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EXISTENCE THEORY FOR NONRESONANT SINGULAR BOUNDARY VALUE PROBLEMS

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We present some existence results for the "nonresonant" singular boundary value problem $\frac{1}{pq}(py')'+\mu y=f(t,y)$ a.e. on $[0,1]$ with $\lim_{t\to 0^+} p(t)y'(t)=y(1)=0$. Here μ is such that $\frac{1}{pq}(pu')'+\mu u=0$ a.e. on $[0,1]$ with $\lim_{t\to 0^+} p(t)u'(t)=u(1)=0$ has only the trivial solution.

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1. Introduction

This paper establishes existence results for the "nonresonant" singular boundary value problem

$$\begin{cases} \frac{1}{p(t)q(t)}(p(t)y'(t))'+\mu y(t)=f(t,y(t)) \quad \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t)=0 \\ y(1)=0 \end{cases} \tag{1.1}$$

where μ is such that

$$\begin{cases} \frac{1}{pq}(py')'+\mu y(t)=0 \quad \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t)=0 \\ y(1)=0 \end{cases} \tag{1.2}$$

has only the trivial solution. Throughout the paper $p\in C[0,1]\cap C^1(0,1)$ together with $p>0$ on $(0,1)$; also q is measurable with $q>0$ a.e. on $[0,1]$ and $\int_0^1 p(x)q(x)\,dx<\infty$.

Remarks. (i). Throughout the condition $y(1)=0$ could be replaced by the more general condition $ay(1)+b\lim_{t\to 1^-} p(t)y'(t)=0, a>0, b\geqq 0$.
(ii). We do **not** assume $\int_0^1 \frac{ds}{p(s)}<\infty$.

In addition $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ will be a *Carathéodory* function. By this we mean:
(i). $t\to f(t,y)$ is measurable for all $y\in\mathbf{R}$
(ii). $y\to f(t,y)$ is a continuous for a.e. $t\in[0,1]$.
For notional purposes let w be a weight function. By $L_w^r[0,1], r\geqq 1$ we mean the space of functions u such that $\int_0^1 w(t)|u(t)|^r dt<\infty$. In particular $L_w^2[0,1]$ denotes the space of functions u such that $\int_0^1 w(t)|u(t)|^2 dt<\infty$; also for $u,v\in L_w^2[0,1]$ define $\langle u,v\rangle=$

$\int_0^1 w(t)u(t)\overline{v(t)}\,dt$. Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

This paper will be divided into three main sections. Section 2 discusses the linear problem i.e. (1.1) with $f\equiv 0$. In Section 3 fixed point methods (in particular a nonlinear alternative of Leray–Schauder type) is used to obtain an existence principle. The final section establishes some existence results for (1.1); these results extend and complement the theory in [**4**, **6**, **21**].

Finally we remark here that problems of type (1.1) occur in many applications in the physical sciences, for example in radially symmetric nonlinear diffusion [**20**, **22**] in the n-dimensional sphere we have $p(t)=t^{n-1}$; these problems involve a homogeneous Neumann condition at zero i.e. $\lim_{t\to 0^+} t^{n-1}y'(t)=0$. Another important example is the Poisson–Boltzmann equation

$$\begin{cases} y''+\frac{\alpha}{t}y'=f(t,y), 0<t<1 \\ y'(0^+)=y(1)=0, \alpha\geqq 1 \end{cases} \tag{1.3}$$

which occurs in the theory of thermal explosions [**3**] and in the theory of electrohydrodynamics [**11**]. The results related to (1.3) in the literature [**4**] usually consider the situation when $\inf\frac{\partial f}{\partial y}$, $\sup\frac{\partial f}{\partial y}$ are bounded and satisfy a "nonresonant" condition; here the infimum and supremum are taken over $\{(t,y):0\leqq t\leqq 1, -\infty<y<\infty\}$. In this paper we improve the above existence result; in fact in our theory the existence of $\frac{\partial f}{\partial y}$ is not assumed.

2. Linear problem

Theorem 2.1. *Suppose*

$$p\in C[0,1]\cap C^1(0,1) \textit{ with } p>0 \textit{ on } (0,1) \tag{2.1}$$

$$q\in L^1_p[0,1] \textit{ with } q>0 \textit{ a.e. on } [0,1] \tag{2.2}$$

and

$$\int_0^1 \frac{1}{p(s)}\left(\int_0^s p(x)q(x)\,dx\right)^{1/\alpha} ds<\infty \textit{ for some constant } \alpha>1 \tag{2.3}$$

are satisfied.

(i) *Then*

$$\begin{cases} \frac{1}{p}(py')'+\mu q y=0 \quad \textit{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t)=0 \\ y(0)=a_0\neq 0 \end{cases} \tag{2.4}$$

has a solution $y_1\in C[0,1]\cap C^1(0,1)$ with $py_1'\in AC[0,1]$. (By a solution to (2.4) we mean a function $y\in C[0,1]\cap C^1(0,1)$, $py'\in AC[0,1]$ which satisfies the differential equation a.e. on $[0,1]$ and the stated conditions.)

(ii) *Then*

$$\begin{cases} \frac{1}{p}(py')' + \mu q y = 0 \quad \textit{a.e. on } [0,1] \\ y(1) = 0 \\ \lim_{t\to 1^-} p(t)y'(t) = 1 \end{cases} \tag{2.5}$$

has a solution $y_2 \in L^{\alpha}_{pq}[0,1]$ *with* $y_2 \in C(0,1] \cap C^1(0,1)$ *and* $py_2' \in AC[0,1]$.

Proof. (i). Let $C[0,1]$ denote the Banach space of continuous functions on $[0,1]$ endowed with the norm

$$|u|_K = \sup_{t\in[0,1]} |e^{-KR(t)}u(t)| \quad \text{where} \quad R(t) = \int_0^t p(x)q(x)\,dx$$

and

$$K = \frac{1}{\beta}\left(|\mu| \int_0^1 \frac{1}{p(s)}\left(\int_0^s p(x)q(x)\,dx\right)^{1/\alpha} ds\right)^{\beta}.$$

Remark. Here $\beta = \frac{\alpha}{\alpha-1}$ i.e. β and α are conjugate exponents.

Solving (2.4) is equivalent to finding $y \in C[0,1]$ which satisfies

$$y(t) = a_0 - \mu \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)y(x)\,dx\,ds.$$

Define the operator $N: C[0,1] \to C[0,1]$ by

$$Ny(t) = a_0 - \mu \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)y(x)\,dx\,ds.$$

Now N is a contraction since

$$\begin{aligned} |Nu - Nv|_K &\leq |\mu|\,|u-v|_K \sup_{t\in[0,1]} e^{-KR(t)} \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)e^{KR(x)}\,dx\,ds \\ &\leq |\mu|\,|u-v|_K \sup_{t\in[0,1]} e^{-KR(t)} \int_0^t \frac{1}{p(s)} \left(\int_0^s pq\,dx\right)^{1/\alpha} \left(\int_0^s pq e^{\beta KR(x)}\,dx\right)^{1/\beta} ds \\ &\leq |\mu|\,|u-v|_K \sup_{t\in[0,1]} e^{-KR(t)} \int_0^t \frac{1}{p(s)} \left(\int_0^s p(x)q(x)\,dx\right)^{1/\alpha} \left(\frac{e^{\beta KR(s)}}{\beta K} - \frac{1}{\beta K}\right)^{1/\beta} ds \\ &\leq \frac{|\mu|}{(\beta K)^{1/\beta}} |u-v|_K \sup_{t\in[0,1]} e^{-KR(t)} (e^1\{\beta KR(t)\} - 1)^{1/\beta} \int_0^t \left(\int_0^s p(x)q(x)\,dx\right)^{1/\alpha} ds \\ &\leq (1 - e^{-\beta KR(1)})^{1/\beta} |u-v|_K \end{aligned}$$

using Hölder's integral inequality. The Banach contraction principle now establishes the result.

(ii). Let $L^{\alpha}_{pq}[0,1]$ denote the Banach space of functions u, with $\int_0^1 pq|u|^{\alpha}\,dt<\infty$, endowed with the norm

$$\|u\|_K=\left(\int_0^1 p(t)q(t)e^{-KQ(t)}|u(t)|^{\alpha}\,dt\right)^{1/\alpha}\quad\text{where } Q(t)=\int_t^1 p(x)q(x)\,dx$$

and

$$K=\frac{\alpha}{\beta}\left(|\mu|^{\alpha}\int_0^1 p(t)q(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\alpha}dt\right)^{\beta/\alpha}\quad\text{where } \beta=\frac{\alpha}{\alpha-1}.$$

Remarks. (i). Notice for example that $\int_{1/2}^1 \frac{ds}{p(s)}<\infty$ since

$$\int_{1/2}^1\frac{ds}{p(s)}=\int_{1/2}^1\frac{(\int_0^s p(x)q(x)\,dx)^{1/\alpha}}{p(s)(\int_0^s p(x)q(x)\,dx)^{1/\alpha}}ds\leqq\frac{1}{(\int_0^{1/2}p(x)q(x)\,dx)^{1/\alpha}}\int_{1/2}^1\frac{1}{p(s)}\left(\int_0^s p(x)q(x)\,dx\right)^{1/\alpha}ds.$$

(ii). Notice (2.3) implies

$$\int_0^1 p(t)q(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\alpha}dt<\infty. \tag{2.6}$$

To see this let

$$g(t)=\left(\int_t^1\frac{ds}{p(s)}\right)^{\alpha-1}$$

and fix $\varepsilon, 0<\varepsilon<1$. Interchange the order of integration and use Hölder's inequality to obtain

$$\int_\varepsilon^1 p(t)q(t)g(t)\int_t^1\frac{ds}{p(s)}dt=\int_\varepsilon^1\frac{1}{p(s)}\int_\varepsilon^s p(t)q(t)g(t)\,dt$$

$$\leqq\left(\int_\varepsilon^1 p(t)q(t)g^{\beta}(t)\,dt\right)^{1/\beta}\int_\varepsilon^1\frac{1}{p(s)}\left(\int_\varepsilon^s p(t)q(t)\,dt\right)^{1/\alpha}ds.$$

Consequently

$$\int_\varepsilon^1 p(t)q(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\alpha}dt\leqq\left(\int_\varepsilon^1 p(t)q(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\alpha}dt\right)^{1/\beta}\int_\varepsilon^1\frac{1}{p(s)}\left(\int_\varepsilon^s p(t)q(t)\,dt\right)^{1/\alpha}ds.$$

We will show that

$$y(t)=-\int_t^1 \frac{ds}{p(s)}-\mu\int_t^1 \frac{1}{p(s)}\int_s^1 p(x)q(x)y(x)\,dx\,ds \tag{2.7}$$

has a solution $y_2\in L^\alpha_{pq}[0,1]$. Also we will show $y_2\in C(0,1]\cap C^1(0,1)$ and $py_2'\in AC[0,1]$ and consequently y_2 will be a solution of (2.5).

Define the operator: $L^\alpha_{pq}[0,1]\to L^\alpha_{pq}[0,1]$ by

$$My(t)=-\int_t^1 \frac{ds}{p(s)}-\mu\int_t^1 \frac{1}{p(s)}\int_s^1 p(x)q(x)y(x)\,dx\,ds.$$

Remark. M is well defined because of (2.6) and

$$\int_0^1 pq\left(\int_t^1 \frac{1}{p}\int_s^1 pq|y|\,dx\,ds\right)^\alpha dt\leqq\left(\int_0^1 pq|y|\,dx\right)^\alpha\int_0^1 pq\left(\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

$$\leqq\left(\int_0^1 pq|y|^\alpha\,dx\right)\left(\int_0^1 pq\,dx\right)^{\alpha/\beta}\int_0^1 pq\left(\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

for any $y\in L^\alpha_{pq}[0,1]$.

Now M is a contraction since

$$\|Mu-Mv\|_K^\alpha\leqq|\mu|^\alpha\int_0^1 pqe^{-KQ(t)}\left(\int_t^1 \frac{1}{p(s)}\int_s^1 p(x)q(x)|u(x)-v(x)|\,dx\,ds\right)^\alpha dt$$

$$\leqq|\mu|^\alpha\int_0^1 pqe^{-KQ(t)}\left(\int_t^1 pqe^{-KQ(x)/\alpha}e^{KQ(x)/\alpha}|u(x)-v(x)|dx\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

$$\leqq|\mu|^\alpha\|u-v\|_K^\alpha\int_0^1 pqe^{-KQ(t)}\left(\int_t^1 pqe^{\beta KQ(x)/\alpha}\,dx\right)^{\alpha/\beta}\left(\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

$$\leqq|\mu|^\alpha\|u-v\|_K^\alpha\int_0^1 pqe^{-KQ(t)}\left(\frac{\alpha}{\beta K}e^{\beta KQ(t)/\alpha}-\frac{\alpha}{\beta K}\right)^{\alpha/\beta}\left(\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

$$\leqq\left(\frac{\alpha}{\beta K}\right)^{\alpha/\beta}|\mu|^\alpha\|u-v\|_K^\alpha\int_0^1 pq\left(1-e^{-\beta KQ(t)/\alpha}\right)^{\alpha/\beta}\left(\int_t^1 \frac{ds}{p(s)}\right)^\alpha dt$$

$$\leqq\left(1-e^{-\beta KQ(0)/\alpha}\right)^{\alpha/\beta}\|u-v\|_K^\alpha.$$

The Banach contraction principle now establishes that (2.7) has a solution $y_2 \in L^{\alpha}_{pq}[0,1]$. Also

$$p(t)y_2'(t) = 1 + \mu \int_t^1 p(x)q(x)y_2(x)\,dx$$

so $py_2' \in AC[0,1]$ since $y_2 \in L^{\alpha}_{pq}[0,1]$ implies $pqy_2 \in L^1[0,1]$. Thus y_2 is a solution of (2.5). □

Consider now

$$\frac{1}{pq}(py')' + \mu y = h(t) \quad \text{a.e. on } [0,1] \tag{2.8}$$

where (2.1), (2.2), (2.3) and

$$h \in L^{\beta}_{pq}[0,1];\ \text{here } \beta = \frac{\alpha}{\alpha - 1} \tag{2.9}$$

hold.

Theorem 2.2. *Suppose* (2.1), (2.2), (2.3) *and* (2.9) *are satisfied. In addition μ is such that* (1.2) *has only the* **trivial** *solution. Then*

$$\begin{cases} \frac{1}{pq}(py')' + \mu y = h(t) \quad \text{a.e. on } [0,1] \\ \lim_{t \to 0^+} p(t)y'(t) = 0 \\ y(1) = 0 \end{cases} \tag{2.10}$$

has exactly one solution y (note $y \in L^{\alpha}_{pq}[0,1]$ with $y \in C(0,1] \cap C^1(0,1)$ and $py' \in AC[0,1]$) given by

$$y(t) = \int_0^1 G(t,s)q(s)h(s)\,ds \tag{2.11}$$

where $G(t,s)$ is the Green's function i.e.

$$G(t,s) = \begin{cases} \dfrac{y_1(s)y_2(t)}{W(s)} = c_0 p(s)y_1(s)y_2(t), 0 < s \leqq t \\[2ex] \dfrac{y_1(t)y_2(s)}{W(s)} = c_0 p(s)y_2(s)y_1(t), 0 \leqq s < 1. \end{cases}$$

Here y_1 and y_2 are as described in Theorem 2.1 *and $W(s)$ is the Wronksian of y_1 and y_2 at s and notice $p(s)W(s) = (1/c_0) \neq 0$ for $s \in [0,1]$.*

Proof. This follows the standard construction of the Green's function; see [**22, 24**] for example. We will just justify that $p(s)W(s)\neq 0$ for $s\in[0,1]$. To see this all one needs to show is that $y_1(1)\neq 0$. If $y_1(1)=0$ then y_1 satisfies (1.2) and consequently $y_1\equiv 0$. This contradicts the fact that $y(0)=a_0\neq 0$. □

Remark. Notice y in (2.11) is in $L^{\alpha}_{pq}[0,1]$ since

$$\int_0^1 p(t)q(t)\left(\int_t^1 p(s)q|y_2(s)y_1(t)h(s)|\,ds\right)^{\alpha} dt \leqq \int_0^1 pq|y_1|^{\alpha}\left(\int_t^1 pq|y_2|^{\alpha}\,ds\right)\left(\int_t^1 pq|h|^{\beta}\,ds\right)^{\alpha/\beta} ds$$

and so

$$\int_0^1 p(t)q(t)\left(\int_0^t p(s)q(s)|y_1(s)y_2(t)h(s)|\,ds\right)^{\alpha} dt<\infty.$$

3. Existence principle

We use a nonlinear alternative of Leray–Schauder type [**9**] to establish our existence principle. By a map being *compact* we mean it is continuous with relatively compact range. A map is *completely continuous* if it is continuous and the image of every bounded set in the domain is contained in a compact set in the range.

Theorem 3.1. *Assume U is a relatively open subset of a convex set K in a Banach space E. Let $N:\bar{U}\to K$ be a compact map with $0\in U$. Then either*
(i) *N has a fixed point in $\bar{U}$; or*
(ii) *there is a $u\in\partial U$ and a $\lambda\in(0,1)$ such that $u=\lambda Nu$.*

Next we gather some well known results [**12**] from the theory of nonlinear integral equations.

Theorem 3.2. *Let $\alpha>1$ be a constant and $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function. Define the operator*

$$Fy(t)=f(t,y(t))$$

and suppose $F:L^{\alpha}_{pq}[0,1]\to L^{\beta}_{pq}[0,1]$; here $\beta=\frac{\alpha}{\alpha-1}$. Then F is continuous and bounded.

Theorem 3.3. *Consider the linear integral operator*

$$Ay(t)=\int_0^1 p(s)q(s)k(t,s)y(s)\,ds$$

with

$$\int_0^1 p(t)q(t)\int_0^1 p(s)q(s)|k(t,s)|^\alpha\,ds\,dt<\infty \text{ for some } \alpha>1. \tag{3.1}$$

Then $A: L^\beta_{pq}[0,1]\to L^\alpha_{pq}[0,1]$, $\beta=\frac{\alpha}{\alpha-1}$ *is completely continuous.*

We next prove an existence principle for (1.1).

Theorem 3.4. *Let* $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ *be a Carethéodory function and suppose* (2.1), (2.2) *and* (2.3) *are satisfied. Also suppose*

$$f(t,y(t))\in L^\beta_{pq}[0,1] \quad \text{whenever} \quad y\in L^\alpha_{pq}[0,1]; \quad \text{here} \quad \beta=\frac{\alpha}{\alpha-1}. \tag{3.2}$$

In addition μ *is such that* (1.2) *has only the* **trivial** *solution. Now suppose there is a constant* M_0, *independent of* λ, *with*

$$\|y\|=\left(\int_0^1 p(t)q(t)|y(t)|^\alpha\,dt\right)^{1/\alpha}\leqq M_0$$

for any solution y (*here* $y\in L^\alpha_{pq}[0,1]$ *with* $y\in C(0,1]\cap C^1(0,1)$ *and* $py'\in AC[0,1]$) *to*

$$\begin{cases}\frac{1}{pq}(py')'+\mu y=\lambda f(t,y) \quad \text{a.e. on } [0,1]\\ \lim_{t\to 0^+}p(t)y'(t)=0\\ y(1)=0\end{cases} \tag{3.3$_\lambda$}$$

for each $\lambda\in(0,1)$. *Then* (1.1) *has at least one solution.*

Proof. Solving $(3.3)_\lambda$ is equivalent to finding $y\in L^\alpha_{pq}[0,1]$ which satisfies

$$y(t)=\lambda\int_0^1 p(s)q(s)k(t,s)f(s,y(s))\,ds \tag{3.4}$$

where

$$k(t,s)=\begin{cases}c_0y_1(s)y_2(t),\ 0<s\leqq t\\ c_0y_2(s)y_1(t),\ t\leqq s<1,\end{cases}$$

and y_1,y_2,c_0 are described in Theorem 2.2. Define the operator $N: L^\alpha_{pq}[0,1]\to L^\alpha_{pq}[0,1]$ by

$$Ny(t)=\int_0^1 p(s)q(s)k(t,s)f(s,y(s)\,ds.$$

Remark. N is well defined since

$$\int_0^1 pq\left(\int_t^1 pq|y_1(t)y_2(s)f(s,y)|\,ds\right)^\alpha dt \leq \int_0^1 pq|y_1|^\alpha\left(\int_t^1 pq|y_2|^\alpha\,ds\right)\left(\int_t^1 pq|f(s,y)|^\beta\,ds\right)^{\alpha/\beta} dt$$

and so

$$\int_0^1 p(t)q(t)\left(\int_0^t p(s)q(s)|y_1(s)y_2(t)f(s,y(s))|\,ds\right)^\alpha dt < \infty.$$

Next define $F: L^\alpha_{pq}[0,1] \to L^\beta_{pq}[0,1]$ by

$$Fy(t) = f(t, y(t))$$

and $A: L^\beta_{pq}[0,1] \to L^\alpha_{pq}[0,1]$ by

$$Ay(t) = \int_0^1 p(s)q(s)k(t,s)y(s)\,ds.$$

Notice (3.2) and Theorem 3.2 implies F is bounded and continuous. A is completely continuous by Theorem 3.3.

Remark. Notice $\int_0^1 p(t)q(t)\int_0^1 p(s)q(s)|k(t,s)|^\alpha\,ds\,dt < \infty$ since

$$\int_0^1 p(t)q(t)\left(\int_0^t p(s)q(s)|y_1(s)y_2(t)|^\alpha\,ds\,dt \leq \int_0^1 p(t)q(t)|y_2(t)|^\alpha \int_0^1 p(s)q(s)|y_1(s)|^\alpha\,ds\,dt < \infty\right.$$

and so

$$\int_0^1 p(t)q(t)\left(\int_t^1 p(s)q(s)|y_2(s)y_1(t)|^\alpha\,ds\,dt < \infty.\right.$$

Consequently $N = AF: L^\alpha_{pq}[0,1] \to L^\alpha_{pq}[0,1]$ is completely continuous. Set

$$U = \{u \in L^\alpha_{pq}[0,1] : \|u\| < M_0 + 1\}, K = E = L^\alpha_{pq}[0,1].$$

Then Theorem 3.1 implies that N has a fixed point i.e. (1.1) has a solution $y \in L^\alpha_{pq}[0,1]$. The fact that $y \in C(0,1] \cap C^1(0,1)$ with $py' \in AC[0,1]$ follows from (3.4) with $\lambda = 1$. □

4. Existence theory

Theorem 4.1. *Let* $f:[0,1] \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function and suppose* (2.1), (2.2) *and* (2.3) *are satisfied. In addition* μ *is such that* (1.2) *has only the* **trivial** *solution. Let* $\beta=\frac{\alpha}{\alpha-1}$. *Now assume*

$$\begin{cases} |f(t,u)|\leqq\phi_1(t)+\phi_2(t)\psi(|u|) \text{ a.e. on } [0,1] \text{ where } \phi_1^\beta,\phi_2\in L^1_{pq}[0,1] \\ \text{and } \psi:[0,\infty)\to[0,\infty) \text{ is a continuous function} \end{cases} \tag{4.1}$$

$$\begin{cases} \text{there exists } Q_0\geqq 0 \text{ and a continuous function } \theta:[0,\infty)\to[0,\infty) \text{ with} \\ \int_0^1 p(s)q(s)\phi_2^\beta(s)\psi^\beta(|y(s)|)\,ds\leqq Q_0\theta(\|y\|) \text{ for any } y\in L^\alpha_{pq}[0,1]; \\ \text{here } \|y\|=(\int_0^1 p(t)q(t)|y(t)|^\alpha\,dt)^{1/\alpha} \end{cases} \tag{4.2}$$

and

$$\begin{cases} A_0\equiv 2^{q_0}c_0^\alpha Q_0^{\alpha/\beta}\|y_2\|^\alpha\|y_1\|^\alpha \limsup\limits_{x\to\infty}\dfrac{(\theta(x))^{\alpha/\beta}}{x^\alpha}<1 \text{ where } y_1,y_2,c_0 \\ \text{are as described in Theorem 2.2 and } q_0=\dfrac{2\alpha^2\beta-\beta^2-\alpha^2+\alpha\beta}{\alpha\beta} \end{cases} \tag{4.3}$$

are satisfied. Then (1.1) *has at least one solution.*

Remarks. (i). Notice (3.2) is automatically satisfied since (4.2) holds and also since $\phi_1^\beta\in L^1_{pq}[0,1]$.

(ii). If $\psi(|u|)=|u|^\gamma$, $0\leqq\gamma<\min\{\frac{\alpha}{\beta},1\}$ and $\phi_2^{\beta\alpha/(\alpha-\beta\gamma)}\in L^1_{pq}[0,1]$ then (4.2) and (4.3) are satisfied since

$$\int_0^1 p(s)q(s)\phi_2^\beta(s)|y(s)|^{\beta\gamma}\,ds\leqq\|y\|^{\beta\gamma}\left(\int_0^1 pq\phi_2^{\beta\alpha/(\alpha-\beta\gamma)}\,ds\right)^{(\alpha-\beta\gamma)/\alpha} \text{ for any } y\in L^\alpha_{pq}[0,1]$$

and so with $\theta(x)=x^{\beta\gamma}$ we have

$$\limsup_{x\to\infty}\frac{(\theta(x))^{\alpha/\beta}}{x^\alpha}=\limsup_{x\to\infty}x^{\alpha(\gamma-1)}=0.$$

Proof. Let y be a solution to $(3.3)_\lambda$ for $0<\lambda<1$. Then

$$y(t)=\lambda c_0y_2(t)\int_0^t p(s)q(s)y_1(s)f(s,y(s))\,ds+\lambda c_0y_1(t)\int_t^1 p(s)q(s)f(s,y(s))\,ds$$

where y_1,y_2,c_0 are as described in Theorem 2.2. Recall $(a_0+b_0)^{r_0}\leqq 2^{r_0-1}(a_0^{r_0}+b_0^{r_0}), a_0\geqq 0, b_0\geqq 0, r_0\geqq 1$ so

$$\|y\|^\alpha\leqq 2^{\alpha-1}c_0^\alpha\int_0^1 p(t)q(t)|y_2(t)|^\alpha\left(\int_0^t p(s)q(s)|y_1(s)||f(s,y(s))|\,ds\right)^\alpha dt$$
$$+2^{\alpha-1}c_0^\alpha\int_0^1 p(t)q(t)|y_1(t)|^\alpha\left(\int_t^1 p(s)q(s)|y_2(s)||f(s,y(s))|\,ds\right)^\alpha dt.$$

This together with Hölder's inequality implies

$$\|y\|^{\alpha} \leqq 2^{\alpha} c_0^{\alpha} \|y_2\|^{\alpha} \|y_1\|^{\alpha} \left(\int_0^1 p(s)q(s)|f(s, y(s))|^{\beta} \, ds \right)^{\alpha/\beta}. \tag{4.4}$$

In addition

$$\int_0^1 p(s)q(s)|f(s,(s))|^{\beta} \, ds \leqq 2^{\beta-1} \int_0^1 p(s)q(s)\phi_1^{\beta}(s) \, ds + 2^{\beta-1} \int_0^1 p(s)q(s)\phi_2^{\beta}(s)\psi^{\beta}(|y(s)|) \, ds$$
$$\leqq 2^{\beta-1} \int_0^1 p(s)q(s)\phi_1^{\beta}(s) \, ds + 2^{\beta-1} Q_0 \theta(\|y\|).$$

This inequality together with $(a_0+b_0)^{1/r_0} \leqq 2^{(r_0-1)/r_0}(a_0^{1/r_0}+b_0^{1/r_0}), a_0 \geqq 0, b_0 \geqq 0, r_0 \geqq 1$ or $(a_0+b_0)^{s_0} \leqq 2^{s_0-1}(a_0^{s_0}+b_0^{s_0}), s_0 \geqq 1$ and (4.4) implies

$$\|y\|^{\alpha} \leqq 2^{\alpha} c_0^{\alpha} \|y_2\|^{\alpha} \|y_1\|^{\alpha} 2^{(\alpha-\beta)/\alpha} \left(2^{\alpha(\beta-1)/\beta} \left(\int_0^1 pq\phi_1^{\beta} \, ds \right)^{\alpha/\beta} + 2^{\alpha(\beta-1)/\beta} Q_0^{\alpha/\beta} (\theta(\|y\|))^{\alpha/\beta} \right). \tag{4.5}$$

Consequently

$$1 \leqq 2^{q_0} c_0^{\alpha} \|y_2\|^{\alpha} \|y_1\|^{\alpha} \left(\frac{(\int_0^1 pq\phi_1^{\beta} \, ds)^{\alpha/\beta}}{\|y\|^{\alpha}} + \frac{Q_0^{\alpha/\beta}(\theta(\|y\|))^{\alpha/\beta}}{\|y\|^{\alpha}} \right). \tag{4.6}$$

Thus there exists a constant M_0, independent of λ, with $\|y\| \leqq M_0$ for any solution y satisfying $(3.3)_{\lambda}$ i.e. $y = \lambda N y$ where N is as described in Theorem 3.4. If this was not true then there exists $u_n = \lambda_n N u_n$ with $\|u_n\| \to \infty$ as $n \to \infty$ and since $\limsup(s_n + t_n) \leqq \limsup s_n + \limsup t_n$ for any sequences $s_n \geqq 0, t_n \geqq 0$ we have from (4.6) that $1 \leqq A_0$, a contradiction (see (4.3)). Thus there exists a constant M_0, independent of λ, with $\|y\| \leqq M_0$ and the result now follows from Theorem 3.4. □

The next two existence results extend in a "particular direction" Theorem 4.1 if certain criteria are fulfilled. To discuss the first result we begin by gathering together some facts on the singular eigenvalue problem

$$\begin{cases} Lu = \lambda u \text{ a.e. on } [0,1] \\ \lim_{t \to 0^+} p(t)u'(t) = 0 \\ u(1) = 0 \end{cases} \tag{4.7}$$

where $Lu = -\frac{1}{pq}(pu')'$. Assume (2.1), (2.2) and

$$\int_0^1 \frac{1}{p(s)} \left(\int_0^s p(x)q(x) \, dx \right)^{1/2} ds < \infty \tag{4.8}$$

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hold.

Remarks. (i). In this case $\alpha=2$ in (2.3).
(ii). Here $t=0$ is a singular point in the limit circle case [**18**, **19**, **24**].

Let

$$D(L)=\left\{\omega\in C[0,1]: w, pw'\in AC[0,1] \text{ with } \frac{1}{pq}(pw')'\in L^2_{pq}[0,1]\right.$$

$$\left.\text{and } \lim_{t\to 0^+} p(t)w'(t)=w(1)=0\right\}.$$

In [**18**, **19**] it was shown that $L^{-1}: L^2_{pq}[0,1]\to D(L)$ and L^{-1} is completely continuous with $\langle L^{-1}u, v\rangle=\langle u, L^{-1}v\rangle$ for $u, v\in L^2_{pq}[0,1]$. Consequently the spectral theorem for compact self adjoint operators [**24**] implies that L has a countably infinite number of real eigenvalues λ_i with corresponding eigenfunctions $\psi_i\in D(L)$. The eigenfunctions ψ_i may be chosen so that they form an orthonormal set and we may arrange the eigenvalues so that

$$\lambda_0<\lambda_1<\lambda_2<\dots.$$

The following Rayleigh–Ritz minimization theorem [**18**, **19**] also holds.

Theorem 4.2. *Suppose* (2.1), (2.2) *and* (4.8) *hold. Then*

$$\lambda_0\int_0^1 p(t)q(t)y^2(t)\,dt\leqq\int_0^1 p(t)[y'(t)]^2\,dt$$

for all functions $y\in D(L)$.

We can improve the result in Theorem 4.1 if (4.8) holds and if $\mu<\lambda_0$; here λ_0 is the first eigenvalue of (4.7). In particular consider

$$\begin{cases}\frac{1}{pq}(py')'=f(t,y) \text{ a.e. on } [0,1]\\ \lim_{t\to 0^+} p(t)y'(t)=0\\ y(1)=0.\end{cases}\tag{4.9}$$

Theorem 4.3. *Let* $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ *be a Carathéodory function and suppose* (2.1), (2.2) *and* (4.8) *are satisfied. Also assume*

$$f(t,y(t))\in L^2_{pq}[0,1] \text{ whenever } y\in L^2_{pq}[0,1]. \tag{4.10}$$

In addition suppose $f(t,u)=g(t,u)+h(t,u)$ *with* $g,h:[0,1]\times\mathbf{R}\to\mathbf{R}$ *Carathéodory functions and*

$$\begin{cases}|uh(t,u)|\leqq\phi_1(t)|u|+\phi_2(t)\rho(|u|) \text{ a.e. on } [0,1] \text{ where } \rho:[0,\infty)\to[0,\infty)\\ \text{is a nondecreasing continuous function}\end{cases} \tag{4.11}$$

$$ug(t,u)\geqq-\mu_0u^2 \text{ for a.e. } t\in[0,1] \text{ and } u\in\mathbf{R}; \text{ here } \mu_0<\lambda_0 \tag{4.12}$$

$$\int_0^1 p(t)q(t)\phi_1(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{1/2}dt<\infty \text{ and } \int_0^1 p(t)q(t)\phi_2(t)\rho\left(\left(\int_t^1\frac{ds}{p(s)}\right)^{1/2}\right)dt<\infty \tag{4.13}$$

$$\begin{cases}\text{there exist constants } Q_1 \text{ (independent of } a_0 \text{ and } b_0) \text{ and } Q_2 \text{ such that for any}\\ a_0\geqq0, b_0\geqq0 \text{ we have } \rho(a_0b_0)\leqq Q_1\rho(a_0)\rho(b_0)+Q_2\rho(b_0)\end{cases} \tag{4.14}$$

and

$$\begin{cases}A_1\equiv Q_1\left(\int_0^1 p(t)q(t)\phi_2(t)\rho\left(\left(\int_t^1\frac{ds}{p(s)}\right)^{1/2}\right)dt\right)\limsup_{x\to\infty}\frac{\rho(x)}{x^2}<\eta_0\\ \text{with } \eta_0=1 \text{ if } \mu_0<0 \text{ whereas } \eta_0=1-\frac{\mu_0}{\lambda_0} \text{ if } 0\leqq\mu_0<\lambda_0\end{cases} \tag{4.15}$$

are satisfied. Then (4.9) *has at least one solution.*

Remark. If $\rho(|u|)=|u|^{\gamma+1}, 0\leqq\gamma<1$ and $\int_0^1 p(t)q(t)\phi_2(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{(\gamma+1)/2}dt<\infty$ then (4.13), (4.14) and (4.15) are satisfied since if $Q_1=1, Q_2=0$ we have $\rho(a_0b_0)=|a_0b_0|^{\gamma+1}=|a_0|^{\gamma+1}|b_0|^{\gamma+1}$ and also

$$\limsup_{x\to\infty}\frac{\rho(x)}{x^2}=\limsup_{x\to\infty}x^{\gamma-1}=0.$$

Proof. Let y be a solution to

$$\begin{cases}\frac{1}{pq}(py')'=\lambda f(t,y) \text{ a.e. on } [0,1]\\ \lim_{t\to0^+}p(t)y'(t)=0\\ y(1)=0\end{cases} \tag{4.16$_\lambda$}$$

for $0<\lambda<1$. Multiply the differential equation by $-y$ and integrate from 0 to 1 to obtain

$$\|y'\|_0^2 \leqq -\lambda \int_0^1 pqyg(t,y)\,dt + \int_0^1 pq|yh(t,y)|\,dt$$

$$\leqq \lambda\mu_0\|y\|^2 + \int_0^1 pq[\phi_1(t)|y(t)| + \phi_1(t)\rho(|y(t)|)]\,dt$$

where for notational purposes $\|u\|^2 = \int_0^1 pq|u|^2\,dt$ and $\|u\|_0^2 = \int_0^1 p|u|^2\,dt$. Apply Theorem 4.2 if $0 \leqq \mu_0 < \lambda_0$ to obtain

$$\eta_0\|y'\|_0^2 \leqq \int_0^1 pq\phi_1|y|\,dt + \int_0^1 pq\phi_2\rho(|y(t)|)\,dt \tag{4.17}$$

where η_0 is as described in (4.15). Also for $t \in (0,1)$ we have from Hölder's inequality that

$$|y(t)| \leqq \|y'\|_0 \left(\int_t^1 \frac{ds}{p(s)}\right)^{1/2} \tag{4.18}$$

and this together with (4.17) and the fact that ρ is nondecreasing yields

$$\eta_0\|y'\|_0^2 \leqq N_1\|y'\|_0 + \int_0^1 p(t)q(t)\phi_2(t)\rho\left(\|y'\|_0\left(\int_t^1 \frac{ds}{p(s)}\right)^{1/2}\right)dt$$

where $N_1 = \int_0^1 pq\phi_1(\int_t^1 \frac{ds}{p(s)})^{1/2}\,dt$. Using (4.14) we obtain

$$\eta_0\|y'\|_0^2 \leqq N_1\|y'\|_0 + Q_1N_2\rho(\|y'\|_0) + Q_2N_2$$

where $N_2 = \int_0^1 pq\phi_2\rho((\int_t^1 \frac{ds}{p(s)})^{1/2})\,dt$. Consequently

$$\eta_0 \leqq \frac{N_1\|y'\|_0 + Q_2N_2}{\|y'\|_0^2} + \frac{Q_1N_2\rho(\|y'\|_0)}{\|y'\|_0^2}.$$

Thus (as in Theorem 4.1) exists a constant M_1, independent of λ, with $\|y'\|_0 \leqq M_1$ for any solution y to $(4.16)_\lambda$. This together with Theorem 4.2 yields

$$\int_0^1 pq|y|^2\,dt \leqq \frac{1}{\lambda_0}M_1^2$$

so the result follows from Theorem 3.4 (with $\mu = 0$ and $\alpha = 2$). □

Finally we examine the boundary value problem (4.9) where in the case $pqf: [0,1] \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function. By this we mean:

(i) $pqf: [0,1] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, and

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(ii) for any $r>0$ there exists $h_r \in L^1[0,1]$ with $|p(t)q(t)f(t,u)| \leqq h_r(t)$ for a.e. $t \in [0,1]$ and for all $|u| \leqq r$.

For the remainder of the paper assume (2.1), (2.2) and

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\,dx\,ds < \infty \tag{4.19}$$

and

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)h_r(x)\,dx\,ds < \infty \text{ for any } r>0\text{; here } h_r \text{ is as described above} \tag{4.20}$$

hold. In [**8**, **18**] we proved the following existence principle.

Theorem 4.4. *Let $pqf:[0,1]\times\mathbf{R}\to\mathbf{R}$ be a L^1-Carathéodory function with* (2.1), (2.2), (4.19) *and* (4.20) *holding. In addition suppose there is a constant M_0, independent of λ, with*

$$|y|_0 = \sup_{[0,1]} |y(t)| \leqq M_0$$

for any solution y (here $y \in C[0,1] \cap C^1(0,1)$ with $py' \in AC[0,1]$) to

$$\begin{cases} \frac{1}{pq}(py')' = \lambda f(t,y) \text{ a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ y(1) = 0 \end{cases} \tag{4.21$_\lambda$}$$

for each $0<\lambda<1$. Then (4.9) *has at least one solution.*

Theorem 4.5. *Let $pqf:[0,1]\times\mathbf{R}\to\mathbf{R}$ be a L^1-Carathéodory function with* (2.1), (2.2) *and* (4.19) *holding. In addition suppose*

$$\begin{cases} |f(t,u)| \leqq \phi_1(t) + \phi_2(t)\Omega(|u|) \text{ a.e. on } [0,1] \text{ where } \Omega:[0,\infty)\to[0,\infty) \\ \text{is a nondecreasing continuous function} \end{cases} \tag{4.22}$$

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\phi_i(x)\,dx\,ds < \infty, i=1,2 \tag{4.23}$$

and

$$\left(\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\phi_2(x)\,dx\,ds\right) \limsup_{x\to\infty} \frac{\Omega(x)}{x} < 1 \tag{4.24}$$

are satisfied. Then (4.9) *has at least one solution.*

Proof. Let y be a solution to $(4.21)_\lambda$ for $0<\lambda<1$. Then for $t\in[0,1]$ we have

$$y(t)=-\int_t^1 \frac{1}{p(s)}\int_0^s p(x)q(x)f(x,y(x))\,dx\,ds$$

and so

$$|y(x)|\leqq\int_t^1 \frac{1}{p(s)}\int_0^s p(x)q(x)\phi_1(x)\,dx\,ds+\int_t^1 \frac{1}{p(s)}\int_0^s p(x)q(x)\phi_2(x)\Omega(|y(x)|)\,dx\,ds.$$

Now $|y(x)|\leqq\sup_{[0,1]}|y(s)|\equiv|y|_0$ and this together with the fact that Ω is nondecreasing yields

$$|y(t)|\leqq\int_0^1 \frac{1}{p(s)}\int_0^s p(x)q(x)\phi_1(x)\,dx\,ds+\Omega(|y|_0)\int_0^1 \frac{1}{p(s)}\int_0^s p(x)q(x)\phi_2(x)\,dx\,ds.$$

Let $K_i=\int_0^1 \frac{1}{p(s)}\int_0^s p(x)q(x)\phi_i(x)\,dx\,ds, i=1,2$ so

$$|y|_0\leqq K_1+K_2\Omega(|y|_0)$$

and consequently

$$1\leqq\frac{K_1}{|y|_0}+\frac{K_2\Omega(|y|_0)}{|y|_0}.$$

Thus (as in Theorem 4.1) there exists a constant M_0, independent of λ, with $|y|_0\leqq M_0$ for any solution y to $(4.21)_\lambda$ The result follows from Theorem 4.4. □

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