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VARIETIES OF LEFT RESTRICTION SEMIGROUPS

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Abstract

Left restriction semigroups are the unary semigroups that abstractly characterize semigroups of partial maps on a set, where the unary operation associates to a map the identity element on its domain. They may be defined by a simple set of identities and the author initiated a study of the lattice of varieties of such semigroups, in parallel with the study of the lattice of varieties of two-sided restriction semigroups. In this work we study the subvariety **B** generated by Brandt semigroups and the subvarieties generated by the five-element Brandt inverse semigroup B_2, its four-element restriction subsemigroup B_0 and its three-element left restriction subsemigroup D. These have already been studied in the 'plain' semigroup context, in the inverse semigroup context (in the first two instances) and in the two-sided restriction semigroup context (in all but the last instance). The author has previously shown that in the last of these contexts, the behavior is pathological: 'almost all' finite restriction semigroups are inherently nonfinitely based. Here we show that this is not the case for left restriction semigroups, by exhibiting identities for the above varieties and for their joins with monoids (the analog of groups in this context). We do so by structural means involving subdirect decompositions into certain primitive semigroups. We also show that each identity has a simple structural interpretation.

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1. Introduction

The left restriction semigroups have received considerable attention in recent years, arising in several different ways and within several historical contexts. Of particular interest is that they abstractly characterize the semigroups of partial mappings of a set, under the unary operation $\alpha \mapsto \alpha^+$ that associates with such a map the identity map on its domain. Regarded as unary semigroups, they form the variety **LR**, which from a different perspective may be viewed as that generated from inverse semigroups $(S, \cdot, ^{-1})$ by forgetting the inverse operation and retaining only the unary operation $x \mapsto x^+ = xx^{-1}$. One set of defining identities is:

$$x^+x = x, \quad (x^+y)^+ = x^+y^+, \quad x^+y^+ = y^+x^+, \quad xy^+ = (xy)^+x.$$

Traditionally, studying classes of semigroups (and algebras in general) through their varieties, in particular through the identities that they satisfy, has proven fruitful. For example, there is an extensive literature on varieties of inverse semigroups and varieties of completely regular semigroups, in each case regarded as unary semigroups.

In [6] the author initiated the study of the lattice of varieties of left restriction semigroups, in parallel with a study of the lattice of varieties of (two-sided and thus biunary) restriction semigroups. In [7], we continued this study in the two-sided case by investigating the variety generated by the Brandt semigroups, a natural step 'up' the lattice and, in addition, relevant to recent intense study of the varieties of 'plain' semigroups generated by completely 0-simple semigroups. The present paper is the 'one-sided' sequel to [6] and a parallel to [7].

As always, the Brandt semigroup of primary interest is $B_2 = \{a, b, ab, ba, 0\}$. Given that it is an inverse semigroup, it may also be regarded both as a restriction semigroup and as a left restriction semigroup. Its subsemigroup $B_0 = \{a, b, ab, 0\}$ is naturally a restriction semigroup and therefore also a left restriction semigroup. The lattice of varieties of *inverse* semigroups generated by B_2, and by Brandt semigroups in general, has long been known to be simply described. (Regarded as 'plain' semigroups, Lee [11] has determined the entire, countably infinite, lattice of subvarieties of the variety generated by B_2.)

The corresponding lattice of varieties of *restriction* semigroups is remarkably complicated. In [7] the author exhibited a necessarily infinite basis of identities for B_2, regarded in this fashion. In fact, it is B_0 that is central to the pathology: it is inherently nonfinitely based and, as a result, the same is true for every finite restriction semigroup in which the left and right unary operations are distinct.

In the context of left restriction semigroups, a general argument in a private communication from M. Jackson (see Section 9 for the terminology and the precise theorem and proof) shows that B_0 and B_2 cannot be inherently nonfinitely based. Quite to the contrary, B_2 is defined by a single identity and B_0 by one further identity. This is shown not by syntactic arguments but by structural ones, based on a subdirect decomposition of semigroups in the variety **B** of left restriction semigroups generated by all Brandt semigroups. In this context, there is a subsemigroup of B_0 of relevance: $D = \{a, ab, 0\}$. This semigroup is left restriction but not restriction and was shown in [6] to play a fundamental role in the lattice of varieties of left restriction semigroups: if such a variety does not consist of unions of monoids, it must contain D.

Bases of identities for the principal varieties of interest are as follows. Here $\mathbf{B}_2$, $\mathbf{B}_0$ and **D** denote the varieties generated by B_2, B_0 and D, respectively. Since the identities themselves are rather unilluminating, we also provide a simple paraphrase of each, in terms of the 'generalized' Green's relation $\mathbb{R} = \{(a, b) : a^+ = b^+\}$ and right identities. Here a *right identity* refers strictly to a projection $e = e^+$ with that property (and elements may or may not possess such right identities). A further characterization of each variety is in terms of subdirect decompositions into 'primitive' left restriction semigroups with a designated 'base', as described in the cited theorems.

THEOREM 1.1 (Theorems 6.6 and 8.11, Corollaries 8.12 and 7.7). *Let S be a left restriction semigroup. Then*

- $S \in \mathbf{B}$ *if and only if satisfies identity (6.1): $(xz)^+(yz^+w)^+ = (yz)^+(xz^+w)^+$; that is, if two $\mathbb{R}$-related elements share a right identity, they share all right identities;*
- $S \in \mathbf{B}_2$ *if and only if it satisfies identity (8.1): $(xz)^+(yz) = (yz)^+(xz)$; that is, no two distinct $\mathbb{R}$-related elements share a right identity;*
- $S \in \mathbf{B}_0$ *if and only if it satisfies the conjunction of (8.1) and identity (8.3): $xyx = (xyx)^+$; that is, $S \in \mathbf{B}_2$ and every regular element is a projection;*
- $S \in \mathbf{D}$ *if and only if S satisfies identity (7.2): $xz^+ = (xz^+)^+$; that is, the only elements with right identities are the projections $e = e^+$.*

In addition to the varieties described in this theorem, in Theorems 8.13 and 7.5 we provide analogous information for the varieties $\mathbf{B}_0 \vee \mathbf{M}$ and $\mathbf{D} \vee \mathbf{M}$, where $\mathbf{M}$ denotes the variety of monoids (regarded as left restriction semigroups), which play the role for restriction semigroups that groups play for inverse semigroups. The join $\mathbf{B}_2 \vee \mathbf{M}$ is just $\mathbf{B}$ itself.

The key tool is Corollary 6.5, in which it is shown that a left restriction semigroup satisfying (6.1) is a subdirect product of monoids and primitive left restriction semigroups, that is, left restriction semigroups with zero in which every projection is 0-minimal. As alluded to above, the primitive factors can be chosen to have a designated 'base', which makes them more amenable to treatment. This decomposition is the one-sided analog of the decomposition of strict restriction semigroups into monoids and primitive restriction semigroups [7].

A second key tool involves various embeddings of these special primitive left restriction semigroups in restriction semigroups, enabling application of the work on the latter cited above. In a sequel [9], we will study the *lattice* of varieties of strict left restriction semigroups by carrying these techniques further, melding them with those of [8]. The latter work studied the lattice of varieties of (two-sided) strict restriction semigroups through an intimate connection with the lattice of varieties of categories, as developed by Tilson [13]. We will show, rather remarkably in the author's view, that these two lattices are 'almost' isomorphic, in a manner made explicit there.

The plan of the paper is as follows. We first review the basics of left and two-sided restriction semigroups and their varieties, including introduction of a key analytical tool involving right identities. In Section 5 we analyze primitive left restriction semigroups and introduce the special class with designated base. Section 6 provides the subdirect decomposition mentioned above, under identity (6.1), and shows that this identity defines $\mathbf{B}$. The special case of $\mathbf{D}$ and $\mathbf{D} \vee \mathbf{M}$ is treated in the following section. The varieties $\mathbf{B}_2$ and $\mathbf{B}_0$ are treated in Section 8 along with their joins with $\mathbf{M}$. These require exceptional treatment, involving a covering theorem and a refinement of the embedding result of Section 5. We leave to Section 9 the argument of Jackson cited above regarding finite basability, since it is does not directly impinge on the rest of the paper.

2. Preliminaries

We briefly review the background on left restriction semigroups themselves and introduce and study some new concepts. For general background on semigroups, see [5]. Since we shall in various places apply results on (two-sided) restriction semigroups, we shall also introduce those here, but leave more specialized background on that topic to other sections. Useful introductions to both left and two-sided restriction semigroups, including different ways in which they have arisen as topics of interest, are [4] and [2]. Our definition of left restriction semigroups is based on the set of identities stated in [2]. In the so-called 'York school', they were originally called weakly [left] E-ample semigroups. We might also note that some of our work on primitive left restriction semigroups could be rephrased in terms of the 'inductive constellations' of the two cited authors [3], but there seems little benefit to the reader in adding another level of abstraction to this work.

From the first two defining identities for left restriction semigroups in Section 1, it follows that if S is such a semigroup, then for all $x \in S$, x^+ is idempotent and, in conjunction with the second identity, $(x^+)^+ = x^+$. These idempotents (there may be others) are the *projections* of S. The set P_S of projections forms a semilattice, ordered in the usual fashion, by virtue of the third identity. The last identity (or a variation of it) is often termed the 'left ample' identity. The following lemma is well known and will find frequent use.

Lemma 2.1. *If S is a left restriction semigroup and $x, y \in S$, then $x^+ \geq (xy)^+$ and $(xy)^+ = (xy^+)^+$. Thus an equivalent form of the left ample identity is: if $e \in P_S$, then $xe = (xe)^+x$.*

A *restriction* semigroup is a biunary semigroup $(S, \cdot, ^+, ^*)$ that is a left restriction semigroup with respect to $^+$, satisfies the 'dual' identities obtained by replacing $^+$ by * and reversing the order of each expression, and further satisfies $(x^+)^* = x^+$ and $(x^*)^+ = x^*$. Thus $P_S = \{x^+ : x \in S\} = \{x^* : x \in S\}$. Every restriction semigroup may be regarded as a left restriction semigroup, by 'forgetting' the second unary operation.

In the context of this work, an inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$ and 'forgetting' the inverse operation. It may also be regarded as a left restriction semigroup by admitting only the former operation. In either case, P_S is just the semilattice of idempotents.

The term *primitive* refers to any (left or two-sided) restriction semigroup with zero in which each nonzero projection is minimal with respect to that property.

For the purposes of this paper, the relevant generalized Green's relations may be defined as follows. In a left restriction semigroup, $\mathbb{R} = \{(a, b) : a^+ = b^+\}$. In a restriction semigroup, the relation $\mathbb{L} = \{(a, b) : a^* = b^*\}$ is defined dually; then $\mathbb{H} = \mathbb{R} \cap \mathbb{L}$ and $\mathbb{D} = \mathbb{R} \vee \mathbb{L}$ (not in general equal to $\mathbb{R} \circ \mathbb{L}$). We have reverted to the notation of [6, 7] after having misguidedly changed notation in [8]. The natural partial order on a left restriction semigroup S is defined by $a \leq b$ if $a = eb$ for some $e \in P_S$ and, by application of the left ample identity, is easily seen to be compatible with the operations. On a restriction semigroup, this partial order is self-dual (and so compatible with both unary operations).

In general, the terms 'homomorphism', 'congruence' and 'divides' will be used appropriate to context; that is, they should respect the unary operation for left restriction semigroups and both unary operations for restriction semigroups. For instance, by Lemma 2.1, $\mathbb{R}$ is a left congruence, in either case. In the case of subsemigroups, we shall generally use the modifier 'left restriction' (or 'unary') or 'restriction' to clarify. If S is a left restriction semigroup and T is a restriction semigroup, when we say that T is a left restriction subsemigroup of S, the second operation on T is forgotten. Note that in this situation, the $\mathbb{R}$-relations coincide and P_T is a subsemilattice of P_S (because the projections are determined by either unary operation).

One congruence of note on left restriction semigroups is the greatest projection-separating congruence, denoted μ. Equivalently, μ is the greatest congruence contained in $\mathbb{R}$. Elements a and b are μ-related if and only if $(ae)^+ = (be)^+$ for all $e \in P_S$.

In the standard terminology, restriction semigroups S with $|P_S| = 1$ are termed *reduced*. Since, in essence, they are just monoids, regarded as restriction semigroups by setting $a^+ = a^* = 1$ for all a, we will generally omit the qualifier 'reduced', except in case of possible ambiguity.

A (unary) subsemigroup T of a left restriction semigroup S is a *submonoid* if it contains a unique projection e. By [6, Lemma 4.6], the maximal submonoids of S have the form $M_e = \{a \in \mathbb{R}_e : a = ae\}$. We present the following lemma to illustrate the subtleties entailed and application of the identities used in the sequel.

Lemma 2.2. *Let $S = (S, \cdot, ^+, ^*)$ be a restriction semigroup. Its maximal submonoids are those of its left restriction reduct $(S, \cdot, ^+)$.*

Proof. Recall that the projections are the same, however S is regarded. Let $e \in P_S$. It suffices to show that if $a \in M_e$, as defined above, then $a^* \in M_e$, for then M_e is a biunary submonoid of S and thus the submonoid $\mathbb{H}_e$. Now $(a^*)^+ = (a^+)^* = e^* = e$, so $a^* \in \mathbb{R}_e$; further, $a^* = (ae)^* = (ae)^* e^* = a^* e$ (using the duals of the identities for left restriction semigroups). That is, $a^* \in M_e$. □

A $^+$-*ideal* I of a left restriction semigroup S is an ideal of S that is also a left restriction subsemigroup. It is easily seen that the Rees quotient semigroup S/I is again a left restriction semigroup. As usual, for technical reasons it is convenient to allow the empty set to be an ideal and, in that case, to put $S/I = S$. A *restriction ideal* (r-ideal in [7, 8]) of a restriction semigroup S is an ideal that is closed under both unary operations.

The following definitions are key tools of this work. Throughout it will be implicit that the term 'right identity' always refers to a projection, rather than an idempotent in general.

In a (two-sided) restriction semigroup, every element a has a least right identity, namely a^*. This is of course no longer true in the left restriction case and it is convenient to introduce notation that concisely distinguishes the existence of right identities. Let S be such a semigroup. Denote by S^{RI} the set of elements of S that *have* a right identity, and by S^{NRI} the set of elements of S that do *not* have a right identity. The following, though obvious, is worth stating.

Lemma 2.3. *For any left restriction semigroup S, S^{RI} is a $^+$-subsemigroup of S that is also a left ideal of S.*

Again, let S be any left restriction semigroup and $e \in P_S$. Put $\mathrm{RF}(e) = \{s \in S : as^+ = a$ for some $a \in \mathbb{R}_e\}$. That is, $\mathrm{RF}(e)$ is the union of the $\mathbb{R}$-classes of those projections of S that are right identities for some element of R_e. Observe that for any $a \in \mathbb{R}_e$ and $f \in P_S$, f is a right identity for a if and only if $af \in \mathbb{R}_e$, since $af = (af)^+a$, the left 'ample' identity.

Note that the submonoid M_e consists of those members of $\mathbb{R}_e$ having e as right identity. The special case whereby no other member of $\mathbb{R}_e$ has a right identity will be of interest in the sequel. In that case, clearly $\mathrm{RF}(e)$ is the union of the $\mathbb{R}$-classes of the projections $f \geq e$.

Lemma 2.4. *If S is a left restriction semigroup and $e \in P_S$, then $\mathrm{RF}(e) = \{s \in S : as \in \mathbb{R}_e$ for some $a \in \mathbb{R}_e\}$. If $x, y \in S$ and $xy \in \mathrm{RF}(e)$ then $x, y \in \mathrm{RF}(e)$.*

Proof. The alternative description of $\mathrm{RF}(e)$ follows from the fact that $as \,\mathbb{R}\, as^+$ (and that if $as^+ \,\mathbb{R}\, a$ then $as^+ = a$, as noted above). To prove the second statement, suppose $xy \in \mathrm{RF}(e)$ and $a \in \mathbb{R}_e$ is such that $axy \in \mathbb{R}_e$. Then $ax \in \mathbb{R}_e$ (since $e = a^+ \geq (ax)^+ \geq (axy)^+ = e$), so that $x \in \mathrm{RF}(e)$; and then $y \in \mathrm{RF}(e)$, from the definition. □

Put $I_e = S \setminus \mathrm{RF}(e)$. By the lemma, unless $S = \mathrm{RF}(e)$, I_e is a $^+$-ideal of S. As usual, we may identify the Rees quotient S/I_e with the union of $\mathrm{RF}(e)$ and 0 when convenient. If $S = \mathrm{RF}(e)$, then e must be the least projection of S and $\mathbb{R}_e = M_e$. By convention, $S/I_e = S$ in that case.

3. Strictness

Call a left restriction semigroup *strict* if it is a subdirect product of primitive left restriction semigroups and monoids. This definition – with the benefit of some hindsight – is a natural generalization of the terminology for inverse semigroups and restriction semigroups, although not the first that sprang to the author's mind: see the concluding remarks of this section. We briefly review how this term was used in those contexts, since it is of great relevance in the sequel.

As defined in [12, II.4], an inverse semigroup is *strict* if it is a subdirect product of Brandt semigroups (that is, completely 0-simple inverse semigroups) and groups. It is easy to see that this is equivalent to being a subdirect product of primitive inverse semigroups and groups. Various alternative descriptions are provided in [12], but [7, Result 2.2] gives a convenient summary. The simplest structural description is that they are the inverse semigroups that satisfy '$\mathcal{D}$-majorization', the property that if idempotents e, f, g satisfy $e > f, g$ and $f \,\mathcal{D}\, g$, then $f = g$.

By analogy, a restriction semigroup is *strict* [7, 8] if it is a subdirect product of monoids and completely 0-r-simple restriction semigroups. Here we shall instead term the latter *connected*, based on the connection with categories that was elucidated in [8] and will be the basis for the sequel [9]. In the terminology of [7, 8], a restriction semigroup S with zero is 0-*r-simple* if $\{0\}$ and S are its only restriction ideals and *completely* 0-*r-simple* (here, *connected*) if, in addition, it is primitive. The semigroup is connected if and only if it is primitive and its nonzero elements form a single $\mathbb{D}$-class.

The basic computational tool is the following. Note that if such a semigroup has only one nonzero projection, then it is a monoid with adjoined zero, and so a semilattice of monoids.

Result 3.1 [8, Lemma 2.1]. If a and b are nonzero elements of a connected restriction semigroup S, then $ab \neq 0$ if and only if $a^* = b^+$, in which case $(ab)^+ = a^+$ and $(ab)^* = b^*$.

Various characterizations of strict restriction semigroups were given in [7, Theorem 8.1], including the varietal one stated in Result 4.6 below, one by biunary identities, and a structural one (cf. strict inverse semigroups, above): a restriction semigroup is strict if and only if it satisfies $\mathbb{D}$-majorization, that is, whenever f, g, h are projections, $f > g, h$ and $g \mathbin{\mathbb{D}} h$, then $g = h$.

Now it is remarked at the beginning of Section 7 that what might at first appear to be the logical left restriction semigroups to consider, in view of the above, are those that are 'completely 0-$\mathbb{R}$-simple': primitive, with a single nonzero $\mathbb{R}$-class (or perhaps by $\mathbb{R}$-majorization, which is, however, meaningless in this context). This class, while of interest (see Section 7), turns out to be too narrow: the appropriate generalization of strictness, rather, is the one above. A simply stated structural description will follow from the characterization via identities provided by Theorem 6.6.

4. Varieties of left restriction semigroups and connections with varieties of restriction semigroups

The elementary material on this topic is extracted from [6]. Again, due to the intimate connection we establish with the two-sided case, we also briefly consider varieties of restriction semigroups. We shall need only elementary universal algebra, which may be found in [1].

Denote by **LR** the variety of all left restriction semigroups, under the operations $\{\cdot, ^+\}$. The subvarieties consisting of trivial semigroups, of monoids, and of semilattices, respectively, are denoted by **T**, **M** and **S** (by **SL** in [6]). Recall that monoids in this context are left restriction semigroups with one projection and so may be defined by the identity $x^+ = y^+$. Note that subvarieties of **M** are essentially varieties of monoids, and we shall treat them as such. It is clear that a variety **V** of left restriction semigroups consists of monoids if and only if $\mathbf{V} \cap \mathbf{S} = \mathbf{T}$.

For any variety **V** of left restriction semigroups, denote by $\mathcal{L}(\mathbf{V})$ its lattice of subvarieties. If $\mathbf{U}, \mathbf{V} \in \mathcal{L}(\mathbf{V})$ and $\mathbf{U} \subseteq \mathbf{V}$, $[\mathbf{U}, \mathbf{V}]$ denotes the interval sublattice

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$\{\mathbf{W} : \mathbf{U} \subseteq \mathbf{W} \subseteq \mathbf{V}\}$. The notation $\mathbf{V} \succ \mathbf{U}$, $\mathbf{V}$ *covers* $\mathbf{U}$, means that the interval consists only of the two given varieties. If X is a set of left restriction semigroups, $\langle X \rangle$ (or sometimes $\langle X \rangle_{LR}$) will denote the variety of left restriction semigroups it generates.

Result 4.1 [6, Theorems 4.1, 4.2]. If $\mathbf{V} \in \mathcal{L}(\mathbf{LR})$, then $\mathbf{V} \vee \mathbf{M} = \{S \in \mathbf{LR} : S/\mu \in \mathbf{V}\}$. Hence the map $\mathbf{V} \longrightarrow \mathbf{V} \vee \mathbf{M}$ is a complete lattice homomorphism. The map $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{M}$ is a lattice homomorphism.

If $\mathbf{N}$ is a variety of monoids, we shall in general abbreviate $\mathbf{S} \vee \mathbf{N}$ to $\mathbf{SN}$. The base of the lattice $\mathcal{L}(\mathbf{LR})$ consists of two 'layers', the intervals $[\mathbf{T}, \mathbf{M}]$ and $[\mathbf{S}, \mathbf{SM}]$, as follows.

Result 4.2 [6, Theorems 4.4, 4.5]. The following are equivalent for a left restriction semigroup S: (a) $S \in \mathbf{SM}$; (b) S satisfies $(xy)^+ = x^+y^+$; (c) S becomes a restriction semigroup under the assignment $a^* = a^+$; (d) S is a (strong) semilattice of monoids.

The sublattice $\mathcal{L}(\mathbf{SM})$ of $\mathcal{L}(\mathbf{LR})$ is isomorphic to the direct product of the two-element lattice $\mathcal{L}(\mathbf{S})$ and the lattice $\mathcal{L}(\mathbf{M})$, under the map $\mathbf{V} \mapsto (\mathbf{V} \cap \mathbf{S}) \vee (\mathbf{V} \cap \mathbf{M})$. If $\mathbf{V}$ is not simply a variety of monoids, then it consists of all (strong) semilattices of monoids from $\mathbf{V} \cap \mathbf{M}$.

If $\mathbf{V}$ is any variety of left restriction semigroups, let $\mathbf{loc}(\mathbf{V})$ consist of the left restriction semigroups S that are 'locally' in $\mathbf{V}$, meaning that $eSe \in \mathbf{V}$ for all $e \in P_S$. It is easily verified that $\mathbf{loc}(\mathbf{V})$ is again a variety. In a related vein, for any variety $\mathbf{N}$ of monoids, let $\mathbf{mon}(\mathbf{N})$ consist of the left restriction semigroups all of whose submonoids belong to $\mathbf{N}$. Note that $\mathbf{loc}(\mathbf{SN}) \subset \mathbf{mon}(\mathbf{N})$. In general $\mathbf{mon}(\mathbf{N})$ is not a variety.

The topic of this paper is the variety $\mathbf{B}$ of left restriction semigroups generated by the Brandt semigroups and the monoids. In view of the discussion in the previous section, it may also be described as that generated by the reducts of strict inverse semigroups, or by the reducts of strict restriction semigroups and monoids, and so on. Since Brandt semigroups and monoids are both locally semilattices of monoids, $\mathbf{B} \subset \mathbf{loc}(\mathbf{SM})$. That the inclusion is proper is demonstrated in Corollary 6.7. In conjunction with the inclusion $\mathbf{loc}(\mathbf{SN}) \subset \mathbf{mon}(\mathbf{N})$, we summarize as follows.

Proposition 4.3 (Cf. [7, Proposition 8.3]). *The inclusion* $\mathbf{B} \subset \mathbf{loc}(\mathbf{SM})$ *holds. For any variety* $\mathbf{N}$ *of monoids,* $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{SN})$ *and so forms a subvariety.*

As remarked in the introduction, the Brandt semigroup of primary interest is the inverse semigroup $B_2 = \{a, b, e = ab, f = ba, 0\}$. Its subsemigroup $B_0 = \{e, a, f, 0\}$ is naturally a restriction semigroup (with $a^+ = e$ and $a^* = f$) and therefore also a left restriction semigroup, forgetting the * operation. The left restriction subsemigroup $D = \{e, a, 0\}$ of B_0 is not a restriction semigroup. The varieties of left restriction semigroups generated by these three semigroups, respectively, are $\mathbf{B}_2$, $\mathbf{B}_0$ and $\mathbf{D}$. Note that they are all contained in $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$. We finish this subsection by reviewing the lattice coverings found in [6].

The variety $\mathbf{L}_2^1$ is generated by the left restriction semigroup L_2^1 that is obtained from the two-element left zero semigroup $L_2 = \{g, h\}$ by adjoining an identity 1, setting $g^+ = g$ and $h^+ = 1^+ = 1$. Note that, when regarded as a 'plain' semigroup, L_2^1 is a

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monoid, but not when regarded as a left restriction semigroup. It is the union of the submonoids $M_1 = \{1, h\}$ and M_g.

Result 4.4 [6, Proposition 4.18, Theorems 4.19 and 4.20]. The coverings $\mathbf{D} \succ \mathbf{S} \succ \mathbf{T}$ hold. Any variety of left restriction semigroups that consists of unions of monoids, but not merely semilattices of monoids, contains L_2^1. The varieties $\mathbf{D}$ and $\mathbf{L}_2^1$ are the two varieties minimal with respect to not being contained in $\mathbf{SM}$.

Proposition 4.5. *Neither L_2^1 nor D^1 belongs to* $\mathbf{B}$.

Proof. Here D^1 is the left restriction semigroup obtained by putting $1^+ = 1$. If either of these semigroups is denoted by S, then $S = 1S1$, where $1 \in P_S$, but $S \notin \mathbf{SM}$, so Proposition 4.3 applies. □

Since $L_2^1 \notin \mathbf{B}$, varieties consisting of unions of monoids play no role in this paper and in our context $\mathbf{D}$ is the unique relevant cover of $\mathbf{S}$.

4.1. Varieties of restriction semigroups. Next we briefly review varieties of restriction semigroups, which were introduced in [6] in parallel with the material above, and explored further in [7] and [8]. The material from the last of these papers dealt with the lattice itself, through a connection with varieties of categories. It will not be needed here, but will be integral to the sequel [9].

Let $\mathbf{R}$ denote the variety of all restriction semigroups. On the occasions where confusion might otherwise arise, we shall distinguish two-sided varieties by using the subscript R. So, for example, $\mathbf{B}_R$ and $(\mathbf{B}_0)_R$ then denote the analogous varieties of restriction semigroups. Recall that monoids may be regarded as both left restriction semigroups and restriction semigroups. In fact, the same is true for the variety $\mathbf{SM}$ of semilattices of monoids, and its subvarieties, since these are precisely the restriction semigroups on which $a^* = a^+$ (also see Result 4.2). So in these cases we shall not use the subscript notation.

Similarly, if X is a set of restriction semigroups, $\langle X \rangle_R$ will denote the variety of restriction semigroups it generates.

First note that direct analogs of Results 4.1 and 4.2 hold in the two-sided case. The notation $\mathbf{l}\mathrm{oc}(\mathbf{V})$ and $\mathbf{m}\mathrm{on}(\mathbf{N})$ will be used as in the one-sided case and the analog of Proposition 4.3 holds.

Recall from Section 3 that a restriction semigroup is *strict* if it is a subdirect product of connected restriction semigroups and monoids. In [7], bases of identities were found for $\mathbf{B}_R$, $(\mathbf{B}_2)_R$, $(\mathbf{B}_0)_R$ and $(\mathbf{B}_0)_R \vee \mathbf{M}$. As these all involve both unary operations, we shall not need them in this work.

Result 4.6 [7, Theorems 8.1, 10.6, 9.3 and 10.3]. A restriction semigroup S belongs to the variety $\mathbf{B}_R$ if and only if it is strict. In that event:

(i) $S \in (\mathbf{B}_2)_R$ if and only if $\mathbb{H}$ is the identical relation;

(ii) if (i) holds, then $S \in (\mathbf{B}_0)_R$ if and only if, further, the only regular elements of S are the projections;

(iii) $S \in (\mathbf{B}_0)_R \vee \mathbf{M}$ if and only if for $e, f \in P_S$, both $\mathbb{R}_e \cap \mathbb{L}_f$ and $\mathbb{L}_e \cap \mathbb{R}_f$ cannot be nonempty.

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Result 4.7 [7, Corollary 10.2]. In the lattice $\mathcal{L}(\mathbf{R})$, the sublattice $\mathcal{L}((\mathbf{B}_2)_R)$ comprises the chain $(\mathbf{B}_2)_R \succ (\mathbf{B}_0)_R \succ \mathbf{S} \succ \mathbf{T}$.

To conclude this section, we make some elementary observations regarding relationships between varieties of left restriction semigroups and varieties of restriction semigroups, based on the association of every restriction semigroup $S = (S, \cdot, ^+, ^*)$ with its left restriction reduct $(S, \cdot, ^+)$. A much deeper analysis will be needed in the sequel [9]. Since there are some ambiguities to be resolved, we begin carefully.

If X is a set of restriction semigroups, the notation $\langle X\rangle_{LR}$, introduced earlier in this section, will perforce denote the variety of left restriction semigroups generated by the reducts of the members of X. Let $\mathbf{V}$ be a variety of left restriction semigroups. Then $\mathbf{V}^R$ will denote the collection of two-sided restriction semigroups whose reducts belong to $\mathbf{V}$. We now collect the results that will be used in the sequel.

Proposition 4.8. *Let* $\mathbf{V}$ *be a variety of left restriction semigroups and let* X *be a set of restriction semigroups. Then*

(1) $\mathbf{V}^R$ *is a variety of restriction semigroups and* $\langle \mathbf{V}^R\rangle_{LR} \subseteq \mathbf{V}$;
(2) $\langle X\rangle_R \subseteq (\langle X\rangle_{LR})^R$;
(3) $\langle X\rangle_{LR} = \langle\langle X\rangle_R\rangle_{LR} = \langle(\langle X\rangle_{LR})^R\rangle_{LR}$;
(4) *as a result,* $\mathbf{B}_R \subset \mathbf{B}^R$ *and* $\mathbf{B} = \langle \mathbf{B}_R\rangle_{LR} = \langle \mathbf{B}^R\rangle_{LR}$.

Proof. Statements (1) and (2) follow from the fact that the products, homomorphisms and biunary subsemigroups of restriction semigroups retain their properties in the reducts.

Statement (3) follows from the sequence $\langle X\rangle_{LR} \subseteq \langle\langle X\rangle_R\rangle_{LR} \subseteq \langle(\langle X\rangle_{LR})^R\rangle_{LR} \subseteq \langle X\rangle_{LR}$, where the first is obtained from the inclusion $X \subseteq \langle X\rangle_R$, the second from (2) and the third from (1), applied to $\mathbf{V} = \langle X\rangle_{LR}$.

Now (4) is simply the case where X consists of the Brandt semigroups and the monoids, so that $\langle X\rangle_{LR} = \mathbf{B}$ and $\langle X\rangle_R = \mathbf{B}_R$. □

The inclusion $\mathbf{B}_R \subset \mathbf{B}^R$ asserts that the reduct of every strict restriction semigroup is strict as a left restriction semigroup (looking ahead to the connection made in Theorem 6.6). That the inclusion is strict will be shown in Proposition 8.15.

5. Primitive left restriction semigroups

In view of the definition of strictness, we first focus attention on primitive left restriction semigroups, in fact a special class of such semigroups.

Let S be a primitive left restriction semigroup and suppose $x \in S^{RI}$, that is, x *has* a right identity. If $x \neq 0$, then x has a *unique* right identity, since the set of right identities for x is a subsemilattice of P_S. Denote this right identity by x^*. Put $0^* = 0$. (This notation will be shown to be consistent with its use in two-sided restriction semigroups.) Clearly $e^* = e$ for every projection e of S. We continue the notation and techniques introduced in Section 2.

Lemma 5.1 (Cf. Result 3.1). *Let S be a primitive left restriction semigroup and let $x, y \in S$, $x, y \neq 0$. Then $xy \neq 0$ if and only if (x^* exists and) $x^* = y^+$, in which case $xy \mathrel{\mathbb{R}} x$ and if xy has a right identity then so does y and $(xy)^* = y^*$.*

Proof. If $xy \neq 0$, then since $(xy^+)^+ = (xy)^+ \leq x^+$, equality holds, so that $xy^+ = x$. Therefore x^* exists and equals y^+. Conversely, if $x^* = y^+$, then $(xy)^+ = (xy^+)^+ = (xx^*)^+ = x^+$ and so $xy \neq 0$. If, further, $xy = xyf$, where $f \in P_S$, then of necessity $yf \neq 0$ and, by the first part of the proof, $y^* = f = (xy)^*$. □

Proposition 5.2. *In any primitive left restriction semigroup S, the left restriction subsemigroup S^{RI} is also a primitive restriction semigroup, under the additional operation * defined above. Equivalently, if every element of a primitive left restriction semigroup S has a right identity, then S is also a primitive restriction semigroup in this way.*

Proof. We verify the defining identities. Clearly $aa^* = a$. If $a, b \in S$, then $(ab^*)^* \neq 0$ if and only if $ab^* \neq 0$, if and only if $a^* = b^*$ (by Lemma 5.1), and thus if and only if $a^*b^* \neq 0$. In that event, $(ab^*)^* = (b^*)^* = b^* = a^*b^*$. Clearly, $a^*b^* = b^*a^*$. To prove the right 'ample' identity, note that $a^*b \neq 0$ if and only if $a^* = b^+$, in which case $a^*b = b$ and $b(a^*b)^* = bb^* = b$. Finally, $(a^*)^+ = a^*$ and $(a^+)^* = a^+$.

The first statement clearly implies the second, since if every element has a right identity, $S = S^{RI}$. Conversely, S^{RI} is a left restriction subsemigroup, which is primitive when S is. □

Call a primitive left restriction semigroup S *primitive with base e* if e is a nonzero projection of S with the property that $S \backslash \{0\} = \mathrm{RF}(e)$, in other words every nonzero projection of S is a right identity for some element of $\mathbb{R}_e$. If g is a right identity for $a \in \mathbb{R}_e$, then $g = a^*$. In general, the element a is not unique. Note that $M_e = \{a \in \mathbb{R}_e : a^* = e\}$.

Lemma 5.3. *A primitive left restriction semigroup S has both e and f as a base if and only if there exist $a, b \in S$ such that $a^+ = e = b^*$ and $b^+ = f = a^*$. If M_e and M_f are actually groups, in particular if M_e and M_f are trivial, then this is the case if and only if $e\mathcal{D}f$.*

Proof. The first statement is immediate from the definition. Note that in that event $ab \mathrel{\mathbb{R}} e$ and $ab = abe$, so $ab \in M_e$; similarly, $ba \in M_f$. If M_e and M_f are subgroups, then $e = abc$ and $f = dba$ for some $c, d \in S$. Thus $e \mathrel{\mathcal{R}} a \mathrel{\mathcal{L}} f$. If, conversely, $e \mathrel{\mathcal{D}} f$, then clearly such a, b exist. □

Proposition 5.4. *Let S be a primitive left restriction semigroup with base e. Then S may be $^+$-embedded in a primitive restriction semigroup S^*, in fact in a connected restriction semigroup. The submonoids of S^* are either those of S, or are trivial.*

Proof. If $S = S^{RI}$, put $S^* = S$. Otherwise, we use Proposition 5.2 by showing that a new projection can be adjoined, incomparable to the nonzero projections of S, that is a common right identity for every element of S^{NRI}.

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Let $S^r = S \cup \{h\}$, where h is an element distinct from each element of S. Extend the binary operation on S to S^r by putting $h^2 = h$, $hs = 0$ for all $s \in S\backslash\{h\}$; $sh = s$ for all $s \in S^{NRI}$; and $sh = 0$ for all $s \in S^{RI}$. Put $h^+ = h$.

Consider nonzero elements $r, s, t \in S^r$. If $r = h$, then $(rs)t$ and $r(st)$ are each nonzero if and only if $s = t = h$, in which case $(rs)t = r(st) = h$. If not, suppose $s = h$. Then $(rs)t = 0$ unless $r \in S^{NRI}$ and $t = h$, in which case $(rs)t = r$; and the same holds for $r(st)$. If neither r nor s is h, suppose $t = h$. Then $(rs)t = 0$ unless $rs \in S^{NRI}$, in which case $(rs)t = rs$. Now $r(st) = 0$ unless $s \in S^{NRI}$, in which case $st = s$ and $r(st) = rs$. But by Lemma 5.1 $s \in S^{NRI}$ if and only if $rs \in S^{NRI}$.

Thus the binary operation on S^r is associative. Since each nonzero projection f of S belongs to S^{RI}, $fh = hf = 0$, so S^r is primitive. Observe that the only 'new' instances of the left 'ample' identity are of the form $(xh)^+x = xh$. Both sides are zero unless $x \in S^{NRI}$, in which case both sides yield x. The other identities for a left restriction semigroup are trivially satisfied.

Clearly every element of S^r now possesses a (unique) right identity and S^r is a restriction semigroup, by Proposition 5.2, with respect to the unary operation *, extended from S by putting $a^* = h$ for all $a \in S^{NRI}$. Since every nonzero projection is $\mathbb{L}$-related to an element of $\mathbb{R}_e$, the nonzero elements form a single $\mathbb{D}$-class. So S^r is connected.

The inclusion map is a $^+$-embedding of S in S^*. □

The following sequel to Proposition 4.3 is a preview of the techniques employed in the sequel [9]. It uses the notation introduced in Subsection 4.1.

Corollary 5.5. *For any variety* $\mathbf{N}$ *of monoids,* $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \langle \mathbf{B}_R \cap \mathbf{mon}(\mathbf{N})\rangle_{LR}$.

Proof. That the right-hand side is included in the left-hand side follows from Proposition 4.8. The opposite inclusion follows from the fact that the embedding in Proposition 5.4 introduces only one new, trivial, submonoid. □

While Proposition 5.4 will serve our initial purpose of describing the members of $\mathbf{B}$, it should be clear that the embedding is rather 'blunt'. For instance, it clearly need not preserve the property (satisfied, as we shall see, by members of $\mathbf{B}_2$) that if distinct $\mathbb{R}$-related elements have right identities, the right identities must be distinct. Rather than prove a more refined general embedding theorem, we shall content ourselves in Section 8 with the situation just mentioned. The question of which identities *are* preserved by the embedding in the proposition will be a key part of [9].

6. A basis of identities for B

We first study properties equivalent to satisfaction of identity (6.1), then study its consequences, yielding the characterization of $\mathbf{B}$ in Theorem 6.6. Here and in the sequel it will be convenient to use the convention in writing identities that letters e, f, g, etc, stand for 'projection variables', that is, for x^+, y^+, z^+, as appropriate, as illustrated in (b) in the following lemma. This identity will turn out to define the variety $\mathbf{B}$.

Lemma 6.1. *In any left restriction semigroup S, the following are equivalent:*

(a) *S satisfies*
$$(xz)^+(yz^+w)^+ = (yz)^+(xz^+w)^+; \tag{6.1}$$

(b) *S satisfies $(xf)^+(yfg)^+ = (yf)^+(xfg)^+$, for all x, y and for all projections f, g;*

(c) *if $x \mathbin{\mathbb{R}} y$ in S and $xf = x$, $yf = y$ for some $f \in P_S$, then for any $g \in P_S$, $xg = x$ if and only if $yg = y$.*

Proof. The equations in (a) and (b) are merely restatements of each other, using Lemma 2.1: $(ab)^+ = (ab^+)^+$.

Now suppose S satisfies (b), $x, y \in S$, $f, g \in P_S$, $x \mathbin{\mathbb{R}} y$, $xf = x$, $yf = y$, and $xg = x$. Then, using (b), $(yg)^+ = x^+(yg)^+ = y^+x^+ = y^+$, that is, $yg \mathbin{\mathbb{R}} y$, so that $yg = y$.

Finally, suppose S satisfies (c) and let $x, y \in S$, $f, g \in P_S$. Put $a = (xf)^+(yfg)$ and $b = (yfg)^+(xf)$. Then $a \mathbin{\mathbb{R}} b$ (by Lemma 2.1), $af = a$ and $bf = b$, and $ag = a$, so by (c), $bg = b$. Therefore $a^+ = (bg)^+$, that is, $(xf)^+(yfg)^+ = (yfg)^+(xfg)^+$. By symmetry, $(yf)^+(xfg)^+ = (xfg)^+(yfg)^+$. Thus (b) holds. □

The statement in (c) may be paraphrased as 'if two $\mathbb{R}$-related elements share any right identity, they share all right identities'. We will generally use this paraphrase, and similar ones below, without further comment.

Proposition 6.2. *Any primitive left restriction semigroup S satisfies (6.1).*

Proof. Referring to Lemma 6.1(b), if S is primitive, then either $fg = 0$, in which case both sides are 0, or $f = g$, in which case the two sides are equal. □

The following somewhat technical lemma will be used in Section 8. Its proof demonstrates use of the paraphrased version of identity (6.1). Recall that $a \mathbin{\mu} b$ in a left restriction semigroup S if $(ae)^+ = (be)^+$ for all $e \in P_S$.

Lemma 6.3. *Let S be a left restriction semigroup that satisfies (6.1). Let $e \in P_S$. Then $e\mu = M_e$.*

Proof. Suppose $a \in e\mu$. Then $a^+ = e$, since μ separates projections, and so $ae = (ae)^+a = e^+a = a$, that is, $a \in M_e$. Conversely, suppose $ae = a$ and $f \in P_S$. To show $(af)^+ = (ef)^+$ for all $f \in P_S$, it suffices to assume that $f \leq e$. Now, on the one hand, $fa \mathbin{\mathbb{R}} fe = f$ and fa and f share the right identity e. But f is a right identity for itself and so by (6.1) is also a right identity for fa, that is, $fa = faf$. It follows that $f = (fa)^+ \leq (af)^+$. On the other hand, $(af)^+$ and af again share the right identity e. But f is a right identity for af and so for $(af)^+$. Therefore $(af)^+ = f$, as required. □

Let S be a left restriction semigroup. For each $e \in P_S$, define a relation ρ_e on S as follows: if $x, y \in S$, then $(x, y) \in \rho_e$ if either both x and y belong to I_e, or both x and y belong to $\mathrm{RF}(e)$ *and* there exists $g \in P_S$ such that $gx, gy \in \mathrm{RF}(e)$ and $gx = gy$.

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PROPOSITION 6.4. *Let S be a left restriction semigroup that satisfies (6.1). Then for each $e \in P_S$, the relation ρ_e is a congruence on S that separates the members of $\mathbb{R}_e$. If e is minimum, then S/ρ_e is isomorphic to the monoid M_e; otherwise, S/ρ_e is primitive with base $e\rho_e$.*

If S itself is primitive with base e, then $S \cong S/\rho_e$.

PROOF. Note that the projection g in the definition may be assumed to be such that $g \leq x^+y^+$, and that assumption will be made throughout. That ρ_e is reflexive and symmetric is clear. The only case of transitivity to be verified is where $x, y, z \in \mathrm{RF}(e)$ and $(x, y), (y, z) \in \rho_e$. Then there exist $g, h \in P_S$ such that $gx, gy, hy, hz \in \mathrm{RF}(e)$ and $gx = gy, hy = hz$. We may assume $g \leq x^+y^+$ and $h \leq y^+z^+$, so that $gx \in \mathbb{R}_g, hy \in \mathbb{R}_h$. Now by definition there exist $a, b \in \mathbb{R}_e$ such that $ag = a$ and $bh = b$. Since $g, h \leq y^+$, $ay^+ = by^+$ and so by (6.1), a and b share all right identities, whence $ah = a$ and so $a(gh) = a$. Since $(gh)x = (gh)z$ and $(gh)x \in \mathbb{R}_{gh}$, it follows that $ghx \in \mathrm{RF}(e)$ and so $(x, z) \in \rho_e$. Therefore ρ_e is an equivalence relation on S.

Since I_e is an ideal of S, compatibility with the operations need only be considered for pairs (x, y) from $\mathrm{RF}(e)$. With g from the definition, again take $a \in \mathbb{R}_e$ such that $ag = a$ and $g \leq x^+y^+$. Let $s \in S$.

Suppose $sx \in \mathrm{RF}(e)$, with $b \in \mathbb{R}_e$ such that $b(sx) \in \mathbb{R}_e$. Then $bs \in \mathbb{R}_e$, $b = bs^+$ and $(bs)x^+ = bs$. Since a and bs share x^+ as a right identity and y^+ is a right identity for a, by (6.1) the same is true of bs. Therefore $b(sy) \in \mathbb{R}_e$ and $sy \in \mathrm{RF}(e)$. Likewise, since g is a right identity for a, $(bs)g = bs$, so $bsgx = bsx \in \mathbb{R}_e$ and therefore $sgx \in \mathrm{RF}(e)$. By the left 'ample' identity, $(sg)^+sx = sgx = sgy = (sg)^+sy$, where $(sg)^+ \in P_S$, so $(sx, sy) \in \rho_e$.

Next suppose $xs \in \mathrm{RF}(e)$, with $c \in \mathbb{R}_e$ such that $c(xs) \in \mathbb{R}_e$. Then $cx^+ = c$ and, similarly to the last paragraph, $cg = c$, so that $c(gxs) = cxs \in \mathbb{R}_e$. Therefore $g(xs) = g(ys) \in \mathrm{RF}(e)$ and $(xs, ys) \in \rho_e$.

Now with g as above, $gx^+ = g = gy^+$, so $(x^+, y^+) \in \rho_e$. Therefore ρ_e is a (unary) congruence on S.

To show that ρ_e separates $\mathbb{R}_e$, let $x, y \in \mathbb{R}_e$ and suppose $gx = gy \in \mathrm{RF}(e)$ for some $g \in P_S$. Then $a(gx) \in \mathbb{R}_e$ for some $a \in \mathbb{R}_e$. Since $x = ex$, $ae \in \mathbb{R}_e$, so $ae = a$. Now e and a share the right identity e and so, by (6.1), from $ag = a$ it follows that $eg = e$. Thus $gx = gex = ex = x$ and, similarly, $gy = y$, as required.

If e is the least projection, then $\mathbb{R}_e = M_e$ and $e\rho_e$ is the only projection in S/ρ_e. Otherwise the image of I_e is the zero element. To prove S/ρ_e is primitive, suppose $x, y \in \mathrm{RF}(e)$, $x\rho_e$ and $y\rho_e$ are projections and $x\rho_e \geq y\rho_e$. Since ρ_e respects the unary operation, we may actually assume that $x, y \in P_S$ with $x \geq y$. But then $yy = yx$ and so $x\rho_e y$, as required. Since $S = \mathrm{RF}(e) \cup I_e$, it is clear that $S/\rho_e = \mathrm{RF}(e\rho_e) \cup \{0\}$, so $e\rho_e$ is then a base for the primitivity.

Finally, suppose that S was initially primitive with base e. Then $S = \mathrm{RF}(e) \cup \{0\}$. Suppose $x, y \neq 0$ and $g \in P_S$ is such that $gx = gy \neq 0$. Then by primitivity $g = x^+ = y^+$, so that $x = y$. □

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Corollary 6.5. *Let S be a left restriction semigroup that satisfies (6.1). Then S is a subdirect product of the left restriction semigroups S/ρ_e, $e \in P_S$, each of which is either a monoid or is primitive with base $e\rho_e$.*

Proof. Let $x, y \in S$ and suppose $(x, y) \in \rho_f$ for all $f \in P_S$. Put $e = x^+$. Then $x \in \mathrm{RF}(e)$, so $y \in \mathrm{RF}(e)$ and there exists $g \in P_S$ such that $gx = gy \in \mathrm{RF}(e)$, where again we may take $g \leq x^+y^+$. But then $g = e$ and so $x = ey$. So $x \leq y$ in the natural partial order and, by symmetry, equality holds. □

The proof of the main theorem of this section may now be completed.

Theorem 6.6. *The following are equivalent for a left restriction semigroup S:*

(i) *$S \in \mathbf{B}$, the variety generated by Brandt semigroups and monoids (or, equivalently, by the reducts of strict inverse semigroups, or by the reducts of strict restriction semigroups and monoids);*
(ii) *S satisfies identity (6.1), namely $(xz)^+(yz^+w)^+ = (yz)^+(xz^+w)^+$ (or either of the equivalent formulations in Lemma 6.1);*
(iii) *S is a subdirect product of monoids and primitive left restriction semigroups with a specified base;*
(iv) *S is strict (that is, a subdirect product of monoids and primitive left restriction semigroups).*

Proof. (i) ⇒ (ii) and (iv) ⇒ (ii). Every primitive left restriction semigroup satisfies (6.1), by Proposition 6.2. Every Brandt semigroup is such a semigroup.

(ii) ⇒ (iii) ⇒ (iv). The first is an immediate consequence of Corollary 6.5 and the second is obvious.

(iii) ⇒ (i). By Proposition 5.4, each primitive factor embeds in a primitive restriction semigroup, which, according to Result 4.6 and the discussion in Section 3 belongs to $\mathbf{B}_R$. Thus $S \in \langle \mathbf{B}_R \rangle_{LR} = \mathbf{B}$, applying Proposition 4.8. □

Corollary 6.7. *There exists a left restriction semigroup that is locally a semilattice but does not belong to $\mathbf{B}$.*

Proof. Let $P = \{0, e, f, g\}$ be the semilattice, with 0 as zero, in which $ef = eg = 0$ and $f > g$. Adjoin distinct elements a and b and put $a^+ = b^+ = e$. Apart from $ea = a$ and $eb = b$, all products xa and xb are 0. Apart from $af = ag = a$ and $bf = b$, all products ax and bx are 0. By checking cases, $S = P \cup \{a, b\}$ is seen to be a left restriction semigroup with P as its semilattice of projections. Since each projection is either a left or a right zero for each of a and b, $S \in \mathbf{loc}(\mathbf{S})$. However, a and b share the right identity f but do not share the right identity g for a. So S does not satisfy (6.1). □

7. Bases of identities for D and D ∨ M

The structural approach to membership in the variety generated by Brandt semigroups, in both the inverse and the restriction cases, devolves to primitive semigroups consisting of a single nonzero $\mathcal{D}$-class or $\mathbb{D}$-class, respectively. The direct

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analog for left restriction semigroups comprises the primitive semigroups with a single nonzero $\mathbb{R}$-class, that is, with precisely one nonzero projection.

We term such semigroups 0-$\mathbb{R}$-*simple*. It is clear from the previous section that these are no longer the appropriate structural analogs, but they are of interest in their own right, as we first show. (The essence of the distinction is the existence of the minimal such example, namely D itself.) In a way, this connection is one analog of the equivalence between primitive restriction semigroups and categories established in [8], with M-acts replacing categories. For background on M-acts in general, see [10]. While no results from this theory are directly used, the connection was helpful in establishing Theorem 7.5.

Let M be a monoid, with identity e, that acts unitarily (or 'monoidally') on the left on a nonempty set X, $(m, x) \mapsto m \cdot x$. Let $A(M, X) = M \cup X \cup \{0\}$ and define a product as follows: if $m, n \in M$, the product is that in M; if $x \in X$, the product mx is $m \cdot x$. All other products are zero. Put $a^+ = e$ for all nonzero a and put $0^+ = 0$. Extend the notation to $A(M, \emptyset) = M \cup \{0\}$ (where one might regard the 'empty' action as including this with the case where X is nonempty).

Proposition 7.1. *Let M be a monoid, with identity e, that acts unitarily on the left on a set X. Then $A(M, X)$, as just defined, is a* 0-$\mathbb{R}$-*simple left restriction semigroup.*

Conversely, any such left restriction semigroup S has this form, where if $P_S = \{0, e\}$, then $M = M_e$, $X = \mathbb{R}_e \backslash M$ and the action of M on X is as follows: for $m \in M, x \in X$, $m \cdot x = mx$.

Proof. Put $A = A(M, X)$, for convenience. If $X = \emptyset$, this is clear. Otherwise, since $MX \subseteq X$ and $XA = \{0\}$, the only case of consequence to consider for associativity is $(mn)x = m(nx)$, where $m, n \in M, x \in X$. Equality follows from the definition of action. Since $e \cdot a = a$ for all $a \in A$, $a^+a = a$ for all a. The remaining identities for left restriction semigroups are straightforwardly verified.

Conversely, if S is as given, then $M = M_e$ is a monoid that acts unitarily on X, if $X \neq \emptyset$, and it is clear that $A(M, X) \cong S$ in any case. □

Write $A(1, X)$ in the case that M is trivial. If X is nonempty, M acts identically on the set and so $A(1, X)$ is essentially the null semigroup on X, together with the left identity and right zero element (other than for itself) 1. In particular, $D = A(1, X)$, where X is a singleton set. If X is empty, then $A(1, X)$ is the two-element semilattice. We leave it to the reader to extend Proposition 7.1 to a categorical equivalence in a natural way, an extension not needed in the sequel.

Proposition 7.2. *Any semigroup $A(1, X)$ belongs to* $\mathbf{D}$.

Proof. We may assume that X is nonempty, since $\mathbf{S} \subset \mathbf{D}$. Let Y be any nonempty set and consider the semigroup D^Y. Denote by g the projection all of whose entries are e. Then $|\mathbb{R}_g| = 2^{|Y|}$ and the Rees quotient modulo its complement (as in Section 2) is isomorphic to $A(1, Z)$, where $|Z| = 2^{|Y|} - 1$. Any semigroup $A(1, X)$ embeds in such a semigroup for suitably large Z. □

Proposition 7.3. *Any semigroup $A(M, X)$ is a homomorphic image of $M \times A(1, X)$ and therefore belongs to $\mathbf{D} \vee \mathbf{M}$.*

Proof. The case $X = \emptyset$ is standard [5, Lemma 4.6.10]. Assume otherwise and define $\theta\colon M \times A(1, X) \longrightarrow A(M, X)$ by $(m, 1)\theta = m$, $(m, 0)\theta = 0$ and $(m, x)\theta = mx$, for all $m \in M$ and $x \in X$. The only nonzero products in $M \times A(1, X)$ are $(m, 1)(n, 1)$ and $(m, 1)(n, x)$ and it is easily verified that they are respected by θ. The final statement is then a consequence of Proposition 7.2. □

We turn now to new identities.

Lemma 7.4. *In any left restriction semigroup S, the following are equivalent, and imply identity (6.1):*

(i) *S satisfies*

$$y^+xz^+ = xy^+z^+; \tag{7.1}$$

(ii) *S satisfies $exf = xef$ for all $e, f \in P_S$;*

(iii) *S satisfies $xf = (xf)(xf)^+$ and $xf = fxf$ for all $f \in P_S$.*

Proof. Clearly (i) and (ii) are equivalent. Note that by interchanging e and f, the identity in (ii) implies that $exf = fxe$. Let $x \in S$ and $f \in P_S$. Then $xf = x^+xf = fxx^+$ and so $xf = fxf$. Now $xf = (xf)f = f(xf)(xf)^+ = xf(xf)^+$. Thus (iii) holds. Conversely, let $x \in S$ and $e, f \in P_S$. Then $exf = (exf)(exf)^+ = (exf)e(exf)^+$, so $exf = exfe = exef = xef$.

Now assuming (iii) holds, we show (6.1) holds, via Lemma 6.1(c). Let $e, f, g \in P_S$, $x, y \in \mathbb{R}_e$ with $xf = x$, $yf = y$ and $xg = x$. Using the first identity in (iii), $x, y \in M_e$. Using the second identity, $x = gxg$, so $e = x^+ \leq g$. Therefore $y = ye = yg$. □

Observe that the first identity in (iii) may be paraphrased as 'any element a with a right identity belongs to the monoid M_{a^+}'.

The identities in (iii) are independent. On the one hand, any union of monoids satisfies the first identity, by virtue of its paraphrase, but by the discussion following Result 4.4, the union of monoids L_2^1 does not belong to $\mathbf{B}$ and so cannot satisfy the second identity. On the other hand, the semigroup $D^1 = \{e, a, 0, 1\}$ is easily seen to satisfy the second identity but does not satisfy the first, since 1 is a right identity for a.

Theorem 7.5. *The following are equivalent for a left restriction semigroup S:*

(i) *$S \in \mathbf{D} \vee \mathbf{M}$;*

(ii) *S satisfies identity (7.1), namely $y^+xz^+ = xy^+z^+$ (or either of its equivalent formulations in Lemma 7.4);*

(iii) *S is a subdirect product of monoids and semigroups of the form $A(M, X)$.*

Proof. That (iii) implies (i) is a consequence of Proposition 7.3. It is clear that every monoid satisfies (7.1) (the only projection being the identity element). That $D = \{e, a, 0\}$ satisfies (7.1) is also trivially verified: the only essential case is where $x = a$, but then $xf = 0$ for all $f \in P_D$. So (i) implies (ii).

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To prove that (ii) implies (iii), suppose that S satisfies (7.1). By Lemma 7.4, it thereby satisfies (6.1) and so by Proposition 6.5 is a subdirect product monoids and primitive left restriction semigroups, each with a specified base, that satisfy (7.1). Let T be such a semigroup, with base e, say. According to the paraphrase following the last lemma, the only elements of $\mathbb{R}_e$ with a right identity belong to M_e, in which case, by primitivity, that right identity is e itself. Thus $T = \mathbb{R}_e \cup \{0\}$ and so is 0-$\mathbb{R}$-simple. Then Proposition 7.1 applies. □

Next we turn to the variety **D**.

Lemma 7.6. *In any left restriction semigroup S, the following are equivalent:*

(i) *S satisfies the identity*

$$xz^+ = (xz^+)^+; \tag{7.2}$$

(ii) *S satisfies $xf = (xf)^+$ for all $f \in P_S$;*

(iii) *S satisfies identity (7.1) and has trivial submonoids.*

Proof. The equivalence of (i) and (ii) is obvious. Since (7.2) implies that xf is always a projection, the identity immediately implies both identities in Lemma 7.4(iii) and therefore (7.1). Also clear is that the submonoids must be trivial. Conversely, if S satisfies (7.1), then the paraphrase of the first identity in (iii) of that lemma, together with triviality of submonoids, implies that $xf = (xf)^+$. □

Identity (7.2) may be paraphrased as 'the only elements with right identities are the projections'.

Corollary 7.7. *The following are equivalent for a left restriction semigroup S:*

(i) *$S \in \mathbf{D}$;*

(ii) *S satisfies the identity (7.2), namely $xz^+ = (xz^+)^+$;*

(iii) *S is a subdirect product of semigroups of the form $A(1, X)$.*

Proof. That D satisfies (7.2) is clear from the paraphrase of the latter. Thus (i) implies (ii). If (ii) holds, then by the previous lemma, (7.1) holds and Theorem 7.5 applies. Again by the lemma, (7.2) implies that monoids are trivial, so (iii) holds. Finally, assume (iii) holds. Again applying Theorem 7.5, since S has trivial submonoids, the same is true for each $A(M, X)$ and so each has the form $A(1, X)$. Applying Proposition 7.2, $S \in \mathbf{D}$. Thus (i) holds. □

Theorem 7.8. *The varieties in the interval $[\mathbf{D}, \mathbf{D} \vee \mathbf{M}]$ are precisely those of the form $\mathbf{D} \vee \mathbf{N}$, for some variety $\mathbf{N}$ of monoids.*

Proof. Let $\mathbf{V}$ be such a variety and let $\mathbf{N} = \mathbf{V} \cap \mathbf{M}$, so $\mathbf{V} \subset \mathbf{mon}(\mathbf{N})$. Clearly $\mathbf{D} \vee \mathbf{N} \subseteq \mathbf{V}$. In the proof of Theorem 7.5, the cited results, Propositions 6.5 and 7.3, preserve the property that the submonoids belong to $\mathbf{N}$, so $\mathbf{V} \subseteq \mathbf{D} \vee \mathbf{N}$. □

Recall from Result 4.4 that $\mathbf{T} \prec \mathbf{S} \prec \mathbf{D}$. Combining this with the second part of Result 4.2 yields the following.

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COROLLARY 7.9. *The lattice* $\mathcal{L}(\mathbf{D} \vee \mathbf{M})$ *is isomorphic to the product of* $\mathcal{L}(\mathbf{M})$ *with the three-element chain* $\mathbf{T} \prec \mathbf{S} \prec \mathbf{D}$.

8. Bases of identities for $\mathbf{B}_2$, $\mathbf{B}_0$ and $\mathbf{B}_0 \vee \mathbf{M}$

Observe first that $\mathbf{B} = \mathbf{B}_2 \vee \mathbf{M}$: for by Result 4.7, $\mathbf{B}_R = (\mathbf{B}_2)_R \vee \mathbf{M}$ and so every Brandt semigroup belongs to $\mathbf{B}_2 \vee \mathbf{M}$. Once Theorem 8.11 has been proven, this equation also follows from Result 4.1 (cf. the argument in the proof of Corollary 8.13). Note that, also by Result 4.1, if S is a finitely generated member of $\mathbf{B}$, then P_S is finite, for there is a projection-separating homomorphism upon S/μ, which belongs to the locally finite variety $\mathbf{B}_2$. (The same argument applies, of course, to members of $\mathbf{B}_R$, finitely generated as restriction semigroups.)

We first introduce the identities relevant to this section and examine them in a similar fashion to those previously considered.

LEMMA 8.1. *In any left restriction semigroup* S*, the following are equivalent, and imply identity (6.1):*

(i) *S satisfies*

$$(xz)^+(yz) = (yz)^+(xz); \tag{8.1}$$

(ii) *S satisfies $(xf)^+(yf) = (yf)^+(xf)$ for all $x, y \in S$, $f \in P_S$;*

(iii) *if $a \,\mathbb{R}\, b$, $af = a$ and $bf = b$ for some $f \in P_S$, then $a = b$.*

PROOF. Clear (i) implies (ii). Conversely, given $x, y, z \in S$, from (ii) it follows that $(xz^+)^+(yz^+) = (yz^+)^+(xz^+)$. Right multiplication by z yields (8.1).

The substitution of a for x and b for y shows that (ii) implies (iii). Conversely, given any $x, y \in S$ and $f \in P_S$, it is clear that $(xf)^+(yf) \,\mathbb{R}\, (yf)^+(xf)$ and these terms each have f as right identity, so (iii) implies (ii).

That each implies (6.1) is immediate from their paraphrased versions (see the next paragraph). □

In view of this lemma, identity (8.1) may be paraphrased as 'for any projection e, distinct elements of $\mathbb{R}_e$ do not share a common right identity'. Note that all submonoids must therefore be trivial and that, in fact, if S is a restriction semigroup, then $\mathbb{H}$ is the identical relation.

From this interpretation and the corresponding interpretation of (6.1), above, it is clear that (8.1) implies (6.1).

LEMMA 8.2. *In any left restriction semigroup* S *that satisfies identity (6.1), the following are equivalent:*

(i) *S satisfies*

$$xyx = xyx(xyx)^+; \tag{8.2}$$

(ii) *if $x, y \in S$, $yx^+ = y$ and $xy^+ = x$, then $x^+ = y^+$ (and, as a result, $x, y \in M_{x^+}$).*

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PROOF. Assume (i) holds and let x, y be as in (ii). Put $e = x^+$, $f = y^+$. Then $(xyx)^+ = (xye)^+ = (xy)^+ = (xf)^+ = x^+ = e$ and, by assumption, $xyx \in M_e$. By (the paraphrase of) (6.1), since f is a right identity for both x and xyx and e is a right identity for xyx, e is also a right identity for x, that is, $x \in M_e$. Since, now, x and e share the latter as a right identity, they likewise share f as a right identity. So $e \leq f$ and, by symmetry, $f \leq e$, giving the stated conclusion.

Before proving the reverse implication, we prove that (6.1) implies the identity $xyx = xyxy^+$. Note that $(xyx)^+$ has both x^+ and $(xy)^+$ as a right identity, since each is a left identity and projections commute. Now $xyx^+ \mathbb{R} (xyx)^+$ and xyx^+ has x^+ as a right identity so that—using the paraphrase of (6.1)—it also has $(xy)^+$ as a right identity, that is, $xyx^+ = (xyx^+)(xy)^+ = xy(xy)^+$. Then, using the 'left ample' identity, $xyx = xyxy^+$.

From this identity we can write $xyx = xyxy^+ = (xy^+)((yxy^+)^+y)(xy^+)$ and, applying the identity to the right-hand product, it follows that $xyx = (xyx)((yxy^+)^+y)^+ = xyx(yxy)^+$. Similarly, $yxy = yxy(xyx)^+$, so (ii) implies that $(xyx)^+ = (yxy)^+$ and hence $xyx = xyx(xyx)^+$. □

COROLLARY 8.3. *If a primitive left restriction semigroup with base e satisfies (8.2), then the base is unique.*

PROOF. Suppose f is also a base and let a, b be as in Lemma 5.3. Then by Lemma 8.2, $e = f$. □

LEMMA 8.4. *In any left restriction semigroup S that satisfies identity (8.1), the following are equivalent:*

(i) *S satisfies*

$$xyx = (xyx)^+; \tag{8.3}$$

(ii) *S satisfies (8.2) and its submonoids are trivial;*

(iii) *every regular element of S is a projection.*

PROOF. If (i) holds, then it clearly satisfies (8.2). Further, if $e \in P_S$ and $x \in M_e$, then $x = exe = x^+ = e$.

If (ii) holds and x is a regular element of S, with inverse y, then $x = xyx$ belongs to a submonoid and is therefore a projection.

Suppose (iii) holds and let $x, y \in S$ be as in Lemma 8.2(ii). Put $e = x^+$ and $f = y^+$. Then $xy \mathbb{R} e = x^+$ and both xy and e have e as a right identity. By (the paraphrase of) (8.1), $xy = e$. Similarly $yx = f$. Therefore, by (iii), x and y are projections and so $x = e = f = y$, using commutativity of projections. By the cited lemma, S satisfies (8.2). Identity (8.1) implies triviality of submonoids. Therefore (ii) holds, and (ii) clearly implies (i). □

Finally, identity (8.3), namely $xyx = (xyx)^+$, may be paraphrased as 'the only regular elements are the projections'; cf. [7, Lemma 10.1] for the variety $(\mathbf{B}_0)_R$.

LEMMA 8.5. *Regarded as left restriction semigroups, B_2 satisfies (8.1) and B_0, in addition, satisfies (8.3).*

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Proof. This is simply checked, using the paraphrased versions of identities (8.1) and (8.3). □

In the remainder of this section we show that every primitive left restriction semigroup with base e that satisfies (8.1) divides a suitable restriction semigroup and therefore belongs to $\mathbf{B}_2$ (and to $\mathbf{B}_0$ if it also satisfies (8.3)). The next lemma and the example that follows demonstrate that the method of proof used for $\mathbf{B}$ (and for $\mathbf{B} \cap \mathbf{mon}(\mathbf{N})$ in general) cannot be followed literally.

Lemma 8.6. *Let S be a primitive left restriction semigroup that satisfies (8.1). If $x \mathbin{\mathbb{R}} y$ and $ax = ay \neq 0$ has a right identity, then $x = y$. Hence if S can be ${}^+$-embedded in a restriction semigroup, then it must satisfy:*

$$\textit{if } x \mathbin{\mathbb{R}} y \textit{ and } ax = ay \neq 0, \textit{ then } x = y. \tag{8.4}$$

Proof. From Lemma 5.1 it follows that if $ax = ay \neq 0$ has a right identity, then so do both x and y, whence, by (8.1), $x = y$. □

Note that if S satisfies (8.1) then, according to this lemma, the implication (8.4) is satisfied in S^{RI}, so it need only be tested for $x, y \in S^{NRI}$. It is not a consequence of (8.1), nor of (8.3), as the following example illustrates. As noted in the introduction (for more details, see [4]), for any nonempty set X, the semigroup $\mathcal{PT}_X$ of partial self-maps of X is a left restriction semigroup under the unary operation that assigns to each such map the identity map on its domain.

Proposition 8.7. *Let $X = \{1, 2, 3, 4, 5, 6\}$ and S the unary subsemigroup of $\mathcal{PT}_X$ generated by $\{a, s, t\}$, where $a\colon 1 \mapsto 2$, $s\colon 2 \mapsto 4$, $3 \mapsto 5$, and $t\colon 2 \mapsto 4$, $3 \mapsto 6$. Then S satisfies (8.3), but $s \mathbin{\mathbb{R}} t$ while $as = at \neq 0$.*

Proof. Since s and t are left zeros and $a^2 = 0$, the 'plain' subsemigroup T generated by $\{a, s, t\}$ is $\{a, s, t, b, 0\}$, where $b = as = at\colon 1 \mapsto 4$. Let e and f be respectively the identity maps on $\{1\}$ and $\{2, 3\}$, so that $e = a^+ = b^+$ and $f = s^+ = t^+$. Now both e and f are either left zeros or left identities for every element of T; and likewise on the right. Since $ef = 0$, it now follows that $S = \{e, f, a, s, t, b, 0\}$, that S is primitive with base e, that S satisfies (8.3) and that (8.4) fails. □

Following Proposition 5.4, it was remarked that that embedding result would not suffice in this section. The next result serves our purposes. Note that although we again use the notation S^*, this new oversemigroup will not in general be isomorphic to that in Proposition 5.4. (In fact there is a range of such possible embeddings, but we do not need more general cases and so have not included them.)

Proposition 8.8. *Let S be a primitive left restriction semigroup with base e that satisfies both identity (8.1) and the implication (8.4). Then S may be ${}^+$-embedded in a restriction semigroup S^* that belongs to $(\mathbf{B}_2)_R$. If, in addition, S satisfies (8.3), then $S^* \in (\mathbf{B}_0)_R$.*

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Proof. Let S^* be obtained from S by the adjunction of *one* right identity for each $a \in \mathbb{R}_e \cap S^{NRI}$. In view of the end result (and consistent with the notation used in Section 5 for those elements of S^{RI}) we may denote this right identity by a^*. For convenience, call these the 'new' a^*s. Extend this unary operation to all of S^{NRI} as follows. If $x \in S^{NRI}$, using the paraphrase of (8.1) following Lemma 8.1, there exists a *unique* $a \in \mathbb{R}_e$ such that $ax \in \mathbb{R}_e$. According to Lemma 5.1, $ax \in S^{NRI}$ (and conversely). Consistent with the conclusions of that lemma, set $x^* = (ax)^*$.

Extend the binary operation on S to S^* as follows. For each new a^* and each $x \in S^*$ such that $x^* = a^*$, put $xa^* = x$; set all other new products equal to 0. In particular, $a^*x = 0$ for all new a^* and all $x \in S^* \backslash \{a^*\}$. Extend the unary operation $s \mapsto s^+$ to S^* by putting $(a^*)^+ = a^*$ for each new a^*. Thus P_{S^*} is the union of P_S with the set of new a^*s.

Consider nonzero elements $r, s, t \in S$. If r is a new a^*, then each of $(rs)t$ and $r(st)$ is nonzero if and only if $s = a^* = t$, in which case each product is a^*. If not, suppose s is a new a^* and consider $(ra^*)t$. Then ra^* is nonzero if and only if ($r \in S^{NRI}$ and) $r^* = a^*$, in which case $ra^* = r$, and then $rt = 0$ unless $t = a^*$, in which case $(ra^*)a^* = r$. Considering instead $r(a^*t)$, a^*t is nonzero if and only if $t = a^*$, in which case $a^*t = a^*$ and the previous calculation again applies. Finally, supposing $r, s \in S$ and t is a new a^*, then $(rs)a^*$ is nonzero if and only if ($rs \in S^{NRI}$ and) $(rs)^* = a^*$, in which case $(rs)a^* = rs$. Since $rs \in S^{NRI}$, $s \in S^{NRI}$ by Lemma 5.1, and (by the definition) $a^* = (b(rs))^*$ for some $b \in \mathbb{R}_e$. Now $(br)s \in \mathbb{R}_e$, so $s^* = ((br)s)^* = (b(rs))^* = a^*$. It follows that $sa^* = s$ and so $r(sa^*) = rs$ once more.

Therefore S^* is a semigroup. That it is a left restriction semigroup is checked easily, as in the proof of Proposition 5.4.

To this point we have shown that S^* is a left restriction semigroup, clearly primitive, in which each element now has a right identity. By Proposition 5.2, S^* is a restriction semigroup. Next we show that S^* satisfies (8.1). Suppose $x \,\mathbb{R}\, y$ and $x^* = y^*$ in S^*. If both x and y are in S^{RI}, then the original hypothesis applies. The case where one of x and y is in S^{RI} and the other is in S^{NRI} cannot occur, by the construction of the new projections. If $x, y \in S^{NRI}$, let $b \in \mathbb{R}_e$ be such that $bx, by \in \mathbb{R}_e$, so that $(bx)^* = x^* = y^* = (by)^*$. By the construction, $bx = by$. But then (8.4) implies that $x = y$, as required.

As noted following Lemma 8.1, $\mathbb{H}$ is the identical relation on S^*. Therefore, by Result 4.6, $S^* \in (\mathbf{B}_2)_R$.

If the only regular elements of S are projections, the same is true in S^*, for the only new elements are themselves projections. The final statement is therefore a consequence of Lemma 8.4 and, as already remarked following that lemma, [7, Lemma 10.1]. □

Proposition 8.9. *Let S be a primitive left restriction semigroup. Then there exists a primitive left restriction semigroup $\hat{S}$ with the property that if $\hat{S}^{RI}$ satisfies the implication (8.4), so does $\hat{S}$ itself. Further, there exists a surjective homomorphism $\theta\colon \hat{S} \longrightarrow S$ that restricts to an isomorphism from $\hat{S}^{RI}$ to S^{RI}. Thus θ is projection-separating and the submonoids of $\hat{S}$ are isomorphic to those of S. If S has base e, then the inverse image of e is a base for $\hat{S}$.*

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If S satisfies identity (8.1), then so does $\hat{S}$. In that event, $\hat{S}$ also satisfies the implication (8.4). Further, if S satisfies (8.3), then so does $\hat{S}$.

Proof. If $S = S^{RI}$, put $\hat{S} = S$. Now assume otherwise. Let $\hat{S} = U \cup S^{RI}$, where $U = \{(a, x) \in S^{RI} \times S^{NRI} : a \neq 0 \text{ and } ax \,\mathbb{R}\, a\}$. Recall that, by virtue of Lemma 5.1, $ax \,\mathbb{R}\, a$ if and only if $a^* = x^+$, where a^* is defined as in Section 5. Note that U contains (x^+, x) for every $x \in S^{NRI}$.

Extend the operations in S^{RI} to $\hat{S}$ as follows. Put $(a, x)s = 0$ for all $(a, x) \in U$ and $s \in \hat{S}$. Thus $U = \hat{S}^{NRI}$. If $s \in S^{RI}$ and $(a, x) \in U$, put $s(a, x) = (sa, x)$, if $s^* = a^+$, and 0 otherwise. For $(a, x) \in U$, put $(a, x)^+ = a^+ \in S^{RI}$.

Define $\theta \colon \hat{S} \longrightarrow S$ by $(a, x)\theta = ax$, for $(a, x) \in U$, and $s\theta = s$ for all $s \in S^{RI}$. If $x \in S^{NRI}$, then $(x^+, x)\theta = x$, so θ is surjective.

Note that if $s(a, x) \neq 0$, then by Lemma 5.1, $sa \,\mathbb{R}\, s$ (so $sa \neq 0$) and $(sa)^* = a^* = x^+$ (so $sa \in S^{RI}$), whereby $(sa)x \,\mathbb{R}\, sa$ and $s(a, x) \in U$. Thus the product is closed. The only consequential case of associativity to be checked is $(rs)(a, x) = r(s(a, x))$, for $r, s \in S^{RI}$ and $(a, x) \in U$. Here the left-hand side is zero unless $r^* = s^+$ (so that $(rs)^* = s^*$) and $(rs)^* = a^+$, in which case the product is $((rs)a, x)$. The right-hand side is zero unless $s^* = a^+$ (so that $(sa)^+ = s^+$) and $r^* = (sa)^+$, in which case the product is $(r(sa), x)$. Thus equality holds.

Clearly $P_{\hat{S}} = P_S$ and so any nonzero element of $P_{\hat{S}}$ is minimal with respect to this property. To check that it is a left restriction semigroup, first observe that if $(a, x) \in U$, then $a^+(a, x) = (a, x)$ is immediate from the definition. If $s \in S^{RI}$, then $(s^+(a, x))^+ = (a, x)^+ = a^+$, if $s^+ = a^+$, and is 0 otherwise, while the same holds for $(s^+(a, x)^+)^+$. Similarly, $(s(a, x))^+ s = (sa)^+ s = sa^+ = s(a, x)^+$, as long as $s^* = a^+$; otherwise both sides are zero. The remaining cases are trivially verified.

If e is a base projection for S, it is likewise for $\hat{S}$, since for any $(a, x) \in U$, $(a, x)^+ = a^+$.

Suppose $\hat{S}^{RI} = S^{RI}$ satisfies the implication (8.4). Let $s \in S^{RI}$ and $(a, x), (b, y) \in U = \hat{S}^{NRI}$. Suppose $(a, x) \,\mathbb{R}\, (b, y)$, so that $a^+ = b^+$; that $s(a, x) \neq 0$, so that $s^* = a^+$ and $s(b, y) \neq 0$; and that $s(a, x) = s(b, y)$. Then $x = y$ and $sa = sb \neq 0$, so that, by (8.4) for S^{RI}, $a = b$.

To check that θ is a homomorphism, the only case of consequence entails $s \in S^{RI}$ and $(a, x) \in U$. Then $(s\theta)(a, x)\theta = s(ax)$ is nonzero in S if and only if $s^* = (ax)^+ = a^+$, in which case $s(ax) = (sa)x = (s(a, x))\theta$. The other stated properties of θ are obvious.

Next suppose S satisfies identity (8.1). Then it holds in $\hat{S}^{RI} = S^{RI}$. But it holds vacuously otherwise on $\hat{S}$, applying Lemma 8.1(iii). Further, it is clear from Lemma 5.1 that any regular element of $\hat{S}$ must belong to $\hat{S}^{RI} = S^{RI}$. Thus if S satisfies (8.3), so does $\hat{S}$. □

The combination of Propositions 8.8 and 8.9, together with the second statement of Proposition 4.8, completes the proof of the following, the essence of the main theorems of this section.

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Corollary 8.10. *Let S be a primitive left restriction semigroup, with base e, that satisfies identity (8.1). Then S $^+$-divides $(\hat{S})^* \in (\mathbf{B}_2)_R$ and so $S \in \mathbf{B}_2$. If, in addition, S satisfies (8.3), then $(\hat{S})^* \in (\mathbf{B}_0)_R$ and so $S \in \mathbf{B}_0$.*

Theorem 8.11. *The following are equivalent for a left restriction semigroup S:*

(i) $S \in \mathbf{B}_2$;

(ii) *S satisfies identity (8.1), namely $(xz)^+(yz) = (yz)^+(xz)$ (or its equivalent formulations given by Lemma 8.1);*

(iii) *S is a subdirect product of primitive left restriction semigroups (with designated base) that satisfy (8.1).*

Proof. Lemma 8.5 states that (i) implies (ii). Conversely, according to Lemma 8.1, (8.1) implies identity (6.1) and so, by Corollary 6.5, S is a subdirect product of primitive left restriction semigroups, with specified base. By Corollary 8.10, each such semigroup belongs to $\mathbf{B}_2$.

The equivalence with (iii) is then immediate from Theorem 6.6 (noting that only trivial monoids satisfy (8.1)). □

Corollary 8.12. *The following are equivalent for a left restriction semigroup S:*

(i) $S \in \mathbf{B}_0$;

(ii) *S satisfies identity (8.1) together with identity (8.3), namely $xyx = (xyx)^+$.*

Proof. Again, Lemma 8.5 states that (i) implies (ii). The converse follows by the same argument as for $\mathbf{B}_2$, using the paraphrase of (8.3). □

Corollary 8.13. *The following are equivalent for a left restriction semigroup S:*

(i) $S \in \mathbf{B}_0 \vee \mathbf{M}$;

(ii) *S satisfies identity (6.1) together with identity (8.2), namely $xyx = (xyx)(xyx)^+$;*

(iii) *S is a subdirect product of monoids and primitive left restriction semigroups, each having a unique base.*

Proof. To prove the equivalence of (i) and (ii) recall first, from the remarks at the beginning of this section, that $\mathbf{B} = \mathbf{B}_2 \vee \mathbf{M}$. Next, by Result 4.1, $S \in \mathbf{B}_2 \vee \mathbf{M}$ if and only if $S/\mu \in \mathbf{B}_2$ and so if and only if S/μ satisfies (8.1). Similarly, $S \in \mathbf{B}_0 \vee \mathbf{M}$ if and only if $S/\mu \in \mathbf{B}_0$ and so if and only if S/μ satisfies both (8.1) and (8.3).

Let $x, y \in S$, with $S \in \mathbf{B}$. Then by Lemma 6.3, $xyx \,\mu\, (xyx)^+$ if and only if $xyx = (xyx)(xyx)^+$. That is, S/μ satisfies both (8.1) and (8.3) if and only if S satisfies (8.2). Therefore (i) and (ii) are equivalent.

That (ii) implies (iii) is the combination of Corollaries 6.5 and 8.3. To prove the converse, we apply Lemma 8.2. Identity (6.1) is already known. Suppose $a, b \in S$ with $e = a^+, f = b^+, a = af, b = be$. Then $f \in \mathrm{RF}(e)$ and $e \in \mathrm{RF}(f)$. Suppose $g \in P_S$ and $cg = c$ for some $c \in \mathbb{R}_e$. Then $bc \,\mathbb{R}\, be = b$ and $(bc)g = bc$. That is, $\mathrm{RF}(e) \subseteq \mathrm{RF}(f)$. From symmetry it follows that $\mathrm{RF}(e) = \mathrm{RF}(f)$ and so $\rho_e = \rho_f = \rho$, say, with the classes of e and f both bases for S/ρ. By (iii), $e\rho = f\rho$. So there exists $g \in P_S$ such that

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$ge, gf \in \mathrm{RF}(e)$ and $ge = gf$. Now since $ge \in \mathrm{RF}(e)$, $c = cge$ for some $c \in \mathbb{R}_e$. Here e is a common right identity for c and e itself, and g is a right identity for c, so by (6.1), $eg = e$ and, therefore, $e \leq f$. By symmetry, $f = e$. □

Intersecting with $\mathbf{B}_2$ provides another option for Corollary 8.12, via Lemma 8.4.

Corollary 8.14. *A left restriction semigroup belongs to $\mathbf{B}_0$ if and only if it is a subdirect product of primitive left restriction semigroups, each having a unique base and satisfying (8.1).*

We may now deliver the examples promised at the end of Section 4.1, using the restriction semigroups Λ_k that played an essential role in showing [7] that $(\mathbf{B}_0)_R$ is inherently nonfinitely based (for that terminology, see the following section). Let Y be the semilattice obtained by adjoining to the antichain $\{e_1, \ldots, e_{k+1}\}$ both a zero element and another element f such that $f > e_1$ and $f > e_{k+1}$, only. For $e \in Y$, the principal ideal generated by e is denoted Ye. For $k \geq 1$, Λ_k is the subset of $\mathcal{PT}(Y)$ consisting of the identity mappings 1_{Ye}, for $e \in Y$, and partial order isomorphisms $\{\alpha_1, \ldots, \alpha_{k+1}\}$, where $\alpha_n \colon Ye_n \to Ye_{n+1}$, for n odd, and $\alpha_n \colon Ye_{n+1} \to Ye_n$ for n even. Then [7, Proposition 6.6, Theorem 8.1] for each $k \geq 1$, Λ_k is a restriction semigroup (in fact a restriction subsemigroup of the Munn semigroup T_Y) that does not belong to $\mathbf{B}_R$.

Note that Λ_1 is isomorphic to B_0^1, which, regarded as a left restriction semigroup, does not belong to $\mathbf{B}$ since it does not belong to $\mathbf{l}$oc$(\mathbf{SM})$.

Proposition 8.15. *For $k \geq 2$, the restriction semigroups Λ_k are not strict, when regarded as such, but are strict when regarded as left restriction semigroups;in fact they belong to $\mathbf{B}_0$. That is (cf. Proposition 4.8 and the comments that follow it), the inclusion $\mathbf{B}_R \subset \mathbf{B}^R$ is strict.*

Proof. In view of the discussion above, it remains to show that $\Lambda_k \in \mathbf{B}_0$ for $k \geq 2$. For convenience, we will write $\epsilon_i = 1_{Ye_i}$, for $i = 1, \ldots, k+1$, and ϵ_f for 1_{Yf}. By direct calculation (or from [7]) the only nontrivial $\mathbb{R}$- and $\mathbb{L}$-relations are given by $\alpha_i \in \mathbb{R}_{\epsilon_i} \cap \mathbb{L}_{\epsilon_{i+1}}$, if i is odd, and $\alpha_i \in \mathbb{L}_{\epsilon_i} \cap \mathbb{R}_{\epsilon_{i+1}}$, if i is even, for $1 \leq i \leq k$.

As in [7, Section 5], the subsemigroup Δ_k of Λ_k obtained by deleting ϵ_f is a connected restriction semigroup that does belong to $(\mathbf{B}_0)_R$, by an application of Result 4.6, and so belongs to $\mathbf{B}_0$, regarded as a left restriction semigroup. We will apply Corollary 8.12 and the paraphrase of (8.1) that follows Lemma 8.1 to deduce that the same is true for Λ_k. Only the ways in which the additional projection ϵ_f acts as a right identity can affect the desired outcome.

Clearly ϵ_f is a right identity for ϵ_1 and ϵ_{k+1}. If k is even, it acts as a right zero otherwise. If k is odd, then ϵ_f is also a right identity for $\alpha_k \mathrel{\mathbb{R}} \epsilon_k$. In any event, no two $\mathbb{R}$-related elements share ϵ_f as a right identity. Thus the conclusion holds. □

It is straightforward to determine the lattice $\mathcal{L}(\mathbf{B}_2)$.

Theorem 8.16 (Cf. Result 4.7). *The lattice of subvarieties of $\mathbf{B}_2$ consists of the chain of coverings $\mathbf{T} \prec \mathbf{S} \prec \mathbf{D} \prec \mathbf{B}_0 \prec \mathbf{B}_2$.*

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Proof. By Section 4, $\mathbf{T} \prec \mathbf{S} \prec \mathbf{D}$ and this is the only sequence of coverings within $\mathbf{B}_\in$ that starts from $\mathbf{T}$.

Suppose $\mathbf{V}$ is a subvariety of $\mathbf{B}_2$ properly containing $\mathbf{D}$. Applying Corollary 7.7 and the paraphrase of identity (7.2), there is a semigroup S, say, in $\mathbf{V}$ containing projections e, f such that f is a right identity for some $a \in \mathbb{R}_e$, $a \neq e$. The variety $\mathbf{V}$ therefore contains the primitive quotient S/ρ_e, as in Proposition 6.4, in which we may identify e, f and a with their congruence classes. In the subset $\{e, f, a, 0\}$, the only nonzero products are $ea = a, af = f, ee = e$ and $ff = f$. Therefore this subset forms a unary subsemigroup isomorphic to B_0, which therefore belongs to $\mathbf{V}$.

So any subvariety of $\mathbf{B}_2$ that properly contains $\mathbf{D}$ must contain $\mathbf{B}_0$.

Now suppose $\mathbf{V}$ is a subvariety of $\mathbf{B}_2$ properly containing $\mathbf{B}_0$. By Corollary 8.12, similar reasoning to the case above shows that $\mathbf{V}$ contains a primitive left restriction semigroup R, say, in which there exist x, y such that $x = xyx$ and $y = yxy$, but $x \notin P_R$. (We could use either $e = x^+$ or $e = y^+$ for ρ_e.) Applying Lemma 5.1, since $xy, yx \neq 0$, $x^* = y^+ = h$, say, and $y^* = x^+ = g$, say, while $(xy)^+ = x^+ = g$. Now $(xy)(xy)^+ = xyg = xyy^* = xy$, so in fact $xy \in M_g$. From (8.1) and the comments following Lemma 8.1, $xy = g$. Similarly, $yx = h$. Since x, y are not projections, $x^2 = y^2 = 0$. The subset $\{x, y, g, h, 0\}$ therefore forms a unary subsemigroup isomorphic to B_2, which therefore belongs to $\mathbf{V}$.

So $\mathbf{B}_2$ covers $\mathbf{B}_0$ and there are no subvarieties other than those listed. □

By Proposition 4.5, both L_2^1 and D^1 generate varieties that are not contained in $\mathbf{B}$. Thus, even in the context of varieties that are not unions of monoids, $\mathbf{B}_0$ is not the only cover of $\mathbf{D}$.

Corollary 8.17. *The interval* $[\mathbf{M}, \mathbf{B}]$ *consists of the chain of coverings* $\mathbf{M} \prec \mathbf{SM} \prec \mathbf{D} \vee \mathbf{M} \prec \mathbf{B}_0 \vee \mathbf{M} \prec \mathbf{B}_2 \vee \mathbf{M} = \mathbf{B}$.

Proof. In combination with Corollary 7.9, this is immediate from Result 4.1 upon confirmation that $B_2 \notin \mathbf{B}_0 \vee \mathbf{M}$ and $B_0 \notin \mathbf{D} \vee \mathbf{M}$, by a simple check of the relevant identities. □

9. Inherent nonfinite basability

A variety of algebras is *finitely based* if it has a finite basis of identities, and is *nonfinitely based* otherwise. An individual algebra A is finitely, or nonfinitely, based if the respective property holds for the variety it generates. The algebra A is *inherently nonfinitely based* (often abbreviated as INFB) if any locally finite variety of algebras that contains A has no finite basis of identities. If A is finite, then the variety it generates is locally finite, and so if A is INFB, then it is necessarily nonfinitely based.

Among other results in this paper, we have therefore shown that the left restriction semigroups B_2 and B_0 are finitely based as left restriction semigroups (and therefore as unary semigroups). This is in stark contrast to the two-sided situation where [7] in each case the semigroup is not only nonfinitely based, but INFB, regarded as a restriction semigroup. As shown in [7], it follows that *any* finite restriction semigroup that is not simply a semilattice of monoids is INFB.

While the theorem below, due to Marcel Jackson in a private communication, plays no other role herein, it demonstrates that, *a priori*, the extreme nonfinite basability demonstrated for two-sided restriction semigroups cannot occur in the one-sided situation.

We first need the following well-known lemma. Although deducible from descriptions of the free left restriction semigroup, it is worthwhile presenting an elementary proof.

LEMMA 9.1. *Let T be a left restriction semigroup, generated as such by the set X. Denote by U the 'plain' subsemigroup generated by X. Then every element of T is expressible in the form $w_1^+ \dots w_n^+ w$, where each $w_i \in U$ and $w \in U^1$. Thus T is generated, as a semigroup, by the union of U with the subsemilattice of projections generated by $\{u^+ : u \in U\}$.*

PROOF. Let V be the set of such products. Let $u_1, \dots, u_m, v_1, \dots, v_n \in U$ and $u, v \in U^1$. With $u \neq 1$ being the only case of interest, repeated application $uv_i^+ = (uv_i)^+ u$ of the left ample identity yields $(u_1^+ \cdots u_m^+ u)(v_1^+ \cdots v_n^+ v) = u_1^+ \cdots u_m^+ (uv_1)^+ \cdots (uv_n)^+ uv$. Also $(u_1^+ \cdots u_n^+ u)^+ = u_1^+ \cdots u_n^+ u^+$, so V is a unary subsemigroup of T that contains X and is therefore all of T. The last statement is then clear. □

THEOREM 9.2 (Jackson). *If a left restriction semigroup $(S, \cdot, ^+)$ is INFB (as a unary semigroup), then its semigroup reduct $(S, \cdot)$ is also INFB (as a 'plain' semigroup).*

PROOF. Suppose the conclusion is false. Then $(S, \cdot)$ belongs to a locally finite, finitely based, variety $\mathbf{V}$ of semigroups. Let $\mathbf{W}$ be the variety of left restriction semigroups defined, as such, by the same set of 'plain' identities that defines $\mathbf{V}$. Clearly $\mathbf{W}$ is finitely based as a unary semigroup variety and contains $(S, \cdot, ^+)$.

Let $(T, \cdot, ^+)$ be a member of $\mathbf{W}$ that is generated, as a unary semigroup, by the finite set X. As in the lemma, let V be the plain subsemigroup of T generated by X. Now V satisfies the identities of $\mathbf{V}$ and so is finite. By the last statement of the lemma, T itself is therefore finite. Hence the variety $\mathbf{W}$ is locally finite and so $(S, \cdot, ^+)$ cannot be INFB. □

The key semigroups D, B_0 and B_2, for instance, are known to be finitely based as plain semigroups, since that is true for all semigroups of order less than six [14]. Therefore, *a priori* they cannot be INFB.

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References

[1] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra* (Springer, Berlin, 1981).

[2] V. A. R. Gould, Notes on restriction semigroups and related structures (unpublished notes, available at www-users.york.ac.uk/~varg1/restriction.pdf).

[3] V. A. R. Gould and C. Hollings, 'Restriction semigroups and inductive constellations', *Comm. Algebra* **38** (2010), 261–287.

[4] C. Hollings, 'From right *PP* monoids to restriction semigroups: a survey', *Eur. J. Pure Appl. Math.* **2** (2009), 21–57.

[5] J. M. Howie, *Fundamentals of Semigroup Theory*, London Mathematical Society Monographs (Clarendon Press, Oxford, 1995).

[6] P. R. Jones, 'On lattices of varieties of restriction semigroups', *Semigroup Forum* **86** (2013), 337–361.

[7] P. R. Jones, 'The semigroups B_2 and B_0 are inherently nonfinitely based, as restriction semigroups', *Int. J. Algebra Comput.* **23** (2013), 1289–1335.

[8] P. R. Jones, 'Varieties of restriction semigroups and varieties of categories', *Comm. Algebra* **45** (2017), 1037–1056.

[9] P. R. Jones, 'The lattice of varieties of strict left restriction semigroups', *J. Aust. Math. Soc.* to appear.

[10] M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, De Gruyter Expositions in Mathematics, 29 (De Gruyter, Berlin, 2000).

[11] E. W. H. Lee, 'Subvarieties of the variety generated by the five-element Brandt semigroup', *Int. J. Algebra Comput.* **16** (2006), 417–441.

[12] M. Petrich, *Inverse Semigroups* (Wiley, New York, 1984).

[13] B. Tilson, 'Categories as algebra: an essential ingredient in the theory of monoids', *J. Pure Appl. Algebra* **48** (1987), 83–198.

[14] A. N. Trahtman, 'Finiteness of identity bases of five-element semigroups', in: *Semigroups and Their Homomorphisms* (ed. E. S. Lyapin) (Leningrad State Pedagogical Institute, Leningrad, 1991), 76–97 (in Russian).

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