explicitly expressed, respectively, by determinants and by Pfaffians of the standard generators of $\mathfrak{o}_n$ (see [HU], [MN]). Recalling the well-known relation that the square of the Pfaffian is equal to the determinant for alternating matrices with commutative entries, we may as well expect that a similar relation holds even for this noncommutative case of the orthogonal Lie algebras. Such a relation was indeed obtained for $n = 4$ in [O], but with all his attempts for the general case, the verification of such relations has been left as an open problem. The main purpose of the present paper is to settle this problem. The explicit relation is given in our main result (Theorem 4) as follows.

THEOREM. *We have a relation between the Pfaffian and the determinant of the alternating matrix* $\boldsymbol{A} = (A_{ij})_{i,j=1}^{2m}$ *consisting of the generators of* $\mathrm{U}(\mathfrak{o}_{2m})$*:*

$$\mathrm{Pf}(\boldsymbol{A})^2 = \det(\boldsymbol{A} + \mathrm{diag}(m-1, m-2, \ldots, -m)),$$

where $A_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}_{2m}$ *are the standard generators of the orthogonal Lie algebra made from the matrix units* E_{ij}*.*

The precise definitions of the Pfaffian and the determinant for noncommutative entries are given in Sections 1.1 and 1.2.

Via the recent result of Molev and Nazarov [MN], our main theorem gives an explicit connection between the two central elements of the universal enveloping algebras $\mathrm{U}(\mathfrak{o}_n)$ of the orthogonal Lie algebras, one expressed by column-determinant in [HU] and the other expressed by the Sklyanin determinant in [M], whose definitions come from quite different origins. For this point and related study, see [I1].

Let us explain the main feature of our study. One key to start with is in the formulation. Roughly speaking, our basic strategy to establish identities we need is to use generating functions as in the papers [NUW], [U1-4]. In these papers, the ring of formal variables is either the exterior algebra or the symmetric algebra, and the number of formal variables is n, the same as the size of the matrix Lie algebra $\mathfrak{gl}_n$ or $\mathfrak{o}_n$. Our new idea is to exploit the exterior algebra with doubled $2n$ variables, in place of these algebras. With this formulation, we gain two advantages. First, it gives a clear and unified treatment for basic facts on the central elements of $\mathrm{U}(\mathfrak{gl}_n)$ and $\mathrm{U}(\mathfrak{o}_n)$. In particular, we see clearly the machanism that an element is central from the invariance based on exterior calculus. Second, it provides us with an easier commutation relations to handle, with which we can complete the computations. We remark that even in our setting the calculations for our main theorem are by no means obvious. One reason to introduce the doubled formal variables is to consider both the covariant and contravariant variables at the same time. In this sense, our approach has a resemblance with that taken in [O] where the Clifford algebra is used. However, it may be fair to say that the commutation relations with the formal variables from the Clifford algebra were not simple enough.

This paper is organized as follows. In Section 1, we prepare the general formal setting in the exterior calculus, where a transparent treatment for the transformation formulas concerning determinants and Pfaffians is given. In the last part of Section 1, as the prototype of the proof of our main result, we give a simple proof of the classical relation that the square of the Pfaffian is equal to the determinant for the alternating matrix with commutative entries. Concrete applications of our formulation to basic facts on the central elements in $\mathrm{U}(\mathfrak{gl}_n)$ and $\mathrm{U}(\mathfrak{o}_n)$ are given in Sections 2 and 3. The main part of this paper is developed in Section 4, where the relation between the two central elements expressed by the Pfaffian and the determinant is given. The calculations to demonstrate our main theorem are not as straightforward as the classical case mentioned above, but have an interesting relationship with the $\mathfrak{sl}_2$-triplet. Section 5 is a supplement to our main result, where further explicit relations on the central elements in $\mathrm{U}(\mathfrak{o}_n)$ are given. In the last section, we make a brief remark on the similar relation between the central elements expressed by Pfaffian and determinant for other realizations of the orthogonal Lie algebra. This is another merit of our formulation.

1. Exterior Calculus for Pfaffians and Determinants

First we prepare some formal set-up in the exterior calculus, which is very useful for the manipulation of Pfaffians and determinants of matrices with noncommutative entries. In the present paper we work over a fixed base field $\mathbb{K}$ of characteristic 0. The exterior algebra Λ_n is an associative algebra generated by the n elements $e_1, e_2, \dots, e_n$ subject to the relations $e_i e_j + e_j e_i = 0$. For an associative algebra $\mathcal{A}$, we form an extended algebra $\Lambda_n \otimes \mathcal{A}$, in which the two subalgebras Λ_n and $\mathcal{A}$ commute. The elements $e_1, e_2, \dots, e_n$ in Λ_n are thus considered to be formal (anti-commuting) variables to make 'generating functions' with coefficients in $\mathcal{A}$. The grading $\Lambda_n = \bigoplus_{r=0}^{n} \Lambda_n^{(r)}$ of Λ_n is naturally extended to that of $\Lambda_n \otimes \mathcal{A}$.

1.1. When we treat an alternating matrix of size n and consider its Pfaffian, we always assume that n is even and consistently use the notation as $n = 2m$. For an alternating matrix $\Phi = (\Phi_{ij})_{i,j=1}^{2m}$ whose entries are in $\mathcal{A}$, we define its Pfaffian $\mathrm{Pf}(\Phi)$ by the formula

$$\begin{aligned}\mathrm{Pf}(\Phi) &= \frac{1}{2^m\, m!} \sum_{\sigma \in \mathfrak{S}_{2m}} \mathrm{sign}(\sigma) \Phi_{\sigma(1)\sigma(2)} \Phi_{\sigma(3)\sigma(4)} \dots \Phi_{\sigma(2m-1)\sigma(2m)} \\ &= \frac{1}{m!} \sum_{\substack{\sigma \in \mathfrak{S}_{2m} \\ \sigma(2i-1)<\sigma(2i)}} \mathrm{sign}(\sigma) \Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(2m-1)\sigma(2m)}.\end{aligned}$$

The expression of the Pfaffian in terms of the exterior calculus is given as follows:

PROPOSITION 1.1. *For the matrix Φ, form an element $\Theta = \Theta_\Phi \in \Lambda_n \otimes \mathcal{A}$ by*

$$\Theta = \sum_{1 \leqslant i,j \leqslant n} e_i e_j \Phi_{ij},$$

and make its mth *power. Then we have*

$$\Theta^m = e_1 e_2 \dots e_n\, 2^m\, m! \mathrm{Pf}(\Phi).$$

Proof. This is easily seen from the definition:

$$\begin{aligned}\Theta^m &= \sum_{\substack{1 \leqslant i_1, i_2, \dots, i_m \leqslant n \\ 1 \leqslant j_1, j_2, \dots, j_m \leqslant n}} e_{i_1} e_{j_1} e_{i_2} e_{j_2} \dots e_{i_m} e_{j_m} \Phi_{i_1 j_1} \Phi_{i_2 j_2} \dots \Phi_{i_m j_m} \\ &= \sum_{\sigma \in \mathfrak{S}_{2m}} e_{\sigma(1)} e_{\sigma(2)} \dots e_{\sigma(n-1)} e_{\sigma(n)} \Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(n-1)\sigma(n)} \\ &= \sum_{\sigma \in \mathfrak{S}_{2m}} e_1 e_2 \dots e_n \mathrm{sign}(\sigma) \Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(n-1)\sigma(n)} \\ &= e_1 e_2 \dots e_n\, 2^m\, m! \mathrm{Pf}(\Phi).\end{aligned}$$

1.2. For a noncommutative version of determinant, we use the following definition:

$$\det(\Phi) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma)\, \Phi_{\sigma(1)1} \Phi_{\sigma(2)2} \dots \Phi_{\sigma(n)n}.$$

Here $\Phi = (\Phi_{ij})_{i,j=1}^{n}$ is an $n \times n$ matrix whose entries are in the algebra $\mathcal{A}$. This should

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be called 'column-determinant' compared with the name 'row-determinant', which is adopted in some literatures (cf. [MN, O]). We note that the row-determinant of Φ is expressed as $\det({}^t\Phi)$, the column-determinant of the transposed matrix. Our determinant is expressed in terms of the exterior calculus as follows. For Φ, form $\eta_j \in \Lambda_n \otimes \mathcal{A}$ from the columns of Φ by $\eta_j = \sum_{i=1}^n e_i \Phi_{ij}$. Then we have

$$\eta_1 \eta_2 \ldots \eta_n = e_1 e_2 \ldots e_n \det(\Phi).$$

Suggested by the definition of the Pfaffian, we can define yet another kind of determinant, a 'double-determinant' or symmetrized determinant (or mixed one), so to speak. Let us put

$$\mathrm{Det}(\Phi) = \frac{1}{n!} \sum_{(\sigma,\sigma') \in \mathfrak{S}_n \times \mathfrak{S}_n} \mathrm{sign}(\sigma)\mathrm{sign}(\sigma') \Phi_{\sigma(1)\sigma'(1)} \Phi_{\sigma(2)\sigma'(2)} \ldots \Phi_{\sigma(n)\sigma'(n)}.$$

To get an expression of this $\mathrm{Det}(\Phi)$ in the framework of the exterior calculus, we now need to double the anti-commuting variables. Let Λ_{2n} be the exterior algebra generated by e_i, e'_i $(i = 1, \ldots, n)$ which are anti-commutative. In the algebra $\Lambda_{2n} \otimes \mathcal{A}$, we form an element $\Xi = \Xi_\Phi$ by

$$\Xi = \sum_{1 \leqslant i,j \leqslant n} e_i e'_j \Phi_{ij}.$$

Then as in the Pfaffian case, it is easy to see that its nth power gives $\mathrm{Det}(\Phi)$:

$$\begin{aligned} \Xi^n &= e_1 e'_1 e_2 e'_2 \ldots e_n e'_n \, n! \mathrm{Det}(\Phi) \\ &= e_1 e_2 \ldots e_n e'_1 e'_2 \ldots e'_n (-)^{\frac{n(n-1)}{2}} n! \mathrm{Det}(\Phi). \end{aligned}$$

This expression of the double-determinant can be generalized to

$$\Xi^r = (-)^{\frac{r(r-1)}{2}} r! \sum_{|I|=|J|=r} e_{i_1} \ldots e_{i_r} e'_{j_1} \ldots e'_{j_r} \mathrm{Det}(\Phi_{IJ}),$$

where Φ_{IJ} is the submatrix $\Phi_{IJ} = (\Phi_{i_p j_q})_{p,q=1}^r$ made from Φ corresponding to the index sets $I = \{i_1, i_2, \ldots, i_r\}$, $J = \{j_1, j_2, \ldots, j_r\} \subset \{1, 2, \ldots, n\}$ of cardinalities $|I| = |J| = r$. In particular, we have the following for $\Phi_I = \Phi_{II}$:

PROPOSITION 1.2. *We have*

$$\Xi^r \tau^{n-r} = e_1 e_2 \ldots e_n e'_1 e'_2 \ldots e'_n (-)^{\frac{n(n-1)}{2}} r!(n-r)! \sum_{|I|=r} \mathrm{Det}(\Phi_I),$$

where τ is the element defined by $\tau = \sum_{i=1}^n e_i e'_i$.

From the relation $\Xi = \sum_{j=1}^n \eta_j e'_j$, we see $\det(\Phi) = \mathrm{Det}(\Phi)$ if η_j's anti-commute. In particular, when the entries of Φ are commutative, this is obviously true. While simple relations between the two determinants $\det(\Phi)$ and $\mathrm{Det}(\Phi)$ may not be expected for noncommutative case in general, we will see as nice relations as commutative case among these $\det(\Phi), \det({}^t\Phi)$ and $\mathrm{Det}(\Phi)$ especially for $\mathcal{A} = \mathrm{U}(\mathfrak{gl}_n)$ and $\mathcal{A} = \mathrm{U}(\mathfrak{o}_n)$.

1.3. *Remark*. When the entries Φ_{ij} of the alternating matrix Φ commute (or more generally Φ_{ij} and Φ_{kl} commute for $\{i, j\} \cap \{k, l\} = \emptyset$), the expression of the Pfaffian is reduced to

$$\mathrm{Pf}(\Phi) = \sum_{\sigma \in \mathfrak{S}_{2m}/B_{2m}} \mathrm{sign}(\sigma)\Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(2m-1)\sigma(2m)},$$

where B_{2m} is the centralizer of the cycle $(12)(34)\dots(2m-1\ 2m)$.

Similar reductions for $\mathrm{Det}(\Phi)$ lead us to the obvious relations $\mathrm{Det}(\Phi) = \det(\Phi) = \det({}^t\Phi)$ for commutative case.

1.4. Throughout the paper, we will keep working with the exterior algebra Λ_{2n} generated by $\{e_i, e'_i;\ 1 \leqslant i \leqslant n\}$. In the exterior algebra Λ_{2n}, we denote by Λ_n and Λ'_n respectively the subalgebras generated by $\{e_i;\ 1 \leqslant i \leqslant n\}$ and $\{e'_i;\ 1 \leqslant i \leqslant n\}$. Let $\alpha \in \mathrm{GL}_{2n} = \mathrm{GL}_{2n}(\mathbb{K})$ be a linear transformation of the vector space $\mathbb{K}^{2n}$ spanned by the standard basis $\{e_i, e'_i;\ 1 \leqslant i \leqslant n\}$. Then α can be extended to an algebra automorphism α_* of Λ_{2n} and hence of $\Lambda_{2n} \otimes \mathcal{A}$.

Recall that we have a special central element $\tau = \sum_{i=1}^n e_i e'_i \in \Lambda_{2n} \otimes \mathcal{A}$. The intrinsic meaning of τ comes from the standard symplectic form B on the vector space $\mathbb{K}^{2n}$ defined by

$$B(e_i, e'_j) = \delta_{ij}, \quad B(e_i, e_j) = 0, \quad B(e'_i, e'_j) = 0.$$

It is checked that $\alpha \in \mathrm{GL}_{2n}$ belongs to the symplectic group $\mathrm{Sp}_{2n} = \mathrm{Sp}(\mathbb{K}^{2n}, B)$ if and only if $\alpha_*(\tau) = \tau$. More generally we see that the group

$$\{\alpha \in \mathrm{GL}_{2n}\,;\ \alpha_*(\tau) = \chi(\alpha)\tau \text{ for some } \chi(\alpha) \in \mathbb{K}^\times\}$$

coincides with the group $\mathrm{GSp}_{2n} = \mathrm{GSp}(\mathbb{K}^{2n}, B)$ of symplectic similitude and that $\det(\alpha) = \chi(\alpha)^n$.

Here are two types of these transformations that we use later. For $g, g' \in \mathrm{GL}_n$, we consider

$$\alpha_{g,g'} = \mathrm{diag}(g, g') = \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \in \mathrm{GL}_{2n},$$

and $\alpha_g = \alpha_{g,{}^tg^{-1}} \in \mathrm{GL}_{2n}$. It is easy to see that $\alpha_g \in \mathrm{Sp}_{2n}$. Note that $\det(\alpha_g) = 1$ and that α_g leaves the subalgebras Λ_n and Λ'_n invariant. For an element $h \in \mathrm{GL}_2$, we consider the element $h \otimes 1_n \in \mathrm{GL}_{2n}$ obtained by the embedding

$$\mathrm{GL}_2 \ni h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto h \otimes 1_n = \begin{pmatrix} a1_n & b1_n \\ c1_n & d1_n \end{pmatrix} \in \mathrm{GL}_{2n}.$$

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It is clear that $h \otimes 1_n \in \mathrm{GSp}_{2n}$ and $\chi(h \otimes 1_n) = \det(h)$. A typical example of this type is the involution $\imath$ defined by $\imath(e_i) = e'_i$, $\imath(e'_i) = e_i$. We will simply write $\imath_*(\varphi) = \varphi'$ for $\varphi \in \Lambda_{2n} \otimes \mathcal{A}$. Observe then, for example, $\tau' = -\tau$ and $\Xi'_\Phi = \Xi_\Phi$ for an alternating matrix Φ.

The following are obviously the relations expressing the determinant in the exterior calculus:

LEMMA 1.4. *The following formulas hold for the actions on the top-degree elements in $\Lambda_{2n} \otimes \mathcal{A}$ and $\Lambda_n \otimes \mathcal{A}$:*

(1) *Let $\varphi \in \Lambda_{2n}^{(2n)} \otimes \mathcal{A}$. For $\alpha \in \mathrm{GL}_{2n}$, we have $\alpha_*(\varphi) = \det(\alpha)\varphi$. In particular, when α is of the form α_g with $g \in \mathrm{GL}_n$, we have $\alpha_*(\varphi) = \varphi$.*
(2) *Let $\varphi \in \Lambda_n^{(n)} \otimes \mathcal{A}$. For $\alpha_g \in \mathrm{GL}_{2n}$ with $g \in \mathrm{GL}_n$, we have $\alpha_{g*}(\varphi) = \det(g)\varphi$.*

1.5. For an $n \times n$ matrix $\Phi = (\Phi_{ij})_{i,j=1}^n$ with entries in $\mathcal{A}$, we consider a linear transformation of the following form:

$$\Phi \longmapsto g\Phi g' = \Big(\sum_{1 \leqslant p,q \leqslant n} g_{ip}\,\Phi_{pq}\,g'_{qj} \Big)_{i,j=1}^n,$$

where $g = (g_{ij})$ and $g' = (g'_{ij})$ are taken from GL_n. The effect of the transformation of this type on the elements $\Xi_\Phi = \sum_{i,j=1}^n e_i e'_j \Phi_{ij}$ and $\Theta_\Phi = \sum_{i,j=1}^n e_i e_j \Phi_{ij}$ is reduced to the transformation on the formal variables:

$$\Xi_{g\Phi g'} = (\alpha_{g,{}^tg'})_*(\Xi_\Phi), \qquad \Theta_{g\Phi{}^tg} = \alpha_{g*}(\Theta_\Phi).$$

In particular, for any polynomial $p(x) \in \mathbb{K}[x]$, we have

$$p(\Xi_{g\Phi g'}) = (\alpha_{g,{}^tg'})_*(p(\Xi_\Phi)), \qquad p(\Theta_{g\Phi{}^tg}) = \alpha_{g*}(p(\Theta_\Phi)).$$

Putting respectively $p(x) = x^n$ or $p(x) = x^m$ in these formulas, we can deduce from Lemma 1.4 the transformation formulas of the double-determinant and the Pfaffian

$$\mathrm{Det}(g\Phi g') = \det(g)\det(g')\mathrm{Det}(\Phi), \qquad \mathrm{Pf}(g\Phi{}^tg) = \det(g)\mathrm{Pf}(\Phi),$$

which are valid even when the entries of Φ are noncommutative. For the latter formula, we of course assume the matrix Φ to be alternating and the size to be $n = 2m$.

In the case $g' = g^{-1}$, since $\alpha_{g,{}^tg^{-1}} = \alpha_g$ leaves the element τ invariant, we have

$$p(\Xi_{g\Phi g^{-1}}) = \alpha_{g*}(p(\Xi_\Phi))$$

for any polynomial $p(x) \in \mathbb{K}[\tau][x]$ with coefficients in $\mathbb{K}[\tau]$. When this $p(\Xi_\Phi) \in \Lambda_{2n} \otimes \mathcal{A}$ is of degree $2n$, it is seen to be invariant under the conjugation also from Lemma 1.4:

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PROPOSITION 1.5. *Let* $p(x) \in \mathbb{K}[\tau][x]$, *and assume* $p(\Xi_\Phi)$ *to be of degree* $2n$. *Then we have*

$$p(\Xi_{g\Phi g^{-1}}) = p(\Xi_\Phi).$$

1.6. To apply the discussion above, we take a typical situation that the transformation of the matrix Φ is caused by an algebra automorphism γ of $\mathcal{A}$. We denote by γ^* the algebra automorphism of $\Lambda_{2n} \otimes \mathcal{A}$ naturally extending γ.

PROPOSITION 1.6. *Suppose we have an* $n \times n$ *matrix* $\Phi = (\Phi_{ij})_{i,j=1}^n$ *with entries in* $\mathcal{A}$ *such that the action of the automorphism* γ *of* $\mathcal{A}$ *on* Φ *is written in a form* $\gamma(\Phi) = g\Phi g^{-1}$ *for some* $g \in \mathrm{GL}_n$. *Then the following hold:*

(1) *For any* φ *written by a homogeneous polynomial of degree* n *in* Ξ_Φ *and* τ, *we have* $\gamma^*(\varphi) = \varphi$.
(2) *Suppose further the matrix* Φ *above is alternating of size* $n = 2m$ *and* g *is orthogonal in the sense that* $g^{-1} = {}^t g$. *Then* $\gamma^*(\Theta_\Phi^m) = \det(g)\Theta_\Phi^m$.

Proof. By definition we see that $\gamma^*(\Xi_\Phi) = \Xi_{g\Phi g^{-1}}$ for (1) and $\gamma^*(\Theta_\Phi) = \Theta_{g\Phi {}^t g}$ for (2), so that $\gamma^*(\Xi_\Phi) = \alpha_{g*}(\Xi_\Phi)$ and $\gamma^*(\Theta_\Phi) = \alpha_{g*}(\Theta_\Phi)$ from the discussion in the previous section. Noting that both γ^* and α_* are automorphisms of $\Lambda_{2n} \otimes \mathcal{A}$, we see that the assertions follow directly from Lemma 1.4. □

Remark. In the proposition above, if the matrix Φ is the identity matrix 1_n, then the condition $\gamma(\Phi) = g\Phi g^{-1}$ is automatically satisfied for any $g \in \mathrm{GL}_n$. This fact reflects that the element of the form α_g belongs to Sp_{2n} and that $\alpha_{g*}(\tau) = \tau$.

1.7. We now demonstrate how efficiently our formulation provides us with the proof of the relation $\mathrm{Pf}(\Phi)^2 = \det(\Phi)$ in case the entries of the alternating matrix Φ are commutative (cf. [JLW]). As introduced above, we consider the following three elements Θ, Θ', Ξ in $\Lambda_{2n} \otimes \mathcal{A}$, where $\mathcal{A}$ is a commutative algebra:

$$\Theta = \sum_{1 \leqslant i,j \leqslant n} e_i e_j \Phi_{ij}, \quad \Theta' = \sum_{1 \leqslant i,j \leqslant n} e'_i e'_j \Phi_{ij}, \quad \Xi = \sum_{1 \leqslant i,j \leqslant n} e_i e'_j \Phi_{ij}.$$

Take here

$$h = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2$$

and compute $(h \otimes 1_n)_*(\Xi)$:

$$(h \otimes 1_n)_*(\Xi) = \sum_{1 \leqslant i,j \leqslant n} (e_i + e'_i)(-e_j + e'_j)\Phi_{ij} = -\Theta + \Theta'.$$

Then by Lemma 1.4 (1) we have, on the one hand,

$$(h \otimes 1_n)_*(\Xi^n) = \det(h \otimes 1_n)\Xi^n = \det(h)^n \Xi^n = 2^n \Xi^n.$$

On the other hand, from the calculation using the explicit form of $(h \otimes 1_n)_*(\Xi)$, we see

$$(h \otimes 1_n)_*(\Xi^n) = (-\Theta + \Theta')^n = (-)^m \binom{n}{m} \Theta^m \Theta'^m.$$

This is seen from the binomial theorem together with the fact that $\Theta^k = 0 = \Theta'^k$ for $k > m = n/2$. Recalling the relations

$$\Xi^n = e_1 \dots e_n e'_1 \dots e'_n (-)^{\frac{n(n-1)}{2}} n! \mathrm{Det}(\Phi),$$
$$\Theta^m = e_1 \dots e_n 2^m m! \mathrm{Pf}(\Phi), \quad \Theta'^m = e'_1 \dots e'_n 2^m m! \mathrm{Pf}(\Phi),$$

we immediately obtain the formula $\mathrm{Pf}(\Phi)^2 = \mathrm{Det}(\Phi)$.

2. Application to $\mathrm{U}(\mathfrak{gl}_n)$

Before going on to our main object, the orthogonal Lie algebra $\mathfrak{o}_n$, we first treat a more basic case of $\mathfrak{gl}_n$. The purpose of this section is to give a clear treatment of the Capelli elements. This presents an application of our formulation of independent interest.

Let E_{ij} be the standard basis of the Lie algebra $\mathfrak{gl}_n$ corresponding to the matrix unit. In the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$, we write $E_{ij}(u) = E_{ij} + u\delta_{ij} \in \mathrm{U}(\mathfrak{gl}_n)$ with a scalar parameter u. We also form an $n \times n$ matrix $\boldsymbol{E} = (E_{ij})_{i,j=1}^n$. Let us consider the following elements in $\Lambda_n \otimes \mathrm{U}(\mathfrak{gl}_n)$:

$$\eta_j = \sum_{i=1}^n e_i E_{ij}, \qquad \eta_j(u) = \eta_j + u e_j = \sum_{i=1}^n e_i E_{ij}(u);$$
$${}^t\eta_j = \sum_{i=1}^n e_i E_{ji}, \qquad {}^t\eta_j(u) = {}^t\eta_j + u e_j = \sum_{i=1}^n e_i E_{ji}(u).$$

The basic commutation relations for these elements are given in the following:

LEMMA 2.1. *For $1 \leqslant i, j \leqslant n$, we have*

(1) $\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = 0,$

(2) ${}^t\eta_i(u){}^t\eta_j(u+1) + {}^t\eta_j(u){}^t\eta_i(u+1) = 0.$

In particular, we see

(3) $\eta_i(u+1)\eta_i(u) = 0, \qquad {}^t\eta_i(u){}^t\eta_i(u+1) = 0.$

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Proof. These can be verified by direct calculations. First we see

$$\begin{aligned}
&\eta_i(u)\eta_j(u) + \eta_j(u)\eta_i(u)\\
&\quad = \sum_{p,q} e_p e_q [E_{pi}(u), E_{qj}(u)] = \sum_{p,q} e_p e_q [E_{pi}, E_{qj}]\\
&\quad = \sum_{p,q} e_p e_q (E_{pj}\delta_{iq} - E_{qi}\delta_{jp})\\
&\quad = -\sum_{p} e_i e_p E_{pj} - \sum_{q} e_j e_q E_{qi}\\
&\quad = -e_i\eta_j - e_j\eta_i = -e_i\eta_j(u) - e_j\eta_i(u).
\end{aligned}$$

This proves the formula (1). The formula (2) is deduced from (1) by applying the anti-automorphism $E_{ij} \longmapsto E_{ji}$ of $\mathrm{U}(\mathfrak{gl}_n)$. □

With these relations, for the matrices with the noncommutative entries from E_{ij}, we see how the three types of determinants, i.e., column-, row- and double-determinants are connected to each other. More precisely, we have the description of the Capelli determinant in terms of the double-determinant (Proposition 2.2), from which some of the basic properties of the Capelli determinants are naturally deduced.

As before we consider the exterior algebra Λ_{2n} generated by $\{e_i, e'_i;\ 1 \leqslant i \leqslant n\}$. In the algebra $\Lambda_{2n} \otimes \mathrm{U}(\mathfrak{gl}_n)$, we consider the elements

$$\begin{aligned}
\Xi = \Xi_{\boldsymbol{E}} &= \sum_{1 \leqslant i,j \leqslant n} e_i e'_j E_{ij} = \sum_{j=1}^{n} \eta_j e'_j = \sum_{i=1}^{n} e_i\, {}^t\eta'_i,\\
\Xi(u) = \Xi_{\boldsymbol{E}}(u) &= \Xi_{\boldsymbol{E}} + u\tau = \sum_{j=1}^{n} \eta_j(u) e'_j = \sum_{i=1}^{n} e_i\, {}^t\eta'_i(u),
\end{aligned}$$

where

$$\tau = \sum_{i=1}^{n} e_i e'_i, \quad {}^t\eta'_i = \sum_{j=1}^{n} e'_j E_{ij}, \quad {}^t\eta'_i(u) = \sum_{j=1}^{n} e'_j E_{ij}(u).$$

We put

$$\Xi^{(n)}(u) = \Xi(u)\Xi(u-1)\ldots\Xi(u-n+1).$$

Also write $\natural = \mathrm{diag}(n-1, n-2, \ldots, 0)$ and $\natural^* = \mathrm{diag}(0, 1, \ldots, n-1)$ for the diagonal shift. The following proposition explains the mechanism that the two expressions, the one given by the column-determinant and the other by the row-determinant, can be converted to each other for the Capelli element:

PROPOSITION 2.2. *We have*

$$\begin{aligned}\Xi^{(n)}(u+n-1) &= \Xi(u+n-1)\Xi(u+n-2)\dots\Xi(u)\\ &= e_1\dots e_n e'_1\dots e'_n\,(-)^{\frac{n(n-1)}{2}}\,n!\det(\boldsymbol{E}+\natural+u)\\ &= e_1\dots e_n e'_1\dots e'_n\,(-)^{\frac{n(n-1)}{2}}\,n!\det({}^t\boldsymbol{E}+\natural^*+u).\end{aligned}$$

From this, the following is immediate:

COROLLARY 2.3. *The equality holds between the determinant of the matrix* $\boldsymbol{E}$ *with the diagonal shift and that of transposed one with the reverse shift:*

$$\det(\boldsymbol{E}+\natural+u)=\det({}^t\boldsymbol{E}+\natural^*+u).$$

Proof of Proposition 2.2. Using the relation $\Xi(u)=\sum_{j=1}^n \eta_j(u)e'_j$ and Lemma 2.1 (1), we have on the one hand,

$$\begin{aligned}&\Xi^{(n)}(u+n-1)\\ &\quad=\sum_{1\leqslant j_1,j_2,\dots,j_m\leqslant n}\varepsilon\,\eta_{j_1}(u+n-1)\eta_{j_2}(u+n-2)\dots\eta_{j_n}(u)e'_{j_1}e'_{j_2}\dots e'_{j_n}\\ &\quad=\varepsilon\sum_{\sigma\in\mathfrak{S}_n}\eta_{\sigma(1)}(u+n-1)\eta_{\sigma(2)}(u+n-2)\dots\eta_{\sigma(n)}(u)e'_{\sigma(1)}e'_{\sigma(2)}\dots e'_{\sigma(n)}\\ &\quad=\varepsilon\sum_{\sigma\in\mathfrak{S}_n}\eta_1(u+n-1)\eta_2(u+n-2)\dots\eta_n(u)e'_1e'_2\dots e'_n\\ &\quad=e_1\dots e_n e'_1e'_2\dots e'_n(-)^{\frac{n(n-1)}{2}}\,n!\det(\boldsymbol{E}+\natural+u).\end{aligned}$$

Here $\varepsilon=(-)^{\frac{n(n-1)}{2}}$ is the signature coming from the transposition of $\eta_j(u)$'s and e'_j's. On the other hand, start from the relation $\Xi(u)=\sum_{i=1}^n e_i\,{}^t\eta'_i(u)$. Reversing the order of the multiplication in $\Xi(u)$ and using Lemma 2.1 (2), we get similarly

$$\Xi^{(n)}(u+n-1)=e_1\dots e_n e'_1e'_2\dots e'_n(-)^{\frac{n(n-1)}{2}}\,n!\det({}^t\boldsymbol{E}+\natural^*+u).$$

Thus proves our assertion. □

Remark. Proposition 2.2 is equivalent to the following expression of $\det(\boldsymbol{E}+\natural+u)$ by the double-determinant with parameters:

$$\det(\boldsymbol{E}+\natural+u)=\frac{1}{n!}\sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n}\operatorname{sign}(\sigma)\operatorname{sign}(\sigma')E_{\sigma(1)\sigma'(1)}(n-1+u)\dots E_{\sigma(n)\sigma'(n)}(u).$$

It is well known that the Capelli determinant is central in $U(\mathfrak{gl}_n)$ and several proofs are known (see, e.g., [Kz, HU, U3]). This fact is also deduced from the following more general result together with the expression in Proposition 2.2:

PROPOSITION 2.4. *Let* φ *be an element written by a homogeneous polynomial of degree* n *in* Ξ *and* τ. *Then the coefficient of* $e_1\dots e_n e'_1\dots e'_n$ *in* φ *belongs to the invariant* *subalgebra* $\mathrm{U}(\mathfrak{gl}_n)^{\mathrm{GL}_n}$ *under the adjoint action of* GL_n*; hence central in* $\mathrm{U}(\mathfrak{gl}_n)$.

This is immediately seen from Proposition 1.6. The following lemma enables us to apply Proposition 1.6 to the proof of Proposition 2.4.

LEMMA 2.5. *The adjoint action of* $g \in \mathrm{GL}_n$ *on the matrix* $\boldsymbol{E}$ *is written as* $(\mathrm{Ad}g)\boldsymbol{E} = {}^t g \boldsymbol{E}\, {}^t g^{-1}$.

Proof. Let us denote by g_{ij} and g^{ij} the entries of the matrices g and g^{-1} respectively. Then the lemma is shown by the direct calculation:

$$(\mathrm{Ad}g)\boldsymbol{E} = (gE_{ij}g^{-1})_{i,j=1}^n = \Bigl(\sum_{1 \leqslant \alpha,\beta \leqslant n} g_{\alpha i}E_{\alpha\beta}g^{j\beta}\Bigr)_{i,j=1}^n = {}^t g E\, {}^t g^{-1}. \qquad \square$$

Remark. From Proposition 2.4, we see that the following elements are central in $\mathrm{U}(\mathfrak{gl}_n)$:

$$J_r = \sum_{|I|=r} \mathrm{Det}(\boldsymbol{E}_I), \qquad C_r(u) = \sum_{|I|=r} \det(\boldsymbol{E}_I + \natural_r + u),$$

where $\natural_r$ is the diagonal shift $\natural_r = \mathrm{diag}(r-1, r-2, \ldots, 0)$. In fact, J_r and $C_r(u)$ are respectively expressed as the coefficients of $\Xi^r\tau^{n-r}$ and $\Xi^{(r)}(u+r-1)\tau^{n-r}$ as seen in Proposition 1.2 and Proposition 2.6 below:

PROPOSITION 2.6. *We have*

$$\begin{aligned}&\Xi^{(r)}(u+r-1)\tau^{n-r}\\&\quad = e_1 \ldots e_n e'_1 \ldots e'_n (-)^{\frac{n(n-1)}{2}} r!(n-r)! \sum_{|I|=r} \det(\boldsymbol{E}_I + \natural_r + u).\end{aligned}$$

This is proved in a similar way as Proposition 1.2 by applying Proposition 2.2 to submatrices $\boldsymbol{E}_I$.

We remark that $C_r = C_r(0)$ is the rth Capelli element, and J_k is the higher Casimir element described in Section 63 of [Z]. Proposition 2.6 also gives the explicit relation between these two series of central elements as follows:

PROPOSITION 2.7. *We have*

$$C_r = \sum_{k=0}^{r} (-)^{r-k} S_k^r \frac{k!(n-k)!}{r!(n-r)!} J_k,$$

where S_k^r *is the Stirling number of the second kind connecting the factorial power* $\lambda^{(r)} = \lambda(\lambda-1)\ldots(\lambda-r+1)$ *with the usual power* $\lambda^{(r)} = \sum_{k=0}^{r} S_k^r \lambda^k$.

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3. Application to $\mathrm{U}(\mathfrak{o}_n)$

Similarly as for the case of $\mathfrak{gl}_n$, we give some basic properties of the central elements of $\mathrm{U}(\mathfrak{o}_n)$ expressed by Pfaffians and determinants.

We consider the orthogonal Lie algebra $\mathfrak{o}_n$ which is realized as the Lie subalgebra of $\mathfrak{gl}_n$ consisting of the alternating matrices. We put $A_{ij} = E_{ij} - E_{ji}$ and arrange a matrix $\boldsymbol{A} = (A_{ij})_{i,j=1}^n$ from these generators of $\mathfrak{o}_n$. As before we write $A_{ij}(u) = A_{ij} + u\delta_{ij}$ with a parameter u. As an analogue of the Capelli determinant, it is shown in [HU] that the determinant $\det(\boldsymbol{A} + \natural + u)$ is central in $\mathrm{U}(\mathfrak{o}_n)$, where $\natural = \mathrm{diag}(n-1, n-2, \ldots, 0)$ (see also [U3]). We give here an alternative proof of this fact as an application of our formulation in Section 1.6.

We put $\psi_j = \sum_{i=1}^n e_i A_{ij} \in \Lambda_n \otimes \mathrm{U}(\mathfrak{o}_n)$ and further $\psi_j(u) = \sum_{i=1}^n e_i A_{ij}(u)$ with the parameter u. The basic commutation relations for these are given as follows:

LEMMA 3.1. *For $1 \leqslant i, j \leqslant n$, we have*

$$\psi_i(u+1)\psi_j(u) + \psi_j(u+1)\psi_i(u) = -\delta_{ij}\Theta,$$

where

$$\Theta = \sum_{1 \leqslant p,q \leqslant n} e_p e_q A_{pq} = -\sum_p e_p \psi_p.$$

Proof. This relation is verified by a direct calculation as Lemma 2.1:

$$\begin{aligned}
\psi_i(u)\psi_j(u) + \psi_j(u)\psi_i(u) &= \sum_{p,q} e_p e_q [A_{pi}(u), A_{qj}(u)] = \sum_{p,q} e_p e_q [A_{pi}, A_{qj}] \\
&= \sum_{p,q} e_p e_q (A_{pj}\delta_{iq} + A_{iq}\delta_{pj} - A_{pq}\delta_{ij} - A_{ij}\delta_{pq}) \\
&= \sum_p e_p e_i A_{pj} + \sum_q e_j e_q A_{iq} - \delta_{ij} \sum_{p,q} e_p e_q A_{pq} \\
&= -e_i \psi_j - e_j \psi_i - \delta_{ij}\Theta \\
&= -e_i \psi_j(u) - e_j \psi_i(u) - \delta_{ij}\Theta.
\end{aligned}$$

This proves our formula. □

From this relation, as Proposition 2.2, we obtain the expression of the determinant $\det(\boldsymbol{A} + \natural + u)$ in terms of the double-determinant. As before we define the elements $\Xi = \Xi_A$ and $\Xi(u) = \Xi_A(u)$ in the algebra $\Lambda_{2n} \otimes \mathrm{U}(\mathfrak{o}_n)$ by

$$\Xi_A = \sum_{1 \leqslant i,j \leqslant n} e_i e'_j A_{ij} = \sum_{j=1}^n \psi_j e'_j, \quad \Xi_A(u) = \Xi_A + u\tau = \sum_{j=1}^n \psi_j(u) e'_j,$$

with $\tau = \sum_{i=1}^n e_i e'_i$.

PROPOSITION 3.2. *We have the relation*

$$\Xi^{(n)}(u+n-1) = e_1 \dots e_n e'_1 \dots e'_n (-)^{\frac{n(n-1)}{2}} n! \det(\boldsymbol{A} + \natural + u).$$

COROLLARY 3.3. *For the determinant of the matrix* $\boldsymbol{A}$*, the following holds:*

$$\det(\boldsymbol{A} + \natural + u) = (-)^n \det(\boldsymbol{A} - \natural^* - u),$$

where $\natural = \mathrm{diag}(n-1, n-2, \dots, 0)$ *and* $\natural^* = \mathrm{diag}(0, 1, \dots, n-1)$.

In particular, we have $\det(\boldsymbol{A} + \mathrm{diag}(v, v-1, \dots, -v)) = 0$ *when the size* $n = 2v+1$ *of the matrix is odd.*

Proof of Proposition 3.2. This is parallel to that of Proposition 2.2. In fact, in the following equality deduced from the product of the relation $\Xi(u) = \sum_{j=1}^n \psi_j(u) e'_j$

$$\begin{aligned}&\Xi^{(n)}(u+n-1)\\ &\quad= \sum_{1 \leqslant j_1, j_2, \dots, j_n \leqslant n} \varepsilon\, \psi_{j_1}(u+n-1)\psi_{j_2}(u+n-2)\dots\psi_{j_n}(u) e'_{j_1} e'_{j_2} \dots e'_{j_n},\end{aligned}$$

the indices $j_1, j_2, \dots, j_n$ may run over only the distinct ones because of the factor $e'_{j_1} e'_{j_2} \dots e'_{j_n}$. Here $\varepsilon = (-)^{\frac{n(n-1)}{2}}$ is the signature coming from the transposition of $\psi_j(u)$'s and e'_j's. Then the commutation relation of $\psi_j(u)$'s in Lemma 3.1 is the same as that of $\eta_j(u)$'s as long as the indices are distinct. The rest of computation is parallel to Proposition 2.2. □

Corollary 3.3 is a consequence of Proposition 3.2 and the equality

$$\begin{aligned}&\Xi(u+n-1)\Xi(u+n-2)\dots\Xi(u)\\ &\quad= (-)^n \Xi(-u-n+1)\Xi(-u-n+2)\dots\Xi(-u),\end{aligned}$$

which holds as a special case of the following:

PROPOSITION 3.4. *We have*

$$\Xi(u_1)\dots\Xi(u_r)\tau^{n-r} = (-)^r \Xi(-u_1)\dots\Xi(-u_r)\tau^{n-r}.$$

Here $u_1, \dots, u_r$ *are scalar parameters.*

Proof. Apply the involution $' = \iota_*$ induced from the replacement $e_i \leftrightarrow e'_i$ to $\varphi = \Xi(u_1)\dots\Xi(u_r)\tau^{n-r}$, and note that $\tau' = -\tau$ and $\Xi(u)' = \Xi(-u)$. Then we have on the one hand

$$\varphi' = (\Xi(u_1)\dots\Xi(u_r)\tau^{n-r})' = \Xi(-u_1)\dots\Xi(-u_r)(-\tau)^{n-r}.$$

On the other hand, since φ is of top-degree, we see $\varphi' = (-)^n \varphi$ by Lemma 1.4. Thus the proposition holds. □

Remarks. (1) Proposition 3.2 is equivalent to the following expression of $\det(\boldsymbol{A}+\natural+u)$:

$$\det(\boldsymbol{A}+\natural+u)=\frac{1}{n!}\sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n}\operatorname{sign}(\sigma)\operatorname{sign}(\sigma')A_{\sigma(1)\sigma'(1)}(n-1+u)\ldots A_{\sigma(n)\sigma'(n)}(u).$$

(2) As seen in the proof of Proposition 3.2, by using the doubled variables, we were able to eliminate the effect of the term Θ for $i=j$ appeared in the right-hand side of the commutation relation in Lemma 3.1. This is one of the most essential points in our formulation. (Compare the proofs given in [U3] with Proposition 3.5 below for the fact that the determinant $\det(\boldsymbol{A}+\natural+u)$ is central.)

(3) Observe the similarity between Corollaries 2.3 and 3.3: if the matrix $\boldsymbol{E}$ in Corollary 2.3 is replaced by $\boldsymbol{A}$, then it yields Corollary 3.3, because ${}^t\boldsymbol{A}=-\boldsymbol{A}$. This is not a mere coincidence. We can actually show the expression of row-determinant

$$\Xi^{(n)}(u+n-1)=e_1\ldots e_n e'_1\ldots e'_n(-)^{\frac{n(n-1)}{2}}n!\det({}^t\boldsymbol{A}+\natural^*+u)$$

in parallel to Proposition 3.2 by starting from the relation $\Xi(u)=\sum_{j=1}^n e_j{}^t\psi'_j(u)$ with ${}^t\psi'_j(u)=\sum_{i=1}^n e'_iA_{ji}(u)$. The commutation relation between these ${}^t\psi'_j(u)$ is seen by applying the anti-automorphism $A_{ij}\mapsto A_{ji}$ to Lemma 3.1.

(4) Corollaries 2.3 and 3.3 are stated in the Remarks in the Appendix of [HU] without proofs.

The determinant $\det(\boldsymbol{A}+\natural+u)$ and Pfaffian $\mathrm{Pf}(\boldsymbol{A})$ of the matrix $\boldsymbol{A}$ are central in $\mathrm{U}(\mathfrak{o}_n)$. As for Proposition 2.4, we can deduce these facts from Proposition 1.6 via Propositions 3.2 and 1.1 as follows.

PROPOSITION 3.5. *Let φ be an element written by a homogeneous polynomial of degree n in Ξ and τ. Then the coefficient of $e_1\ldots e_n e'_1\ldots e'_n$ in φ belongs to the invariant subalgebra $\mathrm{U}(\mathfrak{o}_n)^{\mathrm{O}_n}$ under the adjoint action of O_n; hence central in $\mathrm{U}(\mathfrak{o}_n)$. In particular, we have $\det(\boldsymbol{A}+\natural+u)\in\mathrm{U}(\mathfrak{o}_n)^{\mathrm{O}_n}$.*

PROPOSITION 3.6. *Let the size $n=2m$ of the matrix space be even. The Pfaffian $\mathrm{Pf}(\boldsymbol{A})$, which is expressed as the coefficient of $e_1\ldots e_n$ in Θ^m, belongs to the invariant subalgebra $\mathrm{U}(\mathfrak{o}_n)^{\mathrm{SO}_n}$ under the adjoint action of SO_n; hence central in $\mathrm{U}(\mathfrak{o}_n)$.*

We see these two propositions respectively from Proposition 1.6 (1) and (2). The following lemma assures that the adjoint action of O_n on the matrix $\boldsymbol{A}$ satisfies the assumption of Proposition 1.6:

LEMMA 3.7. *The adjoint action of $g\in\mathrm{O}_n$ on the matrix $\boldsymbol{A}$ is written as $(\mathrm{Ad}g)\boldsymbol{A}={}^tg\boldsymbol{A}\,{}^tg^{-1}={}^tg\boldsymbol{A}g$.*

Proof. We know that $(\mathrm{Ad}g)\boldsymbol{E} = {}^t g \boldsymbol{E}\, {}^t g^{-1}$ by Lemma 2.5, so that $(\mathrm{Ad}g){}^t\boldsymbol{E} = {}^t({}^t g \boldsymbol{E}\, {}^t g^{-1}) = g^{-1}\, {}^t\boldsymbol{E} g$. For $g \in \mathrm{O}_n$, the latter turns out to be ${}^t g\, {}^t\boldsymbol{E}\, {}^t g^{-1}$. Then the assertion is clear, because $\boldsymbol{A} = \boldsymbol{E} - {}^t\boldsymbol{E}$. □

Remark. As in the $\mathfrak{gl}_n$ case, we see that the sums of column- or double-determinants,

$$\sum_{|I|=k} \mathrm{Det}(\boldsymbol{A}_I), \qquad \sum_{|I|=r} \det(\boldsymbol{A}_I + \natural_r + u)$$

are central in $\mathrm{U}(\mathfrak{o}_n)$. In fact, these are expressed as the coefficients of elements in $\Lambda_{2n}^{(2n)} \otimes \mathcal{A}$ as seen in Proposition 1.2 and the following Proposition, which is parallel to Proposition 2.6. The relation between these two will be given in Proposition 5.2.

PROPOSITION 3.8. *We have*

$$\begin{aligned}
&\Xi^{(r)}(u + r - 1)\tau^{n-r} \\
&\quad = e_1 \dots e_n e_1' \dots e_n' (-)^{\frac{n(n-1)}{2}} r!(n-r)! \sum_{|I|=r} \det(\boldsymbol{A}_I + \natural_r + u).
\end{aligned}$$

By Propositions 3.5 and 3.6, we have now two types of central elements in hand respectively expressed with determinant and Pfaffian. In the next section, we give an explicit relation between them.

4. The Relation Between Pfaffian and Determinant

This section is devoted to our main goal:

THEOREM 4. *The following relation holds between the Pfaffian and the determinant of the alternating matrix $\boldsymbol{A}$ consisting of the generators of* $\mathrm{U}(\mathfrak{o}_{2m})$*:*

$$\begin{aligned}
\mathrm{Pf}(\boldsymbol{A})^2 &= \det(\boldsymbol{A} + \mathrm{diag}(m-1, m-2, \dots, -m)) \\
&= \det(\boldsymbol{A} + \mathrm{diag}(m, m-1, \dots, -m+1)).
\end{aligned}$$

The latter equality between the two determinants in the theorem is seen from Corollary 3.3, so that it suffices to show the first equality. For the proof, we introduce the three elements in $\Lambda_{2n} \otimes \mathrm{U}(\mathfrak{o}_n)$ with $n = 2m$:

$$\Theta = \sum_{1 \leqslant i,j \leqslant n} e_i e_j A_{ij}, \quad \Theta' = \sum_{1 \leqslant i,j \leqslant n} e_i' e_j' A_{ij}, \quad \Xi = \sum_{1 \leqslant i,j \leqslant n} e_i e_j' A_{ij}.$$

Then Theorem 4 is immediately deduced from the following identity:

THEOREM 4*. *We have*

$$(-)^m \frac{1}{2^{2m}(m!)^2}\Theta^m \Theta'^m$$
$$= \frac{1}{n!}(\Xi + (m-1)\tau)(\Xi + (m-2)\tau)\dots(\Xi - m\tau).$$

One might expect an easy proof parallel to what we gave in Section 1.7 for the commutative case. We will, however, soon encounter difficulties in such a simple planning, so that we will be forced to make an appropriate detour. Our proof is divided into several steps accordingly.

First we observe the commutation relations between Θ, Θ' and Ξ:

LEMMA 4.1. *We have the commutation relations*

$$[\Theta, \Theta'] = 4\tau\Xi, \quad [\Theta, \Xi] = 2\tau\Theta, \quad [\Theta', \Xi] = -2\tau\Theta'.$$

Remark. These commutation relations are essentially those of the $\mathfrak{sl}_2$-triplet.

Proof. These can be checked by straightforward calculations. First we see

$$\begin{aligned}
[\Theta, \Theta'] &= \sum_{i,j,k,l} e_i e_j e'_k e'_l [A_{ij}, A_{kl}] \\
&= \sum_{i,j,k,l} e_i e_j e'_k e'_l (A_{il}\delta_{jk} - A_{jl}\delta_{ik} - A_{ik}\delta_{jl} + A_{jk}\delta_{il}) \\
&= \sum_{i,l} e_i e'_l A_{il}\tau + \sum_{j,l} e_j e'_l A_{jl}\tau + \sum_{i,k} e_i e'_k A_{ik}\tau + \sum_{j,k} e_j e'_k A_{jk}\tau \\
&= 4\tau\Xi.
\end{aligned}$$

This proves the first formula. Similar calculation derives the second. By applying the involutive automorphism $'$ induced from the replacement $e_i \leftrightarrow e'_i$ to the second formula, we see the third one, because $\tau' = -\tau$ and $\Xi' = \Xi$. □

COROLLARY 4.2. *Put*

$$\Xi(u) = \Xi + u\tau, \qquad \Xi'(u) = \Xi(u)' = \Xi - u\tau = \Xi(-u).$$

Then we have

$$\Xi(u+2)\Theta = \Theta\Xi(u), \quad \Xi'(u+2)\Theta' = \Theta'\Xi'(u).$$

Proof. These follow from the second and the third formulas in Lemma 4.1. □

The commutation relation between Θ and Θ' in Lemma 4.1 itself cannot simply be rewritten in a nice form as in Corollary 4.2. Our idea to handle these commutation relations is to add a catalyst Ξ in them. To be precise, we introduce

$$\theta(u) = \Theta + \Xi(u) = \Theta + \Xi + u\tau, \quad \theta'(u) = \Theta' + \Xi'(u) = \Theta' + \Xi - u\tau.$$

Then we see the following, which is the key to our proof:

LEMMA 4.3. *We have the commutation relation*

$$\theta(u)\theta'(u+2) = \theta'(u)\theta(u+2).$$

Proof. It suffices to show the relation $\theta(u)\theta'(u) - \theta'(u)\theta(u) = 2\tau\theta(u) + 2\tau\theta'(u)$. This is indeed deduced from Lemma 4.1:

$$\begin{aligned}
[\theta(u), \theta'(u)] &= [\Theta + \Xi(u), \Theta' + \Xi'(u)] \\
&= [\Theta, \Theta'] + [\Theta, \Xi'(u)] + [\Xi(u), \Theta'] \\
&= 4\tau\Xi + 2\tau\Theta + 2\tau\Theta' \\
&= 2\tau(\Theta + \Xi(u) + \Theta' + \Xi(-u)) = 2\tau\theta(u) + 2\tau\theta'(u),
\end{aligned}$$

as desired. $\square$

By virtue of Corollary 4.2, we can make the normal ordering of Θ and Ξ in polynomials of $\theta(u)$ (and respectively of Θ' and Ξ in polynomials of $\theta'(u)$) as follows:

LEMMA 4.4. *The following formulas hold:*

$$\begin{aligned}
&\theta(u)\theta(u-2)\dots\theta(u-2i+2) \\
&\quad = \sum_{p=0}^{i} \binom{i}{p} \Xi(u)\Xi(u-2)\dots\Xi(u-2p+2)\Theta^{i-p}, \\
&\theta'(u)\theta'(u-2)\dots\theta'(u-2j+2) \\
&\quad = \sum_{q=0}^{j} \binom{j}{q} \Xi'(u)\Xi'(u-2)\dots\Xi'(u-2q+2)\Theta'^{j-q}.
\end{aligned}$$

Using the notation $F^{(k)}(u;t) = \prod_{l=0}^{k-1} F(u-tl)$ for the factorial power of $F(u)$ with step t, where $F(u) = \theta(u), \Xi(u)$, etc., we can write these formulas as

$$\theta^{(i)}(u;2) = \sum_{p=0}^{i} \binom{i}{p} \Xi^{(p)}(u;2)\Theta^{i-p}, \quad \theta'^{(j)}(u;2) = \sum_{q=0}^{j} \binom{j}{q} \Xi'^{(q)}(u;2)\Theta'^{j-q}.$$

With these preliminary lemmas in hand, we will make the crucial computation for our Theorem. As in Section 1.7, we take

$$h = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2$$

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and consider the transformation $(h \otimes 1_n)_*(\Theta)$. Then we easily see that

$$(h \otimes 1_n)_*(\Theta) = \Theta + 2\Xi + \Theta'.$$

The power of this is expanded as follows:

LEMMA 4.5. *For $k = 0, 1, 2, \ldots,$ we have*

$$\begin{aligned}(h \otimes 1_n)_*(\Theta^k) = (\Theta + 2\Xi + \Theta')^k &= \sum_{p+q+r=k} \frac{k!}{p!q!r!} 2^r \Xi^{(r)}(p-q)\Theta^p \Theta'^q \\ &= \sum_{p+q+r=k} \frac{k!}{p!q!r!} 2^r \Theta^p \Theta'^q \Xi^{(r)}(q-p).\end{aligned}$$

Here

$$\Xi^{(r)}(u) = \Xi(u)\Xi(u-1)\ldots\Xi(u-r+1).$$

Proof. We write for simplicity $\tilde{\Theta} = \Theta + 2\Xi + \Theta'$. Note first that $\tilde{\Theta} = \theta(u) + \theta'(u)$ for any u, so that we have

$$\begin{aligned}\tilde{\Theta}^k &= \prod_{l=0}^{k-1} (\theta(u-2l) + \theta'(u-2l)) \\ &= (\theta(u-2k+2) + \theta'(u-2k+2)) \ldots (\theta(u-2) + \theta'(u-2))(\theta(u) + \theta'(u)).\end{aligned}$$

We use here first Lemma 4.3 to expand the right-hand side, and then Lemma 4.4 for further expansion in the power of Θ, Θ' and the factorial power of Ξ:

$$\begin{aligned}\tilde{\Theta}^k &= \sum_{j=0}^{k} \binom{k}{j} \theta(u-2k+2)\ldots\theta(u-2j-2)\theta(u-2j)\,\theta'(u-2j+2)\ldots\theta'(u-2)\theta'(u) \\ &= \sum_{i+j=k} \binom{k}{j} \theta^{(i)}(u-2j;2)\,\theta'^{(j)}(u;2) \\ &= \sum_{\substack{i+j=k \\ 0 \leqslant p \leqslant i,\, 0 \leqslant q \leqslant j}} \binom{k}{j}\binom{i}{p}\binom{j}{q} \Xi^{(i-p)}(u-2j;2)\Theta^p\, \Xi'^{(j-q)}(u;2)\Theta'^q \\ &= \sum_{\substack{i+j=k \\ 0 \leqslant p \leqslant i,\, 0 \leqslant q \leqslant j}} \binom{k}{j}\binom{i}{p}\binom{j}{q} \Xi^{(i-p)}(u-2j;2)\,\Xi'^{(j-q)}(u-2p;2)\Theta^p\,\Theta'^q \\ &= \sum_{p+q+\mu+\nu=k} \frac{k!}{p!q!\mu!\nu!} \Xi'^{(\mu)}(-u+2k-2p-2;2)\,\Xi'^{(\nu)}(u-2p;2)\Theta^p\,\Theta'^q \\ &= \sum_{p+q+r=k} \frac{k!}{p!q!r!} \left(\sum_{\mu+\nu=r} \binom{r}{\mu} \Xi'^{(\mu)}(-u+2k-2p-2;2)\,\Xi'^{(\nu)}(u-2p;2) \right) \Theta^p\,\Theta'^q.\end{aligned}$$

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Applying the binomial theorem for the factorial power

$$(x+y)^{(r)} = \sum_{\mu=0}^{r} \binom{r}{\mu} x^{(\mu)} y^{(r-\mu)}$$

to the inner summation of the last line, we see that the sum turns out to be

$$\begin{aligned}\prod_{l=0}^{r-1}(2\Xi - (2k-4p-2-2l)\tau) &= 2^r \prod_{l=0}^{r-1}(\Xi - (k-2p-l-1)\tau)\\ &= 2^r \Xi'^{(r)}(k-2p-1) = 2^r \Xi^{(r)}(p-q).\end{aligned}$$

Thus we have proved our assertion. □

To consider a bit more general case, let us take

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2,$$

so that we have

$$(h \otimes 1_n)_*(\Theta) = a^2\Theta + 2ac\Xi + c^2\Theta'.$$

We put

$$\vartheta(u) = a^2\Theta + ac\Xi(u), \quad \vartheta^*(u) = c^2\Theta' + ac\Xi'(u).$$

Then we see that $(h \otimes 1_n)_*(\Theta) = \vartheta(u) + \vartheta^*(u)$ and that the commutation relation

$$\vartheta(u)\vartheta^*(u+2) = \vartheta^*(u)\vartheta(u+2)$$

holds correspondingly to Lemma 4.3. Similar computation for Lemma 4.5 shows the following:

LEMMA 4.6 For $k = 0, 1, 2, \ldots,$ we have

$$\begin{aligned}(a^2\Theta + 2ac\Xi + c^2\Theta')^k &= \sum_{p+q+r=k} \frac{k!}{p!q!r!} a^{2p+r} c^{2q+r} 2^r \Xi^{(r)}(p-q)\Theta^p\,\Theta'^q\\ &= \sum_{p+q+r=k} \frac{k!}{p!q!r!} a^{2p+r} c^{2q+r} 2^r \Theta^p\,\Theta'^q\,\Xi^{(r)}(q-p).\end{aligned}$$

Since the proof is almost the same as Lemma 4.5, we omit it.

Remark. As we noted above, the commutation relations for the three elements Θ, Θ', Ξ are essentially the same as those of the $\mathfrak{sl}_2$-triplet. To be precise, we consider $X, Y, H \in \mathfrak{sl}_2$ subject to the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Then the correspondence is given by

$$X \leftrightarrow -\Theta'/2, \quad Y \leftrightarrow \Theta/2, \quad H \leftrightarrow \Xi.$$

In the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$, Lemma 4.6 is accordingly interpreted as the trinomial expansion

$$(-\lambda^2 X + \lambda\mu H + \mu^2 Y)^k = \sum_{p+q+r=k} \frac{k!}{p!q!r!} \lambda^{2p+r} \mu^{2q+r} Y^p (-X)^q H^{(r)}(q-p),$$

where $H^{(r)}(u) = (H+u)(H+u-1)\dots(H+u-r+1)$. Note here the element $-\lambda^2 X + \lambda\mu H + \mu^2 Y$ in the left-hand side is in the orbit $\mathrm{Ad}(\mathrm{SL}_2)X$ through X under the adjoint action of SL_2, so that it is nilpotent.

Having done the crucial computations, we are now at the final stage of the proof of our Theorem. Notice the obvious relations that $(h \otimes 1_n)_*(\Theta^k) = 0$ for $k = m+1, \dots, n$. By Lemma 4.5 or more generally by Lemma 4.6, these $(h \otimes 1_n)_*(\Theta^k)$ are expressed as the sum of the monomials in Θ, Θ', Ξ arranged in normal order. What we should do for our Theorem is, by making suitable linear combinations, to remove the catalyst from these m relations.

Looking at the coefficient of $a^k c^k$ in the relation $(h \otimes 1_n)_*(\Theta^k) = 0$, we have for $k = m+1, \dots, n$

$$\sum_{0 \leqslant p \leqslant k/2} \frac{1}{p!p!(k-2p)!} \Theta^p \, \Theta'^p \, 2^{-2p} \Xi^{(k-2p)}(0) = 0.$$

In general, introducing an integral parameter s, we define

$$Q_k(s) = \sum_{0 \leqslant p \leqslant k/2} \frac{1}{p!p!(k-2p)!} \Theta^p \, \Theta'^p \, 2^{s-2p} (p-1)^{(s)} \, \Xi^{(k-2p)}(s),$$

where $(p-1)^{(s)} = (p-1)(p-2)\dots(p-s)$. Then we see that $Q_k(0) = 0$ for $k = m+1, \dots, n$, and that

$$\frac{1}{(m-1)!} Q_{2m}(m-1) = \frac{(-)^{m-1}}{(2m)!} 2^{m-1} \Xi^{(2m)}(m-1) + \frac{1}{m!m!} 2^{-m-1} \Theta^m \, \Theta'^m,$$

because the factor $(p-1)^{(m-1)} = (p-1)(p-2)\dots(p-m+1)$ in the summation kills all the terms except the both ends $p = 0$ and $p = m$. Thus Theorem 4 is proved when we show the equality $Q_{2m}(m-1) = 0$, because this is equivalent to Theorem 4*.

LEMMA 4.7. *For $s = 0, 1, 2, \dots, m-1$, we have recursive relations*

$$Q_k(s+1) = (k-2s-2)Q_k(s) - Q_{k-1}(s) \cdot (\Xi + (-k+3s+3)\tau).$$

Proof. For simplicity we put $W_p = \Theta^p \Theta'^p 2^{-2p}/(p!)^2$. We have then by definition

$$Q_k(s) = 2^s \sum_{0 \leqslant p \leqslant k/2} W_p \cdot \frac{\Xi^{(k-2p)}(s)}{(k-2p)!} \cdot (p-1)^{(s)}.$$

Let us introduce an auxiliary element

$$Q_k^\dagger(s) = 2^s \sum_{0 \leqslant p \leqslant k/2} W_p \cdot \frac{\Xi^{(k-2p)}(s+1)}{(k-2p)!} \cdot (p-1)^{(s)}.$$

Then as the difference for the factorial powers, we see

$$\frac{\Xi^{(k-2p)}(s+1) - \Xi^{(k-2p)}(s)}{(k-2p)!} = \frac{\Xi^{(k-1-2p)}(s)}{(k-1-2p)!}\tau.$$

This leads on the one hand to $Q_k^\dagger(s) = Q_k(s) + Q_{k-1}(s)\tau$. On the other hand, from the obvious relation $\Xi(s+1) \cdot \Xi^{(k-1-2p)}(s) = \Xi^{(k-2p)}(s+1)$, we have

$$\begin{aligned} Q_{k-1}(s) \cdot (\Xi + (s+1)\tau) &= 2^s \sum_{0 \leqslant p \leqslant (k-1)/2} W_p \cdot \frac{\Xi^{(k-2p)}(s+1)}{(k-1-2p)!} \cdot (p-1)^{(s)} \\ &= 2^s \sum_{0 \leqslant p \leqslant (k-1)/2} W_p \cdot \frac{\Xi^{(k-2p)}(s+1)}{(k-2p)!} \cdot (k-2p)(p-1)^{(s)}. \end{aligned}$$

Note here $k - 2p = (k-2s-2) - 2(p-s-1)$, so that

$$(k-2p)(p-1)^{(s)} = (k-2s-2)(p-1)^{(s)} - 2(p-1)^{(s+1)}.$$

Then we obtain the relation $Q_{k-1}(s) \cdot (\Xi + (s+1)\tau) = (k-2s-2)Q_k^\dagger(s) - Q_k(s+1)$. Eliminating the auxiliary element $Q_k^\dagger(s)$ from this and $Q_k^\dagger(s) = Q_k(s) + Q_{k-1}(s)\tau$ above, we see our recursive relation in question. □

Proof of Theorem 4. Start from the relations $Q_k(0) = 0$ for $k = m+1, \dots, 2m$. By Lemma 4.7, we have $Q_k(1) = 0$ for $k = m+2, \dots, 2m$. We can repeat this process till we come to $Q_{2m}(m-1) = 0$, which proves our theorem as we saw in the above. □

Remark. It is easy to check that an element of the form $\Theta^p \Theta'^q \varphi$, where φ is a homogeneous polynomial in Ξ and τ of degree r, is zero unless both of the conditions $2p + r \leqslant n$ and $2q + r \leqslant n$ are satisfied. In particular, if $p + q + r = n$, these conditions imply $p = q \leqslant n$. This observation together with Lemma 4.5 will be enough to obtain our conclusion of Theorem 4, though we have used Lemma 4.6 for a clearer reasoning. In fact, through the same process of making $Q_k(s)$ in Lemma 4.7 starting only from the right-hand side of Lemma 4.5, we see that eventually only the terms satisfying the conditions described above survive, so that we get the same result.

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5. Supplementary Formulas

We give here some useful formulas derived from our main result. In this section, except for Proposition 5.6, we assume that n is not necessarily even but arbitrary natural number.

First, replacing $\boldsymbol{A}$ by the submatrices $\boldsymbol{A}_I$ in Theorem 4 and summing up over $|I| = 2k$, we see the following equality in the center of $U(\mathfrak{o}_n)$.

PROPOSITION 5.1. *We have*

$$\sum_{|I|=2k} \mathrm{Pf}(\boldsymbol{A}_I)^2 = \sum_{|I|=2k} \det(\boldsymbol{A}_I + \mathrm{diag}(k, \ldots, -k+1)).$$

The fact that the both sides of Proposition 5.1 belong to the center of $U(\mathfrak{o}_n)$ is seen from Proposition 3.8 and the following formula in $\Lambda_{2n} \otimes U(\mathfrak{o}_n)$, which is similar to Theorem 4*.

PROPOSITION 5.1*. *The following equality holds in $\Lambda_{2n} \otimes U(\mathfrak{o}_n)$:*

$$(-)^k \frac{1}{2^{2k}(k!)^2} \Theta^k \Theta'^k \tau^{n-2k} = \frac{1}{(2k)!} \Xi^{(2k)}(k-1)\tau^{n-2k} = \frac{1}{(2k)!} \Xi^{(2k)}(k)\tau^{n-2k}.$$

This can be deduced by applying Theorem 4* to the submatrices $\boldsymbol{A}_I$, because we have the relations

$$\Theta^k \Theta'^k \tau^{n-2k} = \sum_{|I|=2k} \Theta_I^k \Theta_I'^k \tau_{I^c}^{n-2k}, \quad \Xi^{(2k)}(k)\tau^{n-2k} = \sum_{|I|=2k} \Xi_I^{(2k)}(k)\tau_{I^c}^{n-2k}$$

with

$$\Theta_I = \sum_{i,j \in I} e_i e_j A_{ij}, \quad \Theta'_I = \sum_{i,j \in I} e'_i e'_j A_{ij}, \quad \Xi_I(u) = \sum_{i,j \in I} e'_i e'_j A_{ij}(u), \quad \tau_{I^c}^{n-2k} = \sum_{i \notin I} e_i e'_i.$$

We denote the quantity described in Proposition 5.1 by $C_{2k}^{\mathfrak{o}_n}$. This is related with the sum of double-determinants $J_{2r}^{\mathfrak{o}_n} = \sum_{|I|=2r} \mathrm{Det}(\boldsymbol{A}_I)$ as follows (cf. Proposition 2.7):

PROPOSITION 5.2. *We have the following relation between the central elements in $U(\mathfrak{o}_n)$:*

$$C_{2r}^{\mathfrak{o}_n} = \sum_{k=0}^{r} c_k^r \frac{(2k)!(n-2k)!}{(2r)!(n-2r)!} J_{2k}^{\mathfrak{o}_n},$$

where the coefficient c_k^r is defined by the expansion

$$(\lambda^2 - (r-1)^2)(\lambda^2 - (r-2)^2) \ldots (\lambda^2 - 1)\lambda^2 = \sum_{k=0}^{r} c_k^r \lambda^{2k}.$$

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Proof. We have

$$\begin{aligned}&\Xi(r)\Xi(r-1)\ldots\Xi(-r+1)\tau^{n-2r}\\&\quad=\Xi(r-1)\ldots\Xi(0)\Xi(0)\ldots\Xi(-r+1)\tau^{n-2r}=\sum_{k=0}^{r}c_k^r\Xi^{2k}\tau^{n-2k}\end{aligned}$$

by the equality $\Xi(r-1)\ldots\Xi(-r+1)\tau^{n-2r+1}=0$, which follows from Proposition 3.4. Our assertion is proved from this and the expressions of double-determinant and column-determinant in Propositions 1.2 and 3.8. □

Remark. The coefficient c_k^r in Proposition 5.2 can be expressed with the Stirling number S_l^r of the second kind as

$$c_k^r=\sum_{l=0}^{2k}(-)^{r+l}S_{2k-l}^rS_l^r.$$

The central elements $C_{2k}^{\mathfrak{o}_n}$, as expressed by determinant in the right-hand side of Proposition 5.1, appear as the coefficients in the central element $\det(\boldsymbol{A}+\natural+u)$ with the parameter u. The explicit form is given as follows.

PROPOSITION 5.3. *We have the following expansion:*

$$\det\Big(\boldsymbol{A}+\mathrm{diag}\Big(\frac{n}{2},\frac{n}{2}-1,\ldots,-\frac{n}{2}+1\Big)+u\Big)=\sum_{0\leqslant k\leqslant n/2}C_{2k}^{\mathfrak{o}_n}\cdot\Big(u+\frac{n}{2}-k\Big)^{(n-2k)}.$$

This proof is readily seen from Proposition 3.8 and the following Lemma.

LEMMA 5.4. *We have*

$$\Xi^{(n)}\Big(\frac{n}{2}+u\Big)=\sum_{0\leqslant k\leqslant n/2}\binom{n}{2k}\Xi^{(2k)}(k)\tau^{n-2k}\Big(u+\frac{n}{2}-k\Big)^{(n-2k)}.$$

Proof. We note that the binomial expansion holds:

$$\Xi^{(n)}\Big(\frac{n}{2}+u\Big)=\begin{cases}\displaystyle\sum_{i=0}^{n}\binom{n}{i}\Xi^{(i)}\Big(\Big[\frac{i}{2}\Big]\Big)\tau^{n-i}\Big(u+\Big[\frac{n-i}{2}\Big]\Big)^{(n-i)}, & n\text{: even},\\ \displaystyle\sum_{i=0}^{n}\binom{n}{i}\Xi^{(i)}\Big(\Big[\frac{i}{2}\Big]\Big)\tau^{n-i}\Big(u+\frac{1}{2}+\Big[\frac{n-i-1}{2}\Big]\Big)^{(n-i)}, & n\text{: odd}.\end{cases}$$

Here $[x]$ indicates the greatest integer less than or equal to x. The expansion is proved as a usual polynomial identity by induction on n, as Ξ and τ are commutative. Our assertion follows from this, because we have

$$\Xi^{(2k+1)}(k)\tau^{n-2k-1}=\Xi(k)\ldots\Xi(-k)\tau^{n-2k-1}=0$$

by Proposition 3.4. □

In Lemma 5.4, the terms containing $\Xi^{(2k)}(k)$ in the right-hand side can be rewritten in Θ and Θ' by Proposition 5.1*. Also, the Ξ in the left-hand side of Lemma 5.4 can be transformed into $\Theta' - \Theta$ in the same way as in Section 1.7, so that we have $\Xi^{(n)}(n/2+u) = 2^{-n}\prod_{i=0}^{n-1}(\Theta' - \Theta + (n+2u-2i)\tau)$. Thus, Lemma 5.4 is interpreted as the following expansion for the power of $\Theta' - \Theta$:

PROPOSITION 5.5. *We have*

$$\begin{aligned}&\prod_{i=0}^{n-1}(\Theta' - \Theta + (n+2u-2i)\tau)\\ &\quad = \sum_{0 \leqslant k \leqslant n/2} (-)^k 2^{n-2k}\frac{n!}{k!k!(n-2k)!}\Theta^k\Theta'^k\tau^{n-2k}\left(u+\frac{n}{2}-k\right)^{(n-2k)}.\end{aligned}$$

This is a generalized form of the direct counterpart of the binomial expansion that we used in Section 1.7 for the proof of commutative version of our main theorem. If we could have proved this more directly, the analogy would be much clearer in the commutative case and our $\mathfrak{o}_n$ case. But it does not seem so easy.

We close this section by making a remark on the expressions of the right-hand side of Theorem 4 as follows.

PROPOSITION 5.6. *The following determinants with different shifts are equal to one another:*

$$\det(\boldsymbol{A} + \mathrm{diag}(m-1,\dots,m-k,m-k,\dots,-m+1)),\quad k=0,\dots,2m.$$

Here the dots indicate a sequence of numbers descending by 1.

Proof. It suffices to show

$$\begin{aligned}&\det(\boldsymbol{A} + \mathrm{diag}(m-1,\dots,m-k,m-k,\dots,-m+1))\\ &\quad = \det(\boldsymbol{A} + \mathrm{diag}(m-1,\dots,m-k+1,m-k+1,\dots,-m+1))\end{aligned}$$

for $k = 1,\dots,2m$. For this, observe the equality

$$\begin{aligned}&\psi_1(m-1)\dots\psi_{k-1}(m-k+1)\psi_k(m-k)\psi_{k+1}(m-k)\dots\psi_m(-m+1)\\ &\quad = \psi_1(m-1)\dots\psi_{k-1}(m-k+1)\psi_k(m-k+1)\psi_{k+1}(m-k)\dots\psi_m(-m+1) -\\ &\quad - \psi_1(m-1)\dots\psi_{k-1}(m-k+1)e_k\psi_{k+1}(m-k)\dots\psi_m(-m+1).\end{aligned}$$

Here the latter term in the right-hand side vanishes, because

$$\det(\boldsymbol{A}_I + \mathrm{diag}(m-1,m-2,\dots,-m+1)) = 0$$

for $I = \{1,2,\dots,2m\}\setminus\{k\}$ as seen in Corollary 3.3. Then the resulting equality proves our assertion. $\square$

6. Relation to Other Realizations of the Orthogonal Lie Algebra

So far we have worked within a specially fixed realization of the orthogonal Lie algebra $\mathfrak{o}_n$. However, since the expressions of the Pfaffian and the double-determinant by means of exterior algebra enjoy a certain stability as we saw in Section 1.5, we may transfer our consideration still to other forms.

We take an $n \times n$ nondegenerate symmetric matrix S in $\mathrm{Mat}_n(\mathbb{K})$, and consider the orthogonal Lie algebra $\mathfrak{o}(S)$ with respect to the quadratic form defined by S:

$$\mathfrak{o}(S) = \{X \in \mathfrak{gl}_n\,;\ {}^tXS + SX = 0\}.$$

Via the natural embeddings of $\mathfrak{o}(S)$ and $\mathfrak{o}_n$ in $\mathfrak{gl}_n$, we regard their universal enveloping algebras $\mathrm{U}(\mathfrak{o}(S))$ and $\mathrm{U}(\mathfrak{o}_n)$ as the subalgebras of $\mathrm{U}(\mathfrak{gl}_n)$. We introduce an involution $i_S : X \mapsto S^{-1}\,{}^tXS$ of $\mathfrak{gl}_n$. Then we see $X - i_S(X) \in \mathfrak{o}(S)$ for any $X \in \mathfrak{gl}_n$, and $\mathfrak{o}(S)$ is generated by those elements of the form $F_{ij} = E_{ij} - i_S(E_{ij})$. Arrange the matrix $\boldsymbol{F} = (F_{ij})_{i,j=1}^n$ from these generators. This matrix is expressed as

$$\boldsymbol{F} = \boldsymbol{E} - \mathrm{Ad}(S^{-1})\,{}^t\boldsymbol{E} = \boldsymbol{E} - S\,{}^t\boldsymbol{E}S^{-1}$$

by Lemma 2.5. We see that $\boldsymbol{F}S$ and $S^{-1}\boldsymbol{F}$ are alternating from this expression.

Theorem 4 is rewritten for $\mathfrak{o}(S)$ as follows:

THEOREM 6. *We have*

$$\begin{aligned}
&\mathrm{Pf}(\boldsymbol{F}S)^2\det(S)^{-1}\\
&\quad= \mathrm{Pf}(S^{-1}\boldsymbol{F})^2\det(S)\\
&\quad= \frac{1}{n!}\sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n} \mathrm{sign}(\sigma)\mathrm{sign}(\sigma')F_{\sigma(1)\sigma'(1)}(m)\ldots F_{\sigma(n)\sigma'(n)}(-m+1).
\end{aligned}$$

Proof. Let s be an element in $\mathrm{Mat}_n(\mathbb{K})$ such that $S = {}^tss$ (we may always take such an s by extending the ground field $\mathbb{K}$). Then a natural isomorphism $\mathfrak{o}_n \simeq \mathfrak{o}(S)$ is given as the restriction of the automorphism $\mathrm{Ad}(s^{-1})$: $X \mapsto s^{-1}Xs$ of $\mathfrak{gl}_n$. We remark that the image of our matrix $\boldsymbol{A}$ under this isomorphism is seen to be $\mathrm{Ad}(s^{-1})\boldsymbol{A} = {}^ts^{-1}\boldsymbol{F}\,{}^ts$ from Lemma 2.5. This leads to the equality $\mathrm{Ad}(s^{-1})\boldsymbol{A} = {}^ts^{-1}\boldsymbol{F}Ss^{-1} = sS^{-1}\boldsymbol{F}\,{}^ts$, so that the following relations between Pfaffians hold:

$$\mathrm{Pf}(\mathrm{Ad}(s^{-1})\boldsymbol{A}) = \det(s)^{-1}\mathrm{Pf}(\boldsymbol{F}S) = \det(s)\mathrm{Pf}(S^{-1}\boldsymbol{F}).$$

Since $\mathrm{Ad}(s^{-1})\boldsymbol{A} = {}^ts^{-1}\boldsymbol{F}\,{}^ts$, we have $\Xi^{(2m)}_{\mathrm{Ad}(s^{-1})\boldsymbol{A}}(u) = \Xi^{(2m)}_{\boldsymbol{F}}(u)$ by Proposition 1.5. Hence, transferring Theorem 4 under the isomorphism $\mathrm{Ad}(s^{-1})$: $\mathrm{U}(\mathfrak{o}_n) \simeq \mathrm{U}(\mathfrak{o}(S))$, we obtain the following equality:

$$(-)^m\frac{1}{2^{2m}}\binom{n}{m}\Theta^m_{\mathrm{Ad}(s^{-1})\boldsymbol{A}}\Theta'^m_{\mathrm{Ad}(s^{-1})\boldsymbol{A}} = \Xi^{(2m)}_{\mathrm{Ad}(s^{-1})\boldsymbol{A}}(m) = \Xi^{(2m)}_{\boldsymbol{F}}(m).$$

Comparing the coefficients in both sides, we see

$$\mathrm{Pf}(\mathrm{Ad}(s^{-1})\boldsymbol{A})^2 = \frac{1}{n!} \sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n} \mathrm{sign}(\sigma)\mathrm{sign}(\sigma') F_{\sigma(1)\sigma'(1)}(m)\ldots F_{\sigma(n)\sigma'(n)}(-m+1).$$

Together with the relations of $\mathrm{Pf}(\mathrm{Ad}(s^{-1})\boldsymbol{A})$ to $\mathrm{Pf}(\boldsymbol{F}S)$ and $\mathrm{Pf}(S^{-1}\boldsymbol{F})$ given above, this clearly proves our assertion. □

Remark. We see that the right-hand side of the equality in Theorem 6 is equal to

$$\sum_{l=0}^{m} c_l^m \binom{n}{2l}^{-1} \sum_{|I|=2l} \mathrm{Det}(\boldsymbol{F}_I)$$

as in Proposition 5.2.

In the case $S = 1_n$, Theorem 6 is reduced to the expression by column-determinant as seen in Section 4. However, such a reduction to an alternating sum of $n!$ terms seems rather hard and complicated for general S. In the case of $S = (\delta_{i,n+1-j})_{i,j=1}^{n}$, A. Molev [M] described the square of $\mathrm{Pf}(S^{-1}\boldsymbol{F})$ with Sklyanin determinant in twisted Yangian. This suggests that our double-determinant in Theorem 6 can be reduced to certain 'single' determinant at least this case. Further investigation on this theme will be appearing elsewhere (see [I1]).

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