POLYNOMIAL RINGS OVER RINGS INTEGRAL OVER THEIR CENTRES

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We prove in this paper that every finitely generated critical module over $A[X]$ is compressible where A is a Noetherian ring integral over a subring of its centre. Here $A[X]$ denotes the polynomial ring over A in a commuting indeterminate X.

Introduction

This paper is divided into two sections. Section 1 is essentially devoted to extending results of Wangneo and Tewari ([14] Section 2) to arbitrary right Noetherian rings. We define the support of a finitely generated module M over a right Noetherian ring B and show that if a prime P is a minimal element of the support of M then a prime ideal Q is associated with M if Q lies over $P \cap R$ where R is any subring of the centre of B containing its identity. This result may be used to determine the assassinators of finitely generated modules over right Noetherian rings which are well behaved with respect to their centres. In the present paper we use it to prove in Section 2 that finitely generated critical modules over $A[X]$ are compressible where A is a two-sided Noetherian ring integral over a subring of its centre. This is proved in the uncertainty of fully bounded Noetherianness of $A[X]$. Note that Jategaonker [9] has proved that over a fully bounded Noetherian ring, every finitely generated critical module is compressible. However proving $A[X]$ is fully bounded Noetherian, even in the case when A is a division ring algebraic over its centre, is another version of an open problem due to Amitsur [1].

Also note that in general setting Goodearl and Lenagen ([δ], p. 863) have given an example of a right and left Noetherian ring Q such that all finitely generated critical Q-modules are compressible but such that the polynomial ring $Q[X]$ possesses incompressible finitely generated critical modules.

Finally we conclude this introduction with a few remarks concerning our notation and terminology. All our basic definitions and terminology are given in Section 1 of Wangneo and Tewari [14] except that whenever we refer to a ring A we do not necessarily mean (unless otherwise stated) that A is a right Noetherian ring integral over a subring R of its centre. If B is a ring, then $B[X]$ will mean the polynomial ring over B in a commuting variable X ; the centre of B will be denoted by $Z(B)$ and all rings mentioned in this paper possess an identity. Also, if M is a right B-module, then $E_B(M)$ will denote the injective hull of M over B .

1.

We begin by extending results of Wangneo and Tewari ([14] Section 2) to any right Noetherian ring. We also prove some preliminary results about polynomial rings over right Noetherian rings integral over their centres which will be used later in Section 2.

PROPOSITION 1.1. *Let A be a right Noetherian ring. Then for every cyclic, prime, critical module M with annihilator P and $M \approx \frac{A}{I}$ (I a right ideal of A) we have:*

either (i) M is admissible in which case P, the annihilator of M is a completely prime ideal of A ,

or (ii) I is a prime right ideal of A and P is the bound of I in A .

Proof. If M is admissible, then $I = P$ and so A/I is a prime ring which is also uniform as a right A/I module. Since A/I is also a right Noetherian ring, we have by Goldie ([4] Theorem 2) A/I is a domain and $I = P$ is a completely prime ideal of A . This proves (i). In the case that M is not admissible, then I contains P properly and P is the bound of I in A . I is a prime right ideal, for, if not, then there exists elements $x, y \notin I$ such that $x A y \subseteq I$. Clearly the

image of the right ideal xA in A/I namely $\overline{xA}$ is such that
Ann$(\overline{xA}) = P$. Hence, since $\overline{xA} \neq 0$, so $y \in P \subseteq I$ a contradiction to
what we have assumed. Thus I is a prime right ideal.

Next we consider localization of a noncommutative ring A at a
multiplicatively closed subset S of any subring R of A which contains
the identity of R and which does not contain zero divisors and such that
R is contained in the centre of A . For this localization see Wangneo
and Tewari ([14], p. 1579). Let M be a finitely generated module over
A and consider the localized ring A_S and the localized module M_S .
Then, we have the following:

PROPOSITION 1.2. *Any A_S submodule N of M_S is of the form W_S
for some submodule W of M .*

Proof. This follows exactly as in the commutative case.

Note. We will use this fact in the sequel.

PROPOSITION 1.3. *Let A be a right Noetherian ring and S a
multiplicatively closed subset of R containing no zero divisors and
containing the identity of R , where R is a subring of the centre of
A . If M is a finitely generated right A module, then*
$$f^{-1} (Ass. \ M^1_{A^1}) = Ass. \ (M_A) \cap \{P | P \cap S = \emptyset \} \ , \ where \ \ M^1 = M_S \ , \ A^1 = A_S$$
and f is the usual map $A \rightarrow A_S$.

Proof. First note that the proof of proposition (2.5) of Wangneo and
Tewari [14] can be carried over in the case of any right Noetherian ring.
Together with this result and propositions 1.1 and 1.2 above the whole
proof of proposition (2.6) of Wangneo and Tewari [14] carries over to
yield the required result.

Notation. Consider B any right Noetherian ring with R a subring
of the centre of B containing the identity of B . Let P be a prime
ideal of B . Denote by $F_{P \cap R}$ the set of prime ideals of B lying over
P .

DEFINITION. As on p. 1582 of Wangneo and Tewari [14], we define the
support of a module M over B where B is as in the notation above.
Let P be a prime ideal of B and localize M at $P \cap R$ in R (this

is the usual localization at the prime ideal of a commutative ring).
Denote the localized module of M by $M_{P \cap R}$. We say P is in the
support of M if $M_{P \cap R} \neq 0$. We say P is *minimal supporting* for M
if $P \cap R$ is minimal such that $M_{P \cap R} \neq 0$. Denote the support of M by
supp. M .

THEOREM 1.4. *Let* B *be a right Noetherian ring. Then*
Ass. $M \subseteq$ *supp.* M *for any finitely generated right module* M *over* B *,*
and for any minimal element P *of supp.* M *there is a prime ideal in*
the set $F_{P \cap R}$ *belonging to Ass.* M *, where* R *is a subring of the centre*
of B *containing the identity of* B .

Proof. With the aid of Proposition 1.1 and Proposition 1.3 above
the whole proof of Theorem 2.7 of Wangneo and Tewari [14] carries over
in this case.

Next, suppose that D is a division ring algebraic over a subfield
K of its centre. It is celar that ideals of $D[X]$ lie uniquely over
their contractions in $Z(D)[X]$ where $Z(D)$ is the centre of D . Since
$Z(D)$ is algebraic over K , we get that $Z(D)[X]$ is integral over $K[X]$
and hence, by a well-known result from commutative algebra, we get that
there is lying over, going up and incomparability between the prime
ideals of $D[X]$ and $K[X]$. Invoking the Wedderburn-Artin Theorem, we
can extend this result to any simple Artinian ring S algebraic over a
subfield K of its centre. Thus we have the following proposition:

PROPOSITION 1.5. *Let* S *be a simple Artinian ring algebraic over*
a subfield K *of its centre. Then there is lying over, going up and*
incomparability between the prime ideals of $S[X]$ *and* $K[X]$.

Note. We will henceforth abbreviate lying over, going up and
incomparability by l.o, G.U., and Inc.

THEOREM 1.6. *Let* A *be a right Noetherian ring integral over a*
subring R *of its centre. Then there is l.o, G.U. and Inc. between the*
prime ideals of $R[X]$ *and* $A[X]$.

Proof. We will give the proof for lying over. Let P be a prime
ideal of $R[X]$. Let $P \cap R = P_1$ be a prime ideal of R . By Blair [2]
there exists a prime ideal Q_1 of A lying over P_1 . Consider $\bar{A} = \dfrac{A}{Q_1}$.

Note that $R = \frac{R}{P_1}$ can be identified with a subring of the centre of $\overline{A}$,

where $\overline{A}$ is integral over $\overline{R}$. Localize $\overline{A}$ and $\overline{R}$ at the set S of

regular elements of $\overline{R}$ and denote all the localized objects by suffix S .

Clearly $\overline{R}_S$ is a field and, by Chamarie and Hudry ([3] corollary (1.8)),

$\overline{A}_S$ is a simple Artinian ring. By Proposition 1.5 above, there exists a

prime ideal $\overline{Q}_S$ of $\overline{A}_S[X]$ such that $\overline{P}_S = \overline{Q}_S \cap \overline{R}_S[X]$, where $\overline{Q}$ and $\overline{P}$

are the images of the prime ideals Q and P of A and R respectively

in $\overline{A}$ and $\overline{R}$. Clearly Q is a prime ideal of $A[X]$ lying over P of

$R[X]$.

The proofs for G.U. and Inc. are similar, and are left to the reader.

PROPOSITION 1.7. *Let M be a finitely generated critical right*

module over $A[X]$, where A is a right Noetherian ring integral over a

subring R of its centre. Let $Ann. M = I$, and let P be the assassinator

of M , then $\frac{A[X]}{I}$ is $\frac{P}{I}$ primary.

Proof. Clearly M is finitely generated, faithful and P-primary

module over $\frac{A[X]}{I}$ (denote $\frac{A[X]}{I}$ by A^1). Let Q be a prime ideal

associated with A^1 . Then $Q = Ann. M^1$ where M^1 is a non-zero

submodule of M . So $Q \subseteq \frac{P}{I}$. Next, let $\frac{R[X]}{I \cap R[X]}$ be the ring denoted by

R^1 . Also denote the module M localized at $Q \cap R^1$ by $M_{Q \cap R^1}$. Since,

$M_{Q \cap R^1} \neq 0$, there exists a minimal prime ideal P_1 of R^1 such that

$P_1 \subseteq Q \cap R^1$ and $M_{P_1} \neq 0$ where M_{P_1} has the usual meaning. So there

exists $P_1^{\ 1} \in F_{P_1 \cap R^1}$ such that $P_1^{\ 1}$ is associated with M (see Theorem

1.4). So $P_1^{\ 1} = \frac{P}{I}$. By what we have done above, it is apparent that

$Q \cap R^1 = \frac{P}{I} \cap R^1$. Since $Q \subseteq \frac{P}{I}$ and so by Inc. (Theorem 1.6) $Q = \frac{P}{I}$.

So A^1 is $\frac{P}{I}$ primary.

2.

In this section we prove that every finitely generated critical

module M over $A[X]$, where A is a Noetherian ring integral over a

subring R of its centre, is compressible.

DEFINITION. Let M be a right module over a ring B . Then M is said to be *finitely annihilated* over B if Ann. M is a finite intersection of annihilators of M .

PROPOSITION 2.1 (Goldie). *Let M be a finitely generated module over a right Noetherian ring B with Ass. $M = P$. If every cyclic uniform module W of M is such that Ass. $\frac{B}{\text{Ann. } W} = P^1$ and such that W is P^1-torsionfree, then M is finitely annihilated.*

Proof. With the conditions given above on B and W , the result follows by proposition (8) of Goldie [5].

The following proposition is useful. But before that we give a remark.

REMARK. If P is a prime ideal of a right Noetherian ring B , then we say that the torsion theory cogenerated by $E(B/P)$ is the torsion theory at P . This torsion theory is the same as the torsion theory at $c(P)$ where $c(P)$ is the set of elements of B regular modulo P . For this see Lembek [10].

PROPOSITION 2.2. *Let A be a right Noetherian ring integral over a subring R of its centre. Let P and Q be prime ideals of $A[X]$ lying over a prime ideal P_1 of $R[X]$ where P_1 is of the form $q[X]$ for some prime ideal q of R . Then the torsion theories at P and Q are equivalent.*

Proof. By Mueller ([11], theorem (1.3)) $\frac{A[X]}{P}$ is either torsion or torsionfree with respect to the torsion theory at Q . If $\frac{A[X]}{P}$ is torsion in this torsion theory then by the remark above there exists an element c of $A[X]$, c regular modulo Q such that $c \in P$. Let us denote by I the two-sided ideal generated by c in $A[X]$. Then, it follows that $I \cap R[X] \subseteq P \cap R[X] = P_1$. Also notice that since $P_1 = q[X]$, and P and Q lie over P_1 , we must have $P = P^1[X]$ and $Q = Q^1[X]$ for some prime ideals P^1 and Q^1 of A . Clearly $P^1 \cap R = Q^1 \cap R = q$. Localize A and R at q and denote these localized rings by A_S and R_S . Denote the ideal I localized at q by I_S and suppose the image

of c in I_S is c^1 . Let c have the form $c = a_0 + a_1 X + \ldots + a_n X^n$, $a_n \neq 0$. It is not difficult to see that a_n is regular modulo Q and hence regular modulo Q^1 . Clearly since A is integral over R , the image of a_n in A_S is invertible in A_S and so multiplying c^1 by a suitable element of A_S gives us a monic polynomial c_1 of degree n in I_S which is regular modulo Q_S . Also since c generates I and a_n is regular modulo Q , it follows that c is a polynomial of minimal degree in I . Hence c_1 is a monic polynomial of minimal degree in I_S . For any $r \in A_S$, $f = (rc_1 - c_1 r) \in I_S$ and f is of lower degree than c_1 . Hence $f = 0$. So $c_1 \in Z(A_S)[X]$ and c_1 is regular modulo Q_S . Since $Z(A_S)[X]$ is integral over $R[X]$, it can be easily shown that there exists an element g of $A_S[X]$ such that $0 \neq C_1 g \in R_S[X]$ and $C_1 g \notin P_1$. Hence $I \cap R[X] \not\subset P_1$ a contradiction to what we had in first paragraph of this proposition. Hence $\frac{R}{P}$ is torsionfree with respect to the torsion theory at Q . Since the above argument can be reversed, we get that the torsion theories at P and Q are the same.

DEFINITION. Let B be a ring and M a right module over B , such that $|M| = \alpha$. M is *called smooth* if $|N| = \alpha$, for every non-zero submodule N of M .

Now let A be a right Noetherian ring integral over a subring R of its centre. Let M be a finitely generated critical module over $A[X]$ with Ann. $M = I$. Clearly M is a finitely generated, faithful module over $\frac{A[X]}{I}$ (denote $\frac{A[X]}{I}$ by A^1). If M is P-primary for some prime ideal P of A^1 , then A^1 is P-primary by proposition 1.7. Consider $R^1 = \frac{R[X]}{I \cap R[X]}$. Let $P_1 = P \cap R^1$. Localize M , A^1 and R^1 at P_1 as usual and denote these localized objects by M_S , A^1_S and R^1_S etc. Then we have the following proposition.

PROPOSITION 2.3. *If, in the above notation, we have* $\left| \dfrac{A^1_S}{P_S} \right| = 1$,

then M is either compressible or finitely annihilated.

Proof. Let $P = \dfrac{P^1}{I}$. If P^1 is not of the form $q[X]$ for some prime ideal q of A , then $\dfrac{A[X]}{P^1} \approx (\dfrac{A}{P^1 \cap A})[\overline{X}]$ where $\overline{X}$ is algebraic over $\dfrac{A}{P^1 \cap A}$. If $S =$ complement of $(P^1 \cap R[X])$ in $R[X]$ then, localizing at S yields that $|(\dfrac{A[X]}{P^1})_S| = 0$ because $(\dfrac{A[X]}{P^1})_S$ is a proper homomorphic image of $(\dfrac{A[X]}{P^1 \cap A[X]})_S$ and $|(\dfrac{A[X]}{P^1 \cap A[X]})_S| \leqslant 1$. So, since $|\dfrac{A^1_S}{P_S}| = 1$, we get that P^1 is of the form $q[X]$ for some prime ideal q of A . Now, by Gordon and Robson ([6], Corollary (7.5)) there exists a prime ideal Q^1_S of $(A[X])_S$ such that $|A^1_S| = |(\dfrac{A[X]}{Q^1})_S|$. Clearly $\dfrac{Q^1_S}{I_S}$ contracts to the prime ideal P_S of R^1_S , and it is not difficult to see that Q^1 is of the form $q_1[X]$ for some prime ideal q_1 of A . So $|(\dfrac{A[X]}{Q^1})_S| = |A^1_S| = 1$. Now, if $|M_S| = 0$, then since M is uniform and hence so is M_S , it follows that M_S is simple. Let N be non-zero submodule of M , then $N_S \neq 0$ because M is P-primary and so is N . So $(\dfrac{M}{N})_S = 0$ which means that there exists an element s of R^1 such that $Ms \subseteq N$. The multiplication by s gives a monomorphism of M into N becasue no nonzero element of M can be annihilated by S . So M is compressible. Next consider $|M_S| = 1$. So $|\dfrac{A^1_S}{P_S}| = |M_S| = 1$. Let N_S be a non-zero maximal submodule of M_S such that $|N_S| = 0$. Clearly $|\dfrac{M_S}{N_S}| = 1$ and $\dfrac{M_S}{N_S}$ is smooth. Let B_S be the annihilator of $\dfrac{M_S}{N_S}$ in A^1_S . So $\dfrac{M_S}{N_S}$ is a faithful $\dfrac{A^1_S}{B_S}$-module. Let Q_S be a maximal ideal of A^1_S containing B_S . Then Q_S clearly lies over P_{1S} of R^1_S because R^1_S is a local ring and P_{1S} is also a unique minimal prime ideal of R^1_S for A^1_S is P_S-primary (see proposition 1.7). So $|\dfrac{A^1_S}{Q_S}| = 1$ by the same

argument as given in the beginning of this proposition. Hence $\left|\dfrac{A^1_S}{B_S}\right| = 1$

also. So we have that $\dfrac{M_S}{N_S}$ is a faithful $\dfrac{A^1_S}{B_S}$ module and $\left|\dfrac{A^1_S}{B_S}\right| = 1$.

Hence, by Gordon ([7], lemma (1.7)) $\dfrac{A^1_S}{B_S}$ embeds in a direct sum of copies

of $\dfrac{M_S}{N_S}$. So $\dfrac{A^1_S}{B_S}$ is smooth. Let $U_S \subseteq \dfrac{M_S}{N_S}$ be a nonzero critical prime

submodule of $\dfrac{M_S}{N_S}$ such that Ann. $U_S = J_S$ for some prime ideal J of A^1 .

Note that $|U_S| = \left|\dfrac{M_S}{N_S}\right| = \left|\dfrac{A^1_S}{J_S}\right| = 1$. Also $J = \dfrac{J^1}{I}$ where $J^1 = J_1[X]$ for

some prime ideal J_1 of A . Notice also that J^1 lies over P_1 .

Since $|U_S| = \left|\dfrac{A^1_S}{J_S}\right| = 1$, so U_S is J_S torsionfree. Now P_S and J_S

lie over the same prime ideal of R^1_S . So by proposition 2.2 the torsion

theories at P_S and J_S are the same. Hence U_S is P_S torsionfree

also. Hence U is P-torsionfree and since A^1 is P-primary we are in

the situation of proposition 2.1. So U is a finitely annihilated uniform

module over A^1 . Therefore $|U| = \left|\dfrac{A[X]}{\text{Ann. } U}\right| = \left|\dfrac{A[X]}{J_1[X]}\right| = \left|\dfrac{A[X]}{q[X]}\right| > |M|$,

the second and third equality follow because J_1 and q lie over the

same prime ideal in R . But this is a contradiction because M is

critical and U is a submodule of M/N . So $|N_S| \neq 0$. Hence M_S is

smooth and by the same argument as for $\dfrac{M_S}{N_S}$ and U_S above we get that M

is finitely annihilated over A^1 and hence over $A[X]$.

REMARK. If in the above proposition $\left|\dfrac{A^1_S}{P_S}\right| = 0$, then $|A^1_S| = 0$.

Clearly $|M_S| = 0$ in this case and so M is compressible. Combining

this with the above proposition, we have the following theorem.

THEOREM 2.4. *Let A be a right Noetherian ring integral over a*
subring R of its centre. If M is a finitely generated critical module

with   Ass. M = P   then

either   (i)   M is finitely annihilated
or   (ii) M is compressible.

Also in case (i) we have that P has the form $P_1[X]$ for some prime ideal P_1 of A .

In what follows we will use the concept of an affiliated bi-submodule, and affiliated series for a bimodule B of a Noetherian ring C . One can attach to B a set of primes P_i called the affiliated set of primes of B (or more strictly the affiliated set of primes of an affiliated series). For these definitions we refer the reader to Stafford [13].

Before proving the main result we quote a result of Gordon ([7], Proposition (3.6)) which is actually the crucial result from which our main theorem follows.

PROPOSITION 2.5 (Gordon). *Let B be a Noetherian ring of Krull dimension α such that B/N(B) is smooth, where N(B) is the nilpotent radical of B .*

(i)   A finitely generated module M is N-torsion (that is torsion in the torsion theory cogenerated by $E_B(B/N)$) if and only if $|M| < α$.

(ii) $\dfrac{B}{T_N(B)}$ is an order in an Artinian ring if and only if every finitely generated α-critical module is compressible. ($T_n(B)$ is the N-torsion ideal of B).

THEOREM 2.6. *Let A be a Noetherian ring integral over a subring R of its centre. Let M be a finitely generated critical module over A[X] , then M is compressible.*

Proof. By Theorem 2.4 we have to consider the only case when M is finitely annihilated, and the assassinator P of M is of the form $q[X]$ for some prime ideal q of A . Let $I =$ Ann. M in $A[X]$. Consider $A^1 = \dfrac{A[X]}{I}$ and $R^1 = \dfrac{R[X]}{I \cap R[X]}$ and as in proposition 2.3 let $\dfrac{P}{I} \cap R^1 = J = \dfrac{P_1}{I \cap R[X]}$ for some prime ideal P_1 of $R[X]$. Localize all these objects at P_1 and denote these localized objects as usual by

suffix S . Notice that $\dfrac{P_{1_S}}{(I \cap R[X])_S}$ is a unique maximal and minimal prime

ideal of $R^1{}_S$ and any prime ideal of $A^1{}_S$ lies over P_{1_S} and they are

all finite in number. Clearly all prime ideals of $(A[X])_S$ containing I

are of the form $(Q[X])_S$ for some prime ideal Q_1 of A . Let us

enumerate these prime ideals as $(Q_1[X])_S$, $(Q_2[X])_S$, ..., $(Q_n[X])_S$ for

some prime ideals $Q_1[X], Q_2[X], ..., Q_n[X]$ of $A[X]$, all containing I

and lying over the same prime ideal P_1 of $R[X]$. Consider the

nilpotent radical N of A^1 and let $C(0)$ be the set of regular elements

of A^1 and $C(N)$ be the set of elements of A^1 regular modulo N . It

is well known that $C(0) \subseteq C(N)$. Conversely, let $x \in C(N)$. Denote by

N^1 the image of N in $A^1{}_S$ and by x^1 the image of x in $A^1{}_S$.

Notice that N^1 is the nilpotent radical of $A^1{}_S$. Consider an affiliated

series $A^1{}_S = B_{m_S} \geqslant B_{m-1_S} \geqslant ... \geqslant B_{0_S} = (0)$ of $A^1{}_S$ where clearly B_i ,

$0 \leqslant i \leqslant m$ are bi-submodules of A^1 and B_{i_S} is the usual localization

of B_i at $\dfrac{P_1}{I \cap R[X]}$. The affiliated primes of this series will be primes

of the form $\{(\dfrac{P_i[X]}{I})_S$ for some i , $1 \leqslant i \leqslant n\}$. We can clearly lift this

affiliated series to an affiliated series of A^1 namely $A^1 \geqslant B_m \geqslant ... B_0 = 0$

with affiliated primes of the form $\{\dfrac{P_i[X]}{I}$, for some i , $1 \leqslant i \leqslant n\}$.

Now $\left| \dfrac{A^1}{xA^1 + \dfrac{P_i[X]}{I}} \right| = \left| \dfrac{A[X]}{x_1 A[X] + P_i[X]} \right|$ where $x = x_1 + I$

and $A^1 = \dfrac{A[X]}{I}$. Also $\left| \dfrac{A^1}{xA^1 + \dfrac{P_i[X]}{I}} \right| < \left| \dfrac{A^1}{\dfrac{P_i[X]}{I}} \right|$, $1 \leqslant i \leqslant n$.

So by proposition 2.3 of Stafford [13] we have that x is a right regular

element of A^1 . Since A^1 has left right symmetry, the same argument gives that x is left regular too. Hence $x \in C(0)$.

So $C(0) = C(N)$. Hence A^1 has an Artinian quotient ring by Small's Theorem [12] . Note that M is a faithful finitely annihilated module over A^1 . Therefore $|M| = |A^1|$. Also $\left|\frac{A[X]}{P_i[X]}\right| = \left|\frac{A[X]}{P_j[X]}\right|$, $\{1 \leqslant i, j \leqslant n\}$ because P_i and P_j lie over the same prime ideal of $R[X]$. Therefore $\frac{A}{N}$ is smooth. Hence, by proposition 2.5 above of Gordon, M is compressible over A^1 and hence over $A[X]$. This proves the theorem.

REMARK. Note that Theorem 1.6 can be extended to the case of a polynomial ring in n-commuting indeterminates, $n \geqslant 2$, namely $A[X_1, \ldots, X_n]$. We have not been able to prove whether all finitely generated critical modules over $A[X_1, \ldots, X_n]$ are compressible. The result seems to be not known even for the ring $D[X_1, \ldots, X_n]$ where D is a division ring algebraic over its centre k . However the result is true if $D(X_1, \ldots, X_n) = D[X_1, \ldots, X_n] \otimes_k k(X_1, \ldots, X_n)$ is an artinian ring, where $k(X_1, \ldots, X_n)$ is the quotient field of $k[X_1, \ldots, X_n]$. In the case of one indeterminate over D this is an open problem due to Amitsur; see for example Amitsur [1]. It may be remarked that $D(X_1, \ldots, X_n)$ is artinian if and only if $D[X_1, \ldots, X_n]$ is a fully bounded Noetherian ring.

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