

CHARACTERIZATION OF DOMAINS IN \mathbb{C}^n BY THEIR NONCOMPACT AUTOMORPHISM GROUPS

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Abstract. In this paper, the characterization of domains in \mathbb{C}^n by their noncompact automorphism groups are given.

§1. Introduction

Let Ω be a domain, i.e. connected open subset, in a complex manifold M . Let the *automorphism group* of Ω (denoted $Aut(\Omega)$) be the collection of biholomorphic self-maps of Ω with composition of mappings as its binary operation. The topology on $Aut(\Omega)$ is that of uniform convergence on compact sets (i.e., the compact-open topology).

One of the important problems in several complex variables is to study the interplay between the geometry of a domain and the structure of its automorphism group. More precisely, we wish to see to what extent a domain is determined by its automorphism group.

It is a standard and classical result of H. Cartan that if Ω is a bounded domain in \mathbb{C}^n and the automorphism group of Ω is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in Aut(\Omega)$ such that $\varphi_j(x) \rightarrow p$. In this circumstance we call p a *boundary orbit accumulation point*.

Works in the past twenty years has suggested that the local geometry of the so-called “boundary orbit accumulation point” p in turn gives global information about the characterization of model of the domain. We refer readers to the recent survey [13] and the references therein for the development in related subjects. For instance, B. Wong and J. P. Rosay (see [18], [19]) proved the following theorem.

Received August 28, 2007.

Revised February 6, 2008.

Accepted June 22, 2009.

1991 Mathematics Subject Classification: Primary 32M05; Secondary 32H02, 32H15, 32H50.

WONG-ROSAY THEOREM. *Any bounded domain $\Omega \Subset \mathbb{C}^n$ with a C^2 strongly pseudoconvex boundary orbit accumulation point is biholomorphic to the unit ball in \mathbb{C}^n .*

By using the scaling technique, introduced by S. Pinchuk [16], E. Bedford and S. Pinchuk [2] proved the theorem about the characterization of the complex ellipsoids.

BEDFORD-PINCHUK THEOREM. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded pseudoconvex domain of finite type whose boundary is smooth of class C^∞ , and suppose that the Levi form has rank at least $n - 1$ at each point of the boundary. If $\text{Aut}(\Omega)$ is noncompact, then Ω is biholomorphically equivalent to the domain*

$$E_m = \{(w, z_1, \dots, z_n) \in \mathbb{C}^{n+1} : |w|^2 + |z_1|^{2m} + |z_2|^2 + \dots + |z_n|^2 < 1\},$$

for some integer $m \geq 1$.

We would like to emphasize here that the assumption on boundedness of domains in the above-mentioned theorem is essential in their proofs. It seems to us that some key techniques in their proofs could not use for unbounded domains in \mathbb{C}^n . Thus, there is a natural question that whether the Bedford-Pinchuk theorem is true for any domain in \mathbb{C}^n . In 1994, F. Berteloot [6] gave a partial answer to this question in dimension 2.

BERTELOOT THEOREM. *Let Ω be a domain in \mathbb{C}^2 and let $\xi_0 \in \partial\Omega$. Assume that there exists a sequence (φ_p) in $\text{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\lim \varphi_p(a) = \xi_0$. If $\partial\Omega$ is pseudoconvex and of finite type near ξ_0 then Ω is biholomorphically equivalent to $\{(w, z) \in \mathbb{C}^2 : \text{Re } w + H(z, \bar{z}) < 0\}$, where H is a homogeneous subharmonic polynomial on \mathbb{C} with degree $2m$.*

The main aim in this paper is to show that the above theorems of Bedford-Pinchuk and Berteloot hold for domains (not necessary bounded) in \mathbb{C}^n . Namely, we prove the following.

THEOREM 1.1. *Let Ω be a domain in \mathbb{C}^n and let $\xi_0 \in \partial\Omega$. Assume that*

- (a) $\partial\Omega$ is pseudoconvex, of finite type and smooth of class C^∞ in some neighbourhood of $\xi_0 \in \partial\Omega$.
- (b) The Levi form has rank at least $n - 2$ at ξ_0 .

- (c) *There exists a sequence (φ_p) in $Aut(\Omega)$ such that $\lim \varphi_p(a) = \xi_0$ for some $a \in \Omega$.*

Then Ω is biholomorphically equivalent to a domain of the form

$$M_H = \left\{ (w_1, \dots, w_n) \in \mathbb{C}^n : \operatorname{Re} w_n + H(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\},$$

where H is a homogeneous subharmonic polynomial with $\Delta H \neq 0$.

NOTATIONS.

- $\mathcal{H}(\omega, \Omega)$ is the set of holomorphic mappings from ω to Ω .
- f_p is u.c.c on ω means that the sequence $(f_p), f_p \in \mathcal{H}(\omega, \Omega)$, uniformly converges on compact subsets of ω .
- \mathcal{P}_{2m} is the space of real valued polynomials on \mathbb{C} with degree less than $2m$ and which do not contain any harmonic terms.
- $\mathcal{H}_{2m} = \{H \in \mathcal{P}_{2m} \text{ such that } \deg H = 2m \text{ and } H \text{ is homogeneous and subharmonic}\}$.
- $M_Q = \{z \in \mathbb{C}^n : \operatorname{Re} z_n + Q(z_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\}$ where $Q \in \mathcal{P}_{2m}$.
- $\Omega_1 \simeq \Omega_2$ means that Ω_1 and Ω_2 are biholomorphic equivalent.

The paper is organized as follows. In Section 2, we review some basic notions needed later. In Section 3, we describe the construction of polydiscs around points near the boundary of a domain, and give some of their properties. In particular, we use the Scaling method to show that Ω is biholomorphic to a model M_P with $P \in \mathcal{P}_{2m}$. In Section 4, we end the proof of our theorem by using the Berteloot’s method.

Acknowledgement. We would like to thank Professor François Berteloot for his precious discussions on this material. Especially, we would like to express our gratitude to the referee. His/her valuable comments on the first version of this paper led to significant improvements.

§2. Definitions and results

First of all, we recall the following definition (see [12]).

DEFINITION 2.1. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of open sets in a complex manifold M and Ω_0 be an open set of M . The sequence $\{\Omega_i\}_{i=1}^\infty$ is said to converge to Ω_0 , written $\lim \Omega_i = \Omega_0$ iff

- (i) For any compact set $K \subset \Omega_0$, there is a $i_0 = i_0(K)$ such that $i \geq i_0$ implies $K \subset \Omega_i$, and
- (ii) If K is a compact set which is contained in Ω_i for all sufficiently large i , then $K \subset \Omega_0$.

The following proposition is the generalization of the theorem of H. Cartan (see [12], [17] for more generalizations of this theorem).

PROPOSITION 2.1. *Let $\{A_i\}_{i=1}^\infty$ and $\{\Omega_i\}_{i=1}^\infty$ be sequences of domains in a complex manifold M with $\lim A_i = A_0$ and $\lim \Omega_i = \Omega_0$ for some (uniquely determined) domains A_0, Ω_0 in M . Suppose that $\{f_i : A_i \rightarrow \Omega_i\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_i : A_i \rightarrow M\}$ converges uniformly on compact subsets of A_0 to a holomorphic map $F : A_0 \rightarrow M$ and the sequence $\{g_i := f_i^{-1} : \Omega_i \rightarrow M\}$ converges uniformly on compact subsets of Ω_0 to a holomorphic map $G : \Omega_0 \rightarrow M$. Then one of the following two assertions holds.*

- (i) *The sequence $\{f_i\}$ is compactly divergent, i.e., for each compact set $K \subset \Omega_0$ and each compact set $L \subset \Omega_0$, there exists an integer i_0 such that $f_i(K) \cap L = \emptyset$ for $i \geq i_0$, or*
- (ii) *There exists a subsequence $\{f_{i_j}\} \subset \{f_i\}$ such that the sequence $\{f_{i_j}\}$ converges uniformly on compact subsets of A_0 to a biholomorphic map $F : A_0 \rightarrow \Omega_0$.*

Proof. Assume that the sequence $\{f_i\}$ is not divergent. Then F maps some point p of A_0 into Ω_0 . We will show that F is a biholomorphism of A_0 onto Ω_0 . Let $q = F(p)$. Then

$$G(q) = G(F(p)) = \lim_{i \rightarrow \infty} g_i(F(p)) = \lim_{i \rightarrow \infty} g_i(f_i(p)) = p.$$

Take a neighbourhood V of p in A_0 such that $F(V) \subset \Omega_0$. But then uniform convergence allows us to conclude that, for all $z \in V$, it holds that $G(F(z)) = \lim_{i \rightarrow \infty} g_i(f_i(z)) = z$. Hence $F|_V$ is injective. By the Osgood's theorem, the mapping $F|_V : V \rightarrow F(V)$ is biholomorphic.

Consider the holomorphic functions $J_i : A_i \rightarrow \mathbb{C}$ and $J : A_0 \rightarrow \mathbb{C}$ given by $J_i(z) = \det((df_i)_z)$ and $J(z) = \det((dF)_z)$. Then $J(z) \neq 0$ ($z \in V$) and,

for each $i = 1, 2, \dots$, the function J_i is non-vanishing on A_i . Moreover, the sequence $\{J_i\}_{i=0}^\infty$ converges uniformly on compact subsets of A_0 to J . By Hurwitz's theorem, it follows that J never vanishes. This implies that the mapping $F : A_0 \rightarrow M$ is open and any $z \in A_0$ is isolated in $F^{-1}(F(z))$. According to Proposition 5 in [15], we have $F(A_0) \subset \Omega_0$.

Of course this entire argument may be repeated to see that $G(\Omega_0) \subset A_0$. But then uniform convergence allows us to conclude that, for all $z \in A_0$, it holds that $G \circ F(z) = \lim_{i \rightarrow \infty} g_i(f_i(z)) = z$ and likewise for all $w \in \Omega_0$ it holds that $F \circ G(w) = \lim_{i \rightarrow \infty} f_i(g_i(w)) = w$.

This proves that F and G are each one-to-one and onto, hence in particular that F is a biholomorphic mapping. □

Next, by Proposition 2.1 in [6], we have the following.

PROPOSITION 2.2. *Let M be a domain in a complex manifold X of dimension n and $\xi_0 \in \partial M$. Assume that ∂M is pseudoconvex and of finite type near ξ_0 .*

- (a) *Let Ω be a domain in a complex manifold Y of dimension m . Then every sequence $\{\varphi_p\} \subset \text{Hol}(\Omega, M)$ converges uniformly on compact subsets of Ω to ξ_0 if and only if $\lim \varphi_p(a) = \xi_0$ for some $a \in \Omega$.*
- (b) *Assume, moreover, that there exists a sequence $\{\varphi_p\} \subset \text{Aut}(M)$ such that $\lim \varphi_p(a) = \xi_0$ for some $a \in M$. Then M is taut.*

Proof. Since ∂M is pseudoconvex and of finite type near $\xi_0 \in \partial M$, there exists a local peak plurisubharmonic function at ξ_0 (see [9]). Moreover, since ∂M is smooth and pseudoconvex near ξ_0 , there exists a small ball B centered at ξ_0 such that $B \cap M$ is hyperconvex and therefore is taut. The theorem is deduced from Proposition 2.1 in [6]. □

Remark 2.1. By Proposition 2.2 and by the hypothesis of Theorem 1.1, for each compact subset $K \subset M$ and each neighbourhood U of ξ_0 , there exists an integer p_0 such that $\varphi_p(K) \subset M \cap U$ for every $p \geq p_0$.

Remark 2.2. By Proposition 2.2 and by the hypothesis of Theorem 1.1, M is taut.

The following lemma is a slightly modification of Lemma 2.3 in [6].

LEMMA 2.3. *Let σ_∞ be a subharmonic function of class C^2 on \mathbb{C} such that $\sigma_\infty(0) = 0$ and $\int_{\mathbb{C}} \bar{\partial}\partial\sigma_\infty = +\infty$. Let $(\sigma_k)_k$ be a sequence of subharmonic functions on \mathbb{C} which converges uniformly on compact subsets of \mathbb{C} to σ_∞ . Let ω be any domain in a complex manifold of dimension m ($m \geq 1$) and let z_0 be fixed in ω . Denote by M_k the domain in \mathbb{C}^n defined by*

$$M_k = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \text{Im } z_1 + \sigma_k(z_2) + |z_3|^2 + \dots + |z_n|^2 < 0\}.$$

Then any sequence $h_k \in \text{Hol}(\omega, M_k)$ such that $\{h_k(z_0), k \geq 0\} \subseteq M_\infty$ admits some subsequence which converges uniformly on compact subsets of ω to some element of $\text{Hol}(\omega, M_\infty)$.

§3. Estimates of Kobayashi metric of the domains in \mathbb{C}^n

In this section we use the Catlin’s argument in [8] to study special coordinates and polydiscs. After that, we improve Berteloot’s technique in [7] to construct a dilation sequence, estimate the Kobayashi metric and prove the normality of a family of holomorphic mappings.

3.1. Special coordinates and polydiscs

Let Ω be a domain in \mathbb{C}^n . Suppose that $\partial\Omega$ is pseudoconvex, finite type and is smooth of class C^∞ near a boundary point $\xi_0 \in \partial\Omega$ and suppose that the Levi form has rank at least $n - 2$ at ξ_0 . We may assume that $\xi_0 = 0$ and the rank of Levi form at ξ_0 is exactly $n - 2$. Let r be a smooth defining function for Ω . Note that the type m at ξ_0 is an even integer in this case. We also assume that $\frac{\partial r}{\partial z_n}(z) \neq 0$ for all z in a small neighborhood U about ξ_0 . After a linear change of coordinates, we can find coordinate functions z_1, \dots, z_n defined on U such that

$$(3.1) \quad L_n = \frac{\partial}{\partial z_n}, \quad L_j = \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial z_n}, \quad L_j r \equiv 0, \quad b_j(\xi_0) = 0, \quad j = 1, \dots, n - 1,$$

which form a basis of $\mathbb{C}T^{(1,0)}(U)$ and satisfy

$$(3.2) \quad \partial\bar{\partial}r(q)(L_i, \bar{L}_j) = \delta_{ij}, \quad 2 \leq i, j \leq n - 1,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

We want to show that about each point $z' = (z'_1, \dots, z'_n)$ in U , there is a polydisc of maximal size on which the function $r(z)$ changes by no more than some prescribed small number δ . First, we construct the coordinates

about z' introduced by S. Cho (see also in [9]). These coordinates will be used to define the polydisc.

Let us take the coordinate functions z_1, \dots, z_n about ξ_0 so that (3.2) holds. Therefore $|L_n r(z)| \geq c > 0$ for all $z \in U$, and $\partial \bar{\partial} r(z)(L_i, \bar{L}_j)_{2 \leq i, j \leq n-1}$ has $(n - 2)$ -positive eigenvalues in U where

$$L_n = \frac{\partial}{\partial z_n}, \quad \text{and}$$

$$L_j = \frac{\partial}{\partial z_j} - \left(\frac{\partial r}{\partial z_n} \right)^{-1} \frac{\partial r(z')}{\partial z_j} \frac{\partial}{\partial z_n}, \quad j = 1, \dots, n - 1.$$

For each $z' \in U$, define new coordinate functions u_1, \dots, u_n defined by $z = \varphi_1(u)$

$$z_n = z'_n + u_n - \sum_{j=1}^{n-1} \left[\left(\frac{\partial r}{\partial z_n} \right)^{-1} \frac{\partial r(z')}{\partial z_j} \right] u_j,$$

$$z_j = z'_j + u_j, \quad j = 1, \dots, n - 1.$$

Then L_j can be written as $L_j = \frac{\partial}{\partial u_j} + b'_j \frac{\partial}{\partial u_n}$, $j = 1, \dots, n - 1$, where $b'_j(z') = 0$. In u_1, \dots, u_n coordinates, $A = \left(\frac{\partial^2 r(z')}{\partial u_i \partial \bar{u}_j} \right)_{2 \leq i, j \leq n-1}$ is an hermitian matrix and there is a unitary matrix $P = (P_{ij})_{2 \leq i, j \leq n-1}$ such that $P^* A P = D$, where D is a diagonal matrix whose entries are positive eigenvalues of A .

Define $u = \varphi_2(v)$ by

$$u_1 = v_1, \quad u_n = v_n, \quad \text{and}$$

$$u_j = \sum_{k=2}^{n-1} \bar{P}_{jk} v_k, \quad j = 2, \dots, n - 1.$$

Then $\frac{\partial^2 r(z')}{\partial v_i \partial \bar{v}_j} = \lambda_i \delta_{ij}$, $2 \leq i, j \leq n - 1$, where $\lambda_i > 0$ is an i -th entry of D (we may assume that $\lambda_i \geq c > 0$ in U for all i). Next we define $v = \varphi_3(w)$ by

$$v_1 = w_1, \quad v_n = w_n, \quad \text{and}$$

$$v_j = \lambda_j w_j, \quad j = 2, \dots, n - 1.$$

Then $\frac{\partial^2 r(z')}{\partial w_i \partial \bar{w}_j} = \delta_{ij}$, $2 \leq i, j \leq n - 1$ and $r(w)$ can be written as

$$\begin{aligned}
 (3.3) \quad r(w) &= r(z') + \operatorname{Re} w_n + \sum_{\alpha=2}^{n-1} \sum_{1 \leq j \leq \frac{m}{2}} \operatorname{Re} [(a_j^\alpha w_1^j + b_j^\alpha \bar{w}_1^j) w_\alpha] + \operatorname{Re} \sum_{\alpha=2}^{n-1} c_\alpha w_\alpha^2 \\
 &+ \sum_{2 \leq j+k \leq m} a_{j,k} w_1^j \bar{w}_1^k + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} \operatorname{Re} (b_{j,k}^\alpha w_1^j \bar{w}_1^k w_\alpha) \\
 &+ O(|w_n||w| + |w^*|^2|w| + |w^*|^2|w_1|^{\frac{m}{2}+1} + |w_1|^{m+1}),
 \end{aligned}$$

where $w^* = (0, w_2, \dots, w_{n-1}, 0)$. It is standard to perform the change of coordinates $w = \varphi_4(t)$

$$\begin{aligned}
 w_n &= t_n - \sum_{2 \leq k \leq m} \frac{2}{k!} \frac{\partial^k r(0)}{\partial w_1^k} t_1^k \\
 &\quad - \sum_{\alpha=2}^{n-1} \sum_{1 \leq k \leq \frac{m}{2}} \frac{2}{(k+1)!} \frac{\partial^{k+1} r(0)}{\partial w_\alpha \partial w_1^k} t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} \frac{\partial^2 r(0)}{\partial w_\alpha^2} t_\alpha^2, \\
 w_j &= t_j, \quad j = 1, \dots, n - 1,
 \end{aligned}$$

which serves to remove the pure terms from (3.3), i.e., it removes $w_1^k, \bar{w}_1^k, w_\alpha^2$ terms as well as $w_1^k w_\alpha, \bar{w}_1^k \bar{w}_\alpha$ terms from the summation in (3.3).

We may also perform a change of coordinates $t = \varphi_5(\zeta)$ defined by

$$\begin{aligned}
 t_1 &= \zeta_1, \quad t_n = \zeta_n, \\
 t_\alpha &= \zeta_\alpha - \sum_{1 \leq k \leq \frac{m}{2}} \frac{1}{(k+1)!} \frac{\partial^{k+1} r(0)}{\partial \bar{t}_\alpha \partial t_1^k} \zeta_1^k, \quad \alpha = 2, \dots, n - 1
 \end{aligned}$$

to remove terms of the form $\bar{w}_1^j w_\alpha$ from the summation in (3.3) and hence $r(\zeta)$ has the desired expression as in (3.4) in ζ -coordinates.

Thus, we obtain the following Proposition (see also in [10, Prop. 2.2, p. 806]).

PROPOSITION 3.1. (S. Cho) *For each $z' \in U$ and positive even integer*

m , there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z = \Phi_{z'}^{-1}(\zeta_1, \dots, \zeta_n)$ such that

$$\begin{aligned}
 (3.4) \quad r(\Phi_{z'}^{-1}(\zeta)) &= r(z') + Re \zeta_n + \sum_{\substack{j+k \leq m \\ j,k > 0}} a_{jk}(z') \zeta_1^j \bar{\zeta}_1^k \\
 &+ \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2 + \sum_{\alpha=2}^{n-1} Re \left(\left(\sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} b_{jk}^\alpha(z') \zeta_1^j \bar{\zeta}_1^k \right) \zeta_\alpha \right) \\
 &+ O(|\zeta_n| |\zeta| + |\zeta^*|^2 |\zeta| + |\zeta^*|^2 |\zeta_1|^{\frac{m}{2}+1} + |\zeta_1|^{m+1}),
 \end{aligned}$$

where $\zeta^* = (0, \zeta_2, \dots, \zeta_{n-1}, 0)$.

Remark 3.1. The coordinate changes as above are unique and hence the map $\Phi_{z'}$ is defined uniquely.

We now show how to define the polydisc around z' . Set

$$\begin{aligned}
 (3.5) \quad A_l(z') &= \max\{|a_{j,k}(z')|, j+k=l\} \quad (2 \leq l \leq m), \\
 B_{l'}(z') &= \max\{|b_{j,k}^\alpha(z')|, j+k=l', 2 \leq \alpha \leq n-1\} \quad \left(2 \leq l' \leq \frac{m}{2}\right).
 \end{aligned}$$

For each $\delta > 0$, we define $\tau(z', \delta)$ as follows

$$(3.6) \quad \tau(z', \delta) = \min \left\{ (\delta/A_l(z'))^{1/l}, (\delta^{1/2}/B_{l'}(z'))^{1/l'}, 2 \leq l \leq m, 2 \leq l' \leq \frac{m}{2} \right\}.$$

Since the type of $\partial\Omega$ at ξ_0 equals m and the Levi form has rank at least $n-2$ at ξ_0 , $A_m(\xi_0) \neq 0$. Hence if U is sufficiently small, then $|A_m(z')| \geq c > 0$ for all $z' \in U$. This gives the inequality

$$(3.7) \quad \delta^{1/2} \lesssim \tau(z', \delta) \lesssim \delta^{1/m} \quad (z' \in U).$$

The definition of $\tau(z', \delta)$ easily implies that if $\delta' < \delta''$, then

$$(3.8) \quad (\delta'/\delta'')^{1/2} \tau(z', \delta'') \leq \tau(z', \delta') \leq (\delta'/\delta'')^{1/m} \tau(z', \delta'').$$

Now set $\tau_1(z', \delta) = \tau(z', \delta) = \tau$, $\tau_2(z', \delta) = \dots = \tau_{n-1}(z', \delta) = \delta^{1/2}$, $\tau_n(z', \delta) = \delta$ and define

$$(3.9) \quad R(z', \delta) = \{\zeta \in \mathbb{C}^n : |\zeta_k| < \tau_k(z', \delta), k = 1, \dots, n\}$$

and

$$(3.10) \quad Q(z', \delta) = \{\Phi_{z'}^{-1}(\zeta) : \zeta \in R(z', \delta)\}.$$

In the sequel we denote D_k^l any partial derivative operator of the form $\frac{\partial}{\partial \zeta_k^\mu} \frac{\partial}{\partial \zeta_k^\nu}$, where $\mu + \nu = l$, $k = 1, 2, \dots, n$.

In order to prove the homogeneous property of $Q(z', \delta)$ we need two lemmas.

LEMMA 3.2. ([10, Prop. 2.3, p. 807]) *Let z' be an arbitrary point in U . Then the function $\rho(\zeta) = r(\Phi_{z'}^{-1}(\zeta))$ satisfies*

$$(3.11) \quad \begin{aligned} |\rho(\zeta) - \rho(0)| &\lesssim \delta \\ |D_k^i D_1^l \rho(\zeta)| &\lesssim \delta \tau_1(z', \delta)^{-l} \tau_k(z', \delta)^{-i}, \end{aligned}$$

for $\zeta \in R(z', \delta)$ and $l + \frac{im}{2} \leq m$, $i = 0, 1$; $k = 2, \dots, n - 1$.

LEMMA 3.3. ([10, Cor. 2.8, p. 812]) *Suppose that $z \in Q(z', \delta)$. Then*

$$(3.12) \quad \tau(z, \delta) \approx \tau(z', \delta).$$

We now apply Lemma 3.3 to the question of how the polydiscs $Q(z', \delta)$ and $Q(z'', \delta)$ are related. Let $\Phi_{z'}^{-1}$ be the map associated with z' as in Proposition 3.1. Define ζ'' by $z'' = \Phi_{z'}^{-1}(\zeta'')$. Applying Proposition 3.1 at the point ζ'' with r replaced by $\rho = r \circ \Phi_{z'}^{-1}$, we obtain a map $\Phi_{\zeta''}^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $\Phi_{\zeta''}^{-1} = \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \varphi_4 \circ \varphi_5$ where $z = \varphi_1(u)$ defined by

$$\begin{aligned} z_n &= \zeta''_n + u_n + \sum_{j=1}^{n-1} b_j u_j, \\ z_j &= \zeta''_j + u_j, \quad j = 1, \dots, n - 1, \end{aligned}$$

$u = \varphi_2(v)$ defined by

$$\begin{aligned} u_1 &= v_1, \quad u_n = v_n, \quad \text{and} \\ u_j &= \sum_{k=2}^{n-1} \bar{P}_{jk} v_k, \quad j = 2, \dots, n - 1, \end{aligned}$$

$v = \varphi_3(w)$ defined by

$$\begin{aligned} v_1 &= w_1, \quad v_n = w_n, \quad \text{and} \\ v_j &= \lambda_j w_j, \quad j = 2, \dots, n - 1, \end{aligned}$$

$w = \varphi_4(t)$ defined by

$$w_n = t_n + \sum_{2 \leq k \leq m} d_k t_1^k + \sum_{\alpha=2}^{n-1} \sum_{1 \leq k \leq \frac{m}{2}} d_{\alpha,k} t_\alpha t_1^k + \sum_{\alpha=2}^{n-1} c_\alpha t_\alpha^2,$$

$$w_j = t_j, \quad j = 1, \dots, n-1,$$

and $t = \varphi_5(\xi)$ defined by

$$t_1 = \xi_1, \quad t_n = \xi_n,$$

$$t_\alpha = \xi_\alpha + \sum_{1 \leq k \leq \frac{m}{2}} e_{\alpha,k} \xi_1^k, \quad \alpha = 2, \dots, n-1.$$

$$\begin{aligned} \rho(\Phi_{\zeta''}^{-1}(\xi)) &= \rho(\zeta'') + \operatorname{Re} \xi_n + \sum_{\substack{j+k \leq m \\ j,k > 0}} a_{jk}(\zeta'') \xi_1^j \bar{\xi}_1^k \\ &+ \sum_{\alpha=2}^{n-1} |\xi_\alpha|^2 + \sum_{\alpha=2}^{n-1} \operatorname{Re} \left(\left(\sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} b_{jk}^\alpha(\zeta'') \xi_1^j \bar{\xi}_1^k \right) \xi_\alpha \right) \\ &+ O(|\xi_n| |\xi| + |\xi''|^2 |\xi| + |\zeta''|^2 |\xi_1|^{\frac{m}{2}+1} + |\xi_1|^{m+1}). \end{aligned}$$

Since the composition $\Phi_{z'}^{-1} \circ \Phi_{\zeta''}^{-1}$ gives a map of the same form as $\Phi_{z''}^{-1}$, where $\Phi_{z''}^{-1}$ is obtained by applying Proposition 3.1 to the function r and z'' , we conclude from the uniqueness statement in Proposition 3.1 that

$$(3.13) \quad \Phi_{z''}^{-1} = \Phi_{z'}^{-1} \circ \Phi_{\zeta''}^{-1}.$$

In order to study $Q(z'', \delta)$ we must therefore examine the map $\Phi_{\zeta''}^{-1}$.

LEMMA 3.4. *Suppose that $z'' \in Q(z', \delta)$. Then*

$$(3.14) \quad \begin{aligned} |b_j| &\lesssim \delta \tau_j(z', \delta)^{-1}, \quad |c_\alpha| \lesssim \delta \tau_\alpha(z', \delta)^{-2}, \quad |d_k| \lesssim \delta \tau_1(z', \delta)^{-k}, \\ |d_{\alpha,k}| &\lesssim \delta \tau_1(z', \delta)^{-l} \tau_\alpha(z', \delta)^{-1}, \quad |e_{\alpha,l}| \lesssim \delta \tau_1(z', \delta)^{-l} \tau_\alpha(z', \delta)^{-1}, \end{aligned}$$

for $1 \leq j \leq n-1, 1 \leq k \leq m, 2 \leq \alpha \leq n-1, 1 \leq l \leq m/2$.

Proof. From the proof of Proposition 3.1, we see that

$$\begin{aligned}
 b_j &= -\left(\frac{\partial \rho}{\partial \zeta_1}\right)^{-1} \frac{\partial \rho(\zeta'')}{\partial \zeta_j}, \\
 c_\alpha &= -\frac{\partial^2 \rho(0)}{\partial \zeta_\alpha^2}, \\
 d_k &= -\frac{2}{k!} \frac{\partial^k \rho(0)}{\partial w_1^k}, \\
 d_{\alpha,l} &= -\frac{2}{(l+1)!} \frac{\partial^{l+1} \rho(0)}{\partial w_\alpha \partial w_1^l}, \\
 e_{\alpha,l} &= -\frac{1}{(l+1)!} \frac{\partial^{l+1} \rho(0)}{\partial \bar{t}_\alpha \partial t_1^l},
 \end{aligned}$$

for $1 \leq j \leq n - 1$, $1 \leq k \leq m$, $2 \leq \alpha \leq n - 1$, $1 \leq l \leq m/2$. By Lemma 3.2 and the definition of the biholomorphism $\Phi_{\zeta''}^{-1}$ we conclude that (3.14) holds. □

PROPOSITION 3.5. *There exists a constant C such that if $z'' \in Q(z', \delta)$, then*

$$(3.15) \quad Q(z'', \delta) \subset Q(z', C\delta)$$

and

$$(3.16) \quad Q(z', \delta) \subset Q(z'', C\delta).$$

Proof. Define $S(z'', \delta) = \{\Phi_{\zeta''}^{-1}(\xi) : \xi \in R(z'', \delta)\}$. It is easy to see that $Q(z'', \delta) = \Phi_{z'}^{-1} \circ S(z'', \delta)$. Thus, in order to prove (3.15) it suffices to show that

$$(3.17) \quad S(z'', \delta) \subset R(z', C\delta).$$

Indeed, for each $\xi \in R(z'', \delta)$, set $t = \varphi_5(\xi)$. By Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
 |t_1| &= |\xi_1| \leq \tau_1(z'', \delta) \lesssim \tau_1(z', \delta), \\
 |t_n| &= |\xi_n| \leq \tau_n(z'', \delta) = \tau_n(z', \delta) = \delta, \\
 |t_\alpha| &\leq |\xi_\alpha| + \sum_{k=2}^{n-1} |e_{\alpha,k}| |\xi_1|^k \lesssim \tau_\alpha(z'', \delta) + \delta \tau_1(z', \delta)^{-k} \tau_\alpha(z', \delta)^{-1} \tau_1(z'', \delta)^k \\
 &\lesssim \tau_\alpha(z', \delta), \quad 2 \leq \alpha \leq n - 1.
 \end{aligned}$$

We also set $w = \varphi_4(t)$. By Lemma 3.4, we have

$$\begin{aligned} |w_n| &\leq |t_n| + \sum_{k=2}^m |d_k||t_1|^k + \sum_{\alpha=2}^{n-1} \sum_{k=1}^{m/2} |d_{\alpha,k}||t_\alpha||t_1|^k + \sum_{\alpha=2}^{n-1} |c_\alpha||t_\alpha|^2 \\ &\lesssim \tau_n(z', \delta) + \sum_{k=2}^m \delta \tau_1(z', \delta)^{-k} \tau_1(z', \delta)^k + \sum_{\alpha=2}^{n-1} \delta \tau_\alpha(z', \delta)^{-2} \tau_\alpha(z', \delta)^2 \\ &\quad + \sum_{\alpha=2}^{n-1} \sum_{k=1}^{m/2} \delta \tau_1(z', \delta)^{-k} \tau_\alpha(z', \delta)^{-1} \tau_\alpha(z', \delta) \tau_1(z', \delta)^k \lesssim \delta = \tau_n(z', \delta), \\ |w_j| = |t_j| &\lesssim \tau_j(z', \delta), \quad 1 \leq j \leq n - 1. \end{aligned}$$

Set $v = \varphi_3(w)$, $u = \varphi_2(v)$ and $\zeta = \varphi_1(u)$. It is easy to see that $|v_j| \lesssim \tau_j(z', \delta)$, $|u_j| \lesssim \tau_j(z', \delta)$, $|\zeta_j| \lesssim \tau_j(z', \delta)$, $1 \leq j \leq n$ and hence, (3.17) holds if C is sufficiently large.

To prove (3.16), define $P(z', \delta) = \{\Phi_{\zeta''}(\zeta) : \zeta \in R(z', \delta)\}$, it easy to see that $Q(z', \delta) = \Phi_{z''}^{-1} \circ P(z'', \delta)$. Thus, it suffices to show that

$$(3.18) \quad P(z', \delta) \subset R(z'', C\delta).$$

Indeed, we see that $\Phi_{\zeta''} = \varphi_5^{-1} \circ \varphi_4^{-1} \circ \varphi_3^{-1} \circ \varphi_2^{-1} \circ \varphi_1^{-1}$ and

$$\tau(z', \delta) \lesssim \tau(z'', \delta).$$

Applying (3.14) in the same way as above, we conclude that if $\zeta \in R(z', \delta)$, then $\xi = \Phi_{\zeta''}(\zeta) \in R(z'', C\delta)$, where C is sufficiently large. Hence, (3.18) holds. The proof is completed. □

3.2. Dilation of coordinates

Let Ω be a domain in \mathbb{C}^n . Suppose that $\partial\Omega$ is pseudoconvex, of finite type and is smooth of class C^∞ near a boundary point $\xi_0 \in \partial\Omega$ and suppose that the Levi form has rank at least $n - 2$ at ξ_0 .

We may assume that $\xi_0 = 0$ and the rank of Levi form at ξ_0 is exactly $n - 2$. Let ρ be a smooth defining function for Ω . After a linear change of coordinates, we can find coordinate functions z_1, \dots, z_n defined on a

neighborhood U_0 of ξ_0 such that

$$\begin{aligned} \rho(z) &= Re z_n + \sum_{\substack{j+k \leq m \\ j,k > 0}} a_{j,k} z_1^j \bar{z}_1^k \\ &+ \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} Re((b_{j,k}^\alpha z_1^j \bar{z}_1^k) z_\alpha) \\ &+ O(|z_n||z| + |z^*|^2|z| + |z^*|^2|z_1|^{\frac{m}{2}+1} + |z_1|^{m+1}), \end{aligned}$$

where $z^* = (0, z_2, \dots, z_{n-1}, 0)$.

By Proposition 3.1, for each point η in a small neighborhood of the origin, there exists a unique automorphism Φ_η of \mathbb{C}^n such that

$$\begin{aligned} \rho(\Phi_\eta^{-1}(w)) - \rho(\eta) &= Re w_n + \sum_{\substack{j+k \leq m \\ j,k > 0}} a_{j,k}(\eta) w_1^j \bar{w}_1^k \\ (3.19) \quad &+ \sum_{\alpha=2}^{n-1} |w_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} Re[(b_{j,k}^\alpha(\eta) w_1^j \bar{w}_1^k) w_\alpha] \\ &+ O(|w_n||w| + |w^*|^2|w| + |w^*|^2|w_1|^{\frac{m}{2}+1} + |w_1|^{m+1}), \end{aligned}$$

where $w^* = (0, w_2, \dots, w_{n-1}, 0)$.

We define an anisotropic dilation Δ_η^ϵ by

$$\Delta_\eta^\epsilon(w_1, \dots, w_n) = \left(\frac{w_1}{\tau_1(\eta, \epsilon)}, \dots, \frac{w_n}{\tau_n(\eta, \epsilon)} \right),$$

where $\tau_1(\eta, \epsilon) = \tau(\eta, \epsilon)$, $\tau_k(\eta, \epsilon) = \sqrt{\epsilon}$ ($2 \leq k \leq n-1$), $\tau_n(\eta, \epsilon) = \epsilon$.

For each $\eta \in \partial\Omega$, if we set $\rho_\eta^\epsilon(w) = \epsilon^{-1} \rho \circ \Phi_\eta^{-1} \circ (\Delta_\eta^\epsilon)^{-1}(w)$, then

$$\begin{aligned} \rho_\eta^\epsilon(w) &= Re w_n + \sum_{\substack{j+k \leq m \\ j,k > 0}} a_{j,k}(\eta) \epsilon^{-1} \tau(\eta, \epsilon)^{j+k} w_1^j \bar{w}_1^k + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 \\ (3.20) \quad &+ \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq \frac{m}{2} \\ j,k > 0}} Re(b_{j,k}^\alpha(\eta) \epsilon^{-1/2} \tau(\eta, \epsilon)^{j+k} w_1^j \bar{w}_1^k w_\alpha) + O(\tau(\eta, \epsilon)). \end{aligned}$$

For each $\eta \in U_0$, we define pseudo-balls $Q(\eta, \epsilon)$ by

$$(3.21) \quad \begin{aligned} Q(\eta, \epsilon) &:= \Phi_\eta^{-1}(\Delta_\eta^\epsilon)^{-1}(D \times \dots \times D) \\ &= \Phi_\eta^{-1}\{|w_k| < \tau_k(\eta, \epsilon), 1 \leq k \leq n\}, \end{aligned}$$

where $D_r := \{z \in \mathbb{C} : |z| < r\}$. There exist constants $0 \leq \alpha \leq 1$ and $C_1, C_2, C_3 \geq 1$ such that for $\eta, \eta' \in U_0$ and $\epsilon \in (0, \alpha]$ the following estimates are satisfied with $\eta \in Q(\eta', \epsilon)$

$$(3.22) \quad \rho(\eta) \leq \rho(\eta') + C_1\epsilon,$$

$$(3.23) \quad \frac{1}{C_2}\tau(\eta, \epsilon) \leq \tau(\eta', \epsilon) \leq C_2\tau(\eta, \epsilon),$$

$$(3.24) \quad Q(\eta, \epsilon) \subset Q(\eta', C_3\epsilon) \text{ and } Q(\eta', \epsilon) \subset Q(\eta, C_3\epsilon).$$

Set $\epsilon(\eta) := |\rho(\eta)|$, $\Delta_\eta := \Delta_\eta^{\epsilon(\eta)}$ and $C_4 = C_1 + 1$. By (3.22), we have

$$(3.25) \quad \eta \in Q(\eta', \epsilon(\eta')) \Rightarrow \epsilon(\eta) \leq C_4\epsilon(\eta').$$

Fix neighborhoods W_0, V_0 of the origin with $W_0 \subset V_0 \subset U_0$. Then for sufficiently small constants α_1, α_0 ($0 < \alpha_1 \leq \alpha_0 < 1$), we have

$$(3.26) \quad \eta \in V_0 \text{ and } 0 < \epsilon \leq \alpha_0 \Rightarrow Q(\eta, \epsilon) \subset U_0 \text{ and } \epsilon(\eta) \leq \alpha_0$$

$$(3.27) \quad \eta \in W_0 \text{ and } 0 < \epsilon \leq \alpha_1 \Rightarrow Q(\eta, \epsilon) \subset V_0.$$

Define a pseudo-metric by

$$M(\eta, \vec{X}) := \sum_{k=1}^n \frac{|(\Phi'_\eta(\eta)\vec{X})_k|}{\tau_k(\eta, \epsilon(\eta))} = \|\Delta_\eta \circ \Phi'_\eta(\eta)\vec{X}\|_1$$

on U_0 . By (3.7), one has

$$\frac{\|\vec{X}\|_1}{\epsilon(\eta)^{1/m}} \lesssim M(\eta, \vec{X}) \lesssim \frac{\|\vec{X}\|_1}{\epsilon(\eta)}.$$

LEMMA 3.6. *There exist constants $K \geq 1$ ($K = C_3 \cdot C_4$) and $0 < A < 1$ such that for each integer $N \geq 1$ and each holomorphic $f : D_N \rightarrow U_0$ satisfies $M(f(u), f'(u)) \leq A$ on D_N , we have*

$$f(0) \in W_0 \text{ and } K^{N-1}\epsilon(f(0)) \leq \alpha_1 \Rightarrow \overline{f(D_N)} \subset Q[f(0), K^N\epsilon(f(0))].$$

Proof. Let $\eta_0 \in V_0$ and $\eta \in Q(\eta_0, \epsilon_0)$, where $\epsilon_0 = \epsilon(\eta_0)$. From (3.25), (3.23) and (3.8) one has $\epsilon(\eta) \leq C_4\epsilon_0$ and

$$\tau(\eta, \epsilon(\eta)) \leq \tau(\eta, C_4\epsilon_0) \leq C_2\sqrt{C_4}\tau(\eta_0, \epsilon_0).$$

Thus

$$M(\eta, \vec{X}) \gtrsim \sum_{k=1}^n \frac{|(\Phi'_\eta(\eta)\vec{X})_k|}{\tau_k(\eta_0, \epsilon_0)}.$$

In order to replace $\Phi'_\eta(\eta)$ by $\Phi'_{\eta_0}(\eta)$ in this inequality, we consider the automorphism $\Psi := \Phi_\eta \circ \Phi_{\eta_0}^{-1}$ which equals $\Phi_a^{-1} = \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \varphi_4 \circ \varphi_5$ where $a := \Phi_\eta(\eta_0)$ and φ_j ($1 \leq j \leq 5$) are given in the previous section.

If we set $\Lambda := \Phi'_\eta(\eta) \circ (\Phi'_{\eta_0}(\eta))^{-1} = \Psi'(\Phi_{\eta_0}(\eta))$, then $\Lambda = \varphi'_1 \circ \varphi'_2 \circ \varphi'_3 \circ \varphi'_4 \circ \varphi'_5$. By a simple computation, we have

$$\varphi'_1(w_1, \dots, w_n) = \left(w_1, w_2, \dots, w_n + \sum_{k=1}^{n-1} b_k w_k \right)$$

where $|b_k| \leq C \cdot \frac{\epsilon_0}{\tau_k(\eta_0, \epsilon_0)}$ ($1 \leq k \leq n - 1$) for some constant $C \geq 1$.

Set $\vec{Y} := \Phi'_{\eta_0}(\eta)\vec{X}$, $\vec{Y}^4 := \varphi'_5\vec{Y}$, $\vec{Y}^3 := \varphi'_4\vec{Y}^4$, $\vec{Y}^2 := \varphi'_3\vec{Y}^3$ and $\vec{Y}^1 := \varphi'_2\vec{Y}^2$, since $\Phi'_\eta(\eta)\vec{X} = \Lambda[\vec{Y}] = \varphi'_1\vec{Y}^1$, we have

$$\begin{aligned} M(\eta, \vec{X}) &\gtrsim \frac{|(\Phi'_\eta(\eta)\vec{X})_1|}{\tau_1(\eta_0, \epsilon_0)} + \dots + \frac{|(\Phi'_\eta(\eta)\vec{X})_2|}{\tau_{n-1}(\eta_0, \epsilon_0)} + \frac{|(\Phi'_\eta(\eta)\vec{X})_n|}{2C\epsilon_0} \\ &\gtrsim \sum_{k=1}^{n-1} \left(1 - \frac{|b_k|\tau_k(\eta_0, \epsilon_0)}{2C\epsilon_0} \right) \frac{|Y_k^1|}{\tau_k(\eta_0, \epsilon_0)} + \frac{|Y_n^1|}{2C\epsilon_0} \\ &\gtrsim \sum_{k=1}^n \frac{|Y_k^1|}{\tau_k(\eta_0, \epsilon_0)}. \end{aligned}$$

Because of the definition of the maps φ_2 and φ_3 , it is easy to show that

$$\sum_{k=1}^n \frac{|Y_k^1|}{\tau_k(\eta_0, \epsilon_0)} \gtrsim \sum_{k=1}^n \frac{|Y_k^2|}{\tau_k(\eta_0, \epsilon_0)} \gtrsim \sum_{k=1}^n \frac{|Y_k^3|}{\tau_k(\eta_0, \epsilon_0)}.$$

Next we also have

$$\varphi'_4(w_1, \dots, w_n) = \left(w_1, w_2, \dots, w_n + \sum_{k=1}^{n-1} \gamma_k w_k \right)$$

where

$$\begin{aligned}
 |\gamma_k| &\lesssim \sum_{j=1}^{m/2} |d_{k,j}| \tau_1(\eta_0, \epsilon_0)^j + 2|c_k| \tau_k(\eta_0, \epsilon_0) \leq C \frac{\epsilon_0}{\tau_k(\eta_0, \epsilon_0)}, \\
 |\gamma_1| &\lesssim \sum_{\alpha=2}^{n-1} \sum_{j=1}^{m/2} |d_{\alpha,j}| \tau_\alpha(\eta_0, \epsilon_0) \cdot j \cdot \tau_1(\eta_0, \epsilon_0)^{j-1} + \sum_{j=2}^m |d_j| \cdot j \cdot \tau_1(\eta_0, \epsilon_0)^{j-1} \\
 &\leq C \frac{\epsilon_0}{\tau_1(\eta_0, \epsilon_0)},
 \end{aligned}$$

for $k = 2, \dots, n - 1$ and some constant $C \geq 1$. Using the same argument as above we have

$$\sum_{k=1}^n \frac{|Y_k^3|}{\tau_k(\eta_0, \epsilon_0)} \gtrsim \sum_{k=1}^n \frac{|Y_k^4|}{\tau_k(\eta_0, \epsilon_0)}.$$

The derivative of φ_5 is defined by

$$\varphi'_5(w_1, \dots, w_n) = (w_1, w_2 + \beta_2 w_1, \dots, w_{n-1} + \beta_{n-1} w_1, w_n)$$

where $|\beta_k| \lesssim \sum_{l=1}^{m/2} |e_{k,l}| \cdot l \cdot \tau_1(\eta_0, \epsilon_0)^{l-1} \leq C \frac{\epsilon_0}{\tau_k(\eta_0, \epsilon_0) \tau_1(\eta_0, \epsilon_0)}$ ($2 \leq k \leq n - 1$) for some constant $C \geq 1$.

Since $\vec{Y}^4 = \varphi'_5 \vec{Y}$, we have

$$\begin{aligned}
 \sum_{k=1}^n \frac{|Y_k^4|}{\tau_k(\eta_0, \epsilon_0)} &\gtrsim \frac{|Y_1^4|}{\tau_1(\eta_0, \epsilon_0)} + \sum_{k=2}^{n-1} \frac{|Y_k^4|}{2nC\tau_k(\eta_0, \epsilon_0)} + \frac{|Y_n^4|}{\tau_n(\eta_0, \epsilon_0)} \\
 &\gtrsim \left(1 - \sum_{k=2}^{n-1} \frac{|\beta_k| \tau_1(\eta_0, \epsilon_0)}{2nC\tau_k(\eta_0, \epsilon_0)} \right) \frac{|Y_1|}{\tau_1(\eta_0, \epsilon_0)} \\
 &\quad + \sum_{k=2}^{n-1} \frac{|Y_k|}{2nC\tau_k(\eta_0, \epsilon_0)} + \frac{|Y_n|}{\tau_n(\eta_0, \epsilon_0)} \\
 &\gtrsim \sum_{k=1}^n \frac{|Y_k|}{\tau_k(\eta_0, \epsilon_0)} = \sum_{k=1}^n \frac{|(\Phi'_\eta(\eta) \vec{X})_k|}{\tau_k(\eta_0, \epsilon_0)}.
 \end{aligned}$$

Therefore, there exists a constant $1 \geq A > 0$ such that $M(\eta, \vec{X}) \geq A \|\Delta_{\eta_0} \circ \Phi'_\eta(\eta) \vec{X}\|_1$ for every $\eta_0 \in V_0$ and for every $\eta \in Q(\eta_0, \epsilon(\eta_0))$. By this observation, we can finish the proof.

a) If $N = 1$, the inclusion $f(D_1) \subset Q(\eta_0, \epsilon_0)$ is satisfied as $f(0) \in W_0$. This deduces immediately from the observation that $\|\frac{d}{du} \Delta_{\eta_0} \circ \Phi_{\eta_0} \circ f(u)\|_1 \leq 1$ as $f(u) \in Q(\eta_0, \epsilon_0)$.

b) Suppose now $N \geq 2$ and $f(0) \in W_0$. Fix $\theta_0 \in (0, 2\pi]$ and let $u_j = je^{i\theta_0}$, $\eta_j := f(u_j)$ and $\epsilon_j = \epsilon(\eta_j)$. It is sufficient to show that $\overline{f[D(u_i, 1)]} \subset Q(\eta_0, K^i \epsilon_0)$ for $i \leq N - 1$.

For $i = 1$, this assertion is proved in a). Suppose that these inclusions are satisfied for $i \leq j < N - 1$. Since $\eta_{j+1} \in Q(\eta_0, K^j \epsilon_0)$, we have $\epsilon_{j+1} \leq C_4 K^j \epsilon_0 < \alpha_1$. Moreover, since $\eta_0 \in W_0$, it implies that $\eta_{j+1} \in V_0$ (see (3.27)). We may apply a) to the restriction of f to $D(u_{j+1}, 1)$

$$\begin{aligned} \overline{f[D(u_{j+1}, 1)]} &\subset Q(\eta_{j+1}, \epsilon_{j+1}) \subset Q(\eta_{j+1}, C_4 K^j \epsilon_0) \\ &\subset Q(\eta_0, C_3 C_4 K^j \epsilon_0) = Q(\eta_0, K^{j+1} \epsilon_0). \end{aligned}$$

□

For any sequence $\{\eta_p\}_p$ of points tending to the origin in $U_0 \cap \{\rho < 0\} =: U_0^-$, we associate with a sequence of points $\eta'_p = (\eta_{1p}, \dots, \eta_{mp} + \epsilon_p)$, $\epsilon_p > 0$, η'_p in the hypersurface $\{\rho = 0\}$. Consider the sequence of dilations $\Delta_{\eta'_p}^{\epsilon_p}$. Then $\Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p}(\eta_p) = (0, \dots, 0, -1)$. By (3.20), we see that $\Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p}(\{\rho = 0\})$ is defined by an equation of the form

$$(3.28) \quad \begin{aligned} Re w_n + P_{\eta'_p}(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 + \sum_{\alpha=2}^{n-1} Re(Q_{\eta'_p}^\alpha(w_1, \bar{w}_1)w_\alpha) \\ + O(\tau(\eta'_p, \epsilon_p)) = 0, \end{aligned}$$

where

$$\begin{aligned} P_{\eta'_p}(w_1, \bar{w}_1) &:= \sum_{\substack{j+k \leq m \\ j, k > 0}} a_{j,k}(\eta'_p) \epsilon_p^{-1} \tau(\eta'_p, \epsilon_p)^{j+k} w_1^j \bar{w}_1^k, \\ Q_{\eta'_p}^\alpha(w_1, \bar{w}_1) &:= \sum_{\substack{j+k \leq \frac{m}{2} \\ j, k > 0}} b_{j,k}^\alpha(\eta'_p) \epsilon_p^{-1/2} \tau(\eta'_p, \epsilon_p)^{j+k} w_1^j \bar{w}_1^k. \end{aligned}$$

Note that from (3.5) we know that the coefficients of $P_{\eta'_p}$ and $Q_{\eta'_p}^\alpha$ are bounded by one. But the polynomials $Q_{\eta'_p}^\alpha$ are less important than $P_{\eta'_p}$. In [10], S. Cho proved the following lemma.

LEMMA 3.7. ([10, Lem. 2.4, p. 810]) $|Q_{\eta'_p}^\alpha(w_1, \bar{w}_1)| \leq \tau(\eta'_p, \epsilon_p)^{1/10}$ for all $\alpha = 2, \dots, n - 1$ and $|w_1| \leq 1$.

By Lemma 3.7, it follows that after taking a subsequence, $\Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p}(U_0^-)$ converges to the following domain

$$(3.29) \quad M_P := \left\{ \hat{\rho} := \operatorname{Re} w_n + P(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\}.$$

where $P(w_1, \bar{w}_1)$ is a polynomial of degree $\leq m$ without harmonic terms.

Since M_P is a smooth limit of the pseudoconvex domains $\Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p}(U_0^-)$, it is pseudoconvex. Thus the function $\hat{\rho}$ in (3.29) is plurisubharmonic, and hence P is a subharmonic polynomial whose Laplacian does not vanish identically.

LEMMA 3.8. *The domain M_P is Brody hyperbolic.*

Proof. If $\varphi : \mathbb{C} \rightarrow M_P$ is holomorphic, then the subharmonic functions $\operatorname{Re} \varphi_n + P \circ \varphi_1 + \sum_{\alpha=2}^{n-1} |\varphi_\alpha|^2$ and $\operatorname{Re} \varphi_n + P \circ \varphi_1$ are negative on \mathbb{C} . Consequently, they are constant. This implies that $P \circ \varphi_1$ is harmonic. Hence $\varphi_1, \operatorname{Re} \varphi_n$ and φ_n are constant. In addition, the function $\sum_{\alpha=2}^{n-1} |\varphi_\alpha|^2$ is also constant and hence φ_α ($2 \leq \alpha \leq n - 1$) are constant. \square

3.3. Estimates of Kobayashi metric

Recall that the Kobayashi metric K_Ω of Ω is defined by

$$K_\Omega(\eta, \vec{X}) := \inf \left\{ \frac{1}{R} \mid \exists f : D \rightarrow \Omega \text{ such that } f(0) = \eta, f'(0) = R\vec{X} \right\}.$$

By the same argument as in [5] page 93, there exists a neighborhood U of the origin with $U \subset U_0$ such that

$$K_\Omega(\eta, \vec{X}) \leq K_{\Omega \cap U_0}(\eta, \vec{X}) \leq 2K_\Omega(\eta, \vec{X}) \text{ for all } \eta \in U \cap \Omega.$$

We need the following lemma (see [7]).

LEMMA 3.9. *Let (X, d) be a complete metric space and let $M : X \rightarrow \mathbb{R}^+$ be a locally bounded function. Then, for all $\sigma > 0$ and for all $u \in X$ satisfying $M(u) > 0$, there exists $v \in X$ such that*

- (i) $d(u, v) \leq \frac{2}{\sigma M(u)}$
- (ii) $M(v) \geq M(u)$

(iii) $M(x) \leq 2M(v)$ if $d(x, v) \leq \frac{1}{\sigma M(v)}$.

Proof. If v does not exist, one constructs a sequence (v_j) such that $v_0 = u$, $M(v_{n+1}) \geq 2M(v_j) \geq 2^{n+1}M(u)$ and $d(v_{n+1}, v_j) \leq \frac{1}{\sigma M(v_j)} \leq \frac{1}{\sigma M(u)2^n}$. This sequence is Cauchy. \square

THEOREM 3.10. *Let Ω be a domain in \mathbb{C}^n . Suppose that $\partial\Omega$ is pseudoconvex, of finite type and is smooth of class C^∞ near a boundary point $p \in \partial\Omega$ and suppose that the Levi form has rank at least $n - 2$ at ξ_0 . Then, there exists a neighborhood V of ξ_0 such that*

$$M(\eta, \vec{X}) \lesssim K_\Omega(\eta, \vec{X}) \lesssim M(\eta, \vec{X}) \text{ for all } \eta \in V \cap \Omega.$$

Proof of Theorem 3.10. The second inequality is obvious, by the definition. We are going to prove the first inequality. We may also assume that $\xi_0 = (0, \dots, 0)$. It suffices to show that for η near 0 and \vec{X} is not zero, we have

$$K_\Omega\left(\eta, \frac{\vec{X}}{M(\eta, \vec{X})}\right) \gtrsim 1.$$

Suppose that this is not true. Then there exist $f_p : D \rightarrow \Omega \cap U$ such that $f_p(0) = \eta_p$ tends to the origin and $f_p'(0) = R_p \frac{\vec{X}_p}{M(\eta_p, \vec{X}_p)}$, where $R_p \rightarrow \infty$ as $p \rightarrow \infty$. We may assume that $R_p \geq p^2$. Then, one has

$$M(f_p(0), f_p'(0)) = M\left(\eta_p, R_p \frac{\vec{X}_p}{M(\eta_p, \vec{X}_p)}\right) = R_p \geq p^2.$$

Apply Lemma 3.9 to $M_p(t) := M(f_p(t), f_p'(t))$ on $\bar{D}_{1/2}$ with $u = 0$ and $\sigma = 1/p$. This gives $\tilde{a}_p \in \bar{D}_{1/2}$ such that $|\tilde{a}_p| \leq \frac{2p}{M_p(0)}$ and $M_p(\tilde{a}_p) \geq M_p(0) \geq p^2$. Moreover,

$$M_p(t) \leq 2M_p(\tilde{a}_p) \text{ on } D\left(\tilde{a}_p, \frac{p}{M_p(\tilde{a}_p)}\right).$$

We define a sequence $\{g_p\} \subset Hol(D_p, \Omega)$ by $g_p(t) := f_p(\tilde{a}_p + \frac{At}{2M_p(\tilde{a}_p)})$. This sequence satisfies the estimates

$$M[g_p(t), g_p'(t)] \leq A \text{ on } D_p.$$

Since $\tilde{a}_p \rightarrow 0$, the series $g_p(0) = f_p(\tilde{a}_p)$ tends to the origin. Choose a subsequence, if necessary, we may assume that $K^p \epsilon(g_p(0)) \leq \alpha_1$, where K , A and α_1 are the constants in Lemma 3.6. It follows from Lemma 3.6 that

$$(3.30) \quad g_p(D_N) \subset Q[g_p(0), K^N \epsilon(g_p(0))] \text{ for } N \leq p.$$

We may now apply the method of dilation of the coordinates. Set $\eta_p := g_p(0)$ and $\eta'_p := \eta_p + (0, \dots, 0, \epsilon_p)$, where $\epsilon_p > 0$ and $\rho(\eta'_p) = 0$. It is easy to see that $\epsilon_p \approx \epsilon(\eta_p)$ and $\eta_p \in Q(\eta'_p, c\epsilon_p)$ for $c \geq 1$ is some constant. It follows from (3.30) and (3.24) that, for some constant $C \geq 1$,

$$(3.31) \quad g_p(D_N) \subset Q[\eta'_p, CK^N \epsilon_p] \quad \text{for } N \leq p.$$

Set $\varphi_p := \Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p} \circ g_p$. The inclusions (3.24) imply that

$$\varphi_p(D_N) \subset D_{\sqrt{CK^N}} \times \dots \times D_{\sqrt{CK^N}} \times D_{CK^N}.$$

By using the Montel's theorem and a diagonal process, there exists a subsequence $\{\varphi_{p_k}\}$ of $\{\varphi_p\}$ which converges on compact subsets of \mathbb{C} to an entire curve $\varphi : \mathbb{C} \rightarrow M_P$. Since M_P is Brody hyperbolic, φ must be constant.

On the other hand, we have

$$\frac{A}{2} = M[g_p(0), g_p'(0)] = \sum_{k=1}^n \frac{|(\Phi'_{\eta_p}(\eta_p)g_p'(0))_k|}{\tau_k(\eta_p, \epsilon(\eta_p))}.$$

Since $\epsilon_p \approx \epsilon(\eta_p)$, $\eta_p \in Q(\eta'_p, c\epsilon_p)$ and $\Phi'_{\eta_p}(\eta_p) \circ (\Phi'_{\eta'_p}(\eta_p))^{-1}$ approaches to Id as $p \rightarrow \infty$, we have

$$\frac{A}{2} \lesssim \sum_{k=1}^n \frac{|(\Phi'_{\eta'_p}(\eta_p)g_p'(0))_k|}{\tau_k(\eta'_p, \epsilon_p)} = \|\varphi_p'(0)\|_1.$$

Thus $\|\varphi'(0)\|_1 = \lim_{p_k \rightarrow \infty} \|\varphi_{p_k}'(0)\|_1 \gtrsim A/2$. □

3.4. Normality of the families of holomorphic mappings

First of all, we prove the following theorem.

THEOREM 3.11. *Let Ω be a domain in \mathbb{C}^n . Suppose that $\partial\Omega$ is pseudoconvex, of finite type and is smooth of class C^∞ near a boundary point $(0, \dots, 0) \in \partial\Omega$. Suppose that the Levi form has rank at least $n - 2$ at $(0, \dots, 0)$. Let ω be a domain in \mathbb{C}^k and $\varphi_p : \omega \rightarrow \Omega$ be a sequence of holomorphic mappings such that $\eta_p := \varphi_p(a)$ converges to $(0, \dots, 0)$ for some point $a \in \omega$. Let $(T_p)_p$ be a sequence of automorphisms of \mathbb{C}^n which associates with the sequence $(\eta_p)_p$ by the method of the dilation of coordinates*

(i.e., $T_p = \Delta_{\eta'_p}^{\epsilon_p} \circ \Phi_{\eta'_p}$). Then $(T_p \circ \varphi_p)_p$ is normal and its limits are holomorphic mappings from ω to the domain of the form

$$M_P = \left\{ (w_1, \dots, w_n) \in \mathbb{C}^n : \operatorname{Re} w_n + P(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\},$$

where $P \in \mathcal{P}_{2m}$.

Proof. Let $f : D \rightarrow \Omega$ be a holomorphic map with $f(0)$ near $(0, \dots, 0)$. By Theorem 3.10, we have

$$M[f(u), f'(u)] \lesssim K_\Omega(f(u), f'(u)) \leq K_D\left(u, \frac{\partial}{\partial u}\right).$$

Suppose $0 < r_0 < 1$ such that $r_0 \sup_{|u| \leq r_0} K_D(u, \frac{\partial}{\partial u}) \leq A$, where A is the constant in Lemma 3.6. Set $f_{r_0}(u) := f(r_0 u)$. Then

$$M[f_{r_0}(u), f_{r_0}'(u)] \leq A.$$

By Lemma 3.6, we have $f(D_{r_0}) = f_{r_0}(D) \subset Q[f(0), \epsilon(f(0))]$.

This inclusion is also true if D is replaced by the unit ball in C^k . Let $f : \omega \rightarrow \Omega$ be a holomorphic map such that $f(a)$ near $(0, \dots, 0)$ for some point $a \in \omega$. For any compact subset K of ω , by using a finite covering of balls of radius r_0 and by the property (3.24), we have

$$f(K) \subset Q[f(a), C(K)\epsilon(f(a))],$$

where $C(K)$ is a constant which depends on K .

Since $\eta_p := \varphi_p(a)$ converges to the origin, it implies that

$$\varphi_p(K) \subset Q[\eta'_p, C(K)\epsilon(\eta_p)].$$

Thus $T_p \circ \varphi_p(K) \subset D_{\sqrt{C(K)}} \times \dots \times D_{\sqrt{C(K)}} \times D_{C(K)}$. By the Montel's theorem and a diagonal process, the sequence $T_p \circ \varphi_p$ is normal and its limits are holomorphic mappings from ω to the domain of the form

$$M_P = \left\{ (w_1, \dots, w_n) \in \mathbb{C}^n : \operatorname{Re} w_n + P(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\}.$$

□

§4. Proof of Theorem 1.1

In this section, we use the Berteloot’s method (see [6]) to complete the proof of Theorem 1.1. First of all, for a domain Ω in \mathbb{C}^n and $z \in \Omega$ we shall denote by $\mathcal{P}(\Omega, z)$ the set of polynomials $Q \in \mathcal{P}_{2m}$ such that Q is subharmonic and there exists a biholomorphism $\psi : \Omega \rightarrow M_Q$ with $\psi(z) = (0', -1)$. By the similar argument as in the proof of Proposition 3.1 of [6] (also by using Theorem 3.11 and Lemma 2.3), one also obtains that, if Ω satisfies the assumptions of our theorem, then $\mathcal{P}(\Omega, z)$ is never empty. Moreover, there are choices of z such that every element of $\mathcal{P}(\Omega, z)$ is of degree $2m$. More precisely, we have the following.

PROPOSITION 4.1. *Let Ω be a domain in \mathbb{C}^n such that:*

- (1) $\exists \xi_0 \in \partial\Omega$ such that $\partial\Omega$ is of class C^∞ , pseudoconvex and of finite type in a neighbourhood of ξ_0 .
- (2) The Levi form has rank at least $n - 2$ at ξ_0 .
- (3) $\exists z_0 \in \Omega, \exists \varphi_p \in \text{Aut}(\Omega)$ such that $\lim \varphi_p(z_0) = \xi_0$.

Then

- (a) $\forall z \in \Omega : \mathcal{P}(\Omega, z) \neq \emptyset$.
- (b) $\exists \tilde{z}_0 \in \Omega$ such that if $Q \in \mathcal{P}(\Omega, \tilde{z}_0)$, then $\deg Q = 2m$, where $2m$ is the type of $\partial\Omega$ at ξ_0 .
- (c) $\exists Q \in \mathcal{P}(\Omega, \tilde{z}_0)$ such that $Q = H + R$, where $H \in \mathcal{H}_{2m}$ and $\deg R < 2m$.

The control of sequence of dilations associated to the “orbit” $(\varphi_p(\tilde{z}_0))$ is closely related to the asymptotic behaviour of $(\varphi_p(\tilde{z}_0))$ in Ω . Unfortunately, the direct investigation of this behaviour seems impossible. Our aim is therefore to study the image of $(\varphi_p(\tilde{z}_0))$ in some rigid polynomial realization M_Q of Ω . The proof of our theorem follows from the following proposition which summarizes the different possibilities.

PROPOSITION 4.2. *Let Ω be a domain in \mathbb{C}^n satisfying the following assumptions:*

- (1) $\partial\Omega$ is smoothly pseudoconvex in a neighbourhood of $\xi_0 \in \partial\Omega$ and of finite type $2m$ at ξ_0 .

- (2) $\exists z_0 \in \Omega, \exists \varphi_p \in \text{Aut}(\Omega)$ s.t. $\lim \varphi_p(z_0) = \xi_0$. Let $\tilde{z}_0 \in \Omega$ and $Q \in \mathcal{P}(\Omega, \tilde{z}_0)$ be given by Proposition 4.1 and let ψ denote a biholomorphism between Ω and M_Q which maps \tilde{z}_0 onto $(0', -1)$; denote $\psi \circ \varphi_p(\tilde{z}_0)$ as $a_p = (a_{1p}, \dots, a_{np})$ and $|\text{Re } \psi_n \circ \varphi_p(\tilde{z}_0) + Q[\psi_1 \circ \varphi_p(\tilde{z}_0)] + |\psi_2 \circ \varphi_p(\tilde{z}_0)|^2 + \dots + |\psi_{n-1} \circ \varphi_p(\tilde{z}_0)|^2|$ as ϵ_p . Let H be the homogeneous part of highest degree in Q .

Then three possibilities may occur

- (i) $\lim \epsilon_p = 0$ and $\liminf |a_{1p}| < +\infty$.
Then $Q(z) = H(z-a) + 2 \text{Re} \sum_{j=0}^{2m} \frac{Q_j(a)}{j!} (z-a)^j$ ($a \in \mathbb{C}$) and $\Omega \simeq M_H$.
- (ii) $\lim \epsilon_p = 0$ and $\liminf |a_{1p}| = +\infty$.
Then $Q(z) = H = \lambda[(2 \text{Re}(e^{i\nu} z))^{2m} - 2 \text{Re}(e^{i\nu} z)^{2m}]$ ($\lambda > 0, \nu \in [0, 2\pi)$) and $\Omega \simeq M_H$
- (iii) $\limsup \epsilon_p > 0$. Then $H = \lambda|z|^{2m}$ ($\lambda > 0$) and $\Omega \simeq M_H$.

Proof. We may assume that $\deg Q > 2$. Otherwise $Q = |z|^2$ and the theorem already follows from Proposition 4.1. Let us first consider the case where $\lim \epsilon_p = 0$. Define a sequence of polynomials Q_p by

$$(4.1) \quad Q_p = \frac{1}{\epsilon_p} \sum_{j,q>0} \frac{Q_{j,\bar{q}}(a_{1p})}{(j+q)!} \tau_p^{j+q} z_1^j \bar{z}_1^q$$

where $\tau_p > 0$ is chosen in order to achieve $\|Q_p\| = 1$. Taking a sequence we may assume that $\lim Q_p = Q_\infty$ where $Q_\infty \in \mathcal{P}_{2m}$ and $\|Q_\infty\| = 1$.

Let us consider the sequence of automorphisms of \mathbb{C}^n

$$\begin{aligned} \phi_p : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ z &\longmapsto z', \end{aligned}$$

where z' is given by

$$(4.2) \quad \begin{cases} z'_n = \frac{1}{\epsilon_p} \left[z_n - a_{np} - \epsilon_p + 2 \sum_{j=1}^{2m} \frac{Q_j(a_{1p})}{j!} (z_1 - a_{1p})^j + 2 \sum_{j=2}^{n-1} \bar{a}_{jp} (z_j - a_{jp}) \right] \\ z'_1 = \frac{1}{\tau_p} [z_1 - a_{1p}] \\ z'_2 = \frac{1}{\sqrt{\epsilon_p}} [z_2 - a_{2p}] \\ \dots \\ z'_{n-1} = \frac{1}{\sqrt{\epsilon_p}} [z_{n-1} - a_{n-1p}] \end{cases}$$

It is easy to check that ϕ_p maps biholomorphically M_Q onto M_{Q_p} and a_p to $(0', -1)$.

i) and ii) are now obtained with a slightly modification of the proof of Proposition 4.1 in [6]. We are going to prove iii).

We now consider the case where $\limsup \epsilon_p > 0$. After taking some subsequence we may assume that $\epsilon_p \geq c > 0$ for all p . We shall study the real action (g_t) defined on M by

$$(4.3) \quad \begin{cases} g : \mathbb{R} \times \Omega \rightarrow \Omega \\ (t, z) \mapsto g_t(z) \\ g_t(z) = \psi^{-1}[\psi(z) + (0', it)]. \end{cases}$$

Modifying the proof of Lemma 4.3 of [6], we also conclude that this action is a parabolicity, that is

$$(4.4) \quad \forall z \in \Omega : \lim_{t \rightarrow \pm\infty} g_t(z) = \xi_0.$$

According to [2], the action $(g_t)_t$ itself is of class C^∞ . Thus, we may now consider the holomorphic tangent vector field \vec{X} defined on some neighbourhood of ξ_0 in $\partial\Omega$ by

$$\vec{X} = \left(\frac{d}{dt} \right)_{t=0} g_t(z).$$

The analysis of this vector field is given in the papers of E. Bedford and S. Pinchuk [1], [2]. It yields the conclusion that $H = |z|^{2m}$. It is then possible to study the scaling process more precisely for showing that Ω is biholomorphic to $M_{|z|^{2m}}$. This ends the proof of Proposition 4.2. □

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