INTEGRALS OF DYNAMICAL SYSTEMS LINEAR IN THE VELOCITIES

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1. Introduction

Kilmister (1) has discussed the existence of linear integrals of a dynamical system specified by generalized coordinates $q^\alpha (\alpha = 1, 2, ..., n)$ and a Lagrangian

$$L = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_\alpha\dot{q}^\alpha + a,$$

repeated indices being summed from 1 to n. He derived covariant conditions for the existence of such an integral, conditions which do *not* imply the existence of an ignorable coordinate. Boyer (2) discussed the conditions and found the most general Lagrangian satisfying the conditions for the case of two degrees of freedom ($n = 2$).

In this note it is shown that Kilmister's conditions are too restrictive and a set of necessary and sufficient conditions are obtained. The most general Lagrangian satisfying these conditions is found, without any restriction on the number of degrees of freedom of the system. Finally a comment is made on the " conditional integrals " discussed by Birkhoff (3).

2. Derivation of the Conditions

The equations of motion can be written in the covariant form

$$D\dot{q}^\alpha/Dt + a^{\alpha\rho}(a_{\rho;\beta} - a_{\beta;\rho})\dot{q}^\beta - a^{\alpha\rho}a_{;\rho} = 0, \qquad (1)$$

where the semi-colon denotes covariant differentiation, and D/Dt denotes the absolute derivative with respect to the " metric " $a_{\alpha\beta}$ and $a^{\alpha\rho}a_{\rho\beta} = \delta^\alpha_\beta$. The tensors $a_{\alpha\beta}$ and a^α will be used to raise and lower indices in the usual manner. Assume the existence of just one linear integral

$$b_\alpha\dot{q}^\alpha = 1. \qquad (2)$$

The condition for $b_\alpha\dot{q}^\alpha$ to be constant can be written in the covariant form

$$b_\alpha D\dot{q}^\alpha/Dt + \tfrac{1}{2}(b_{\alpha;\beta} + b_{\beta;\alpha})\dot{q}^\alpha\dot{q}^\beta = 0.$$

Using (1) to eliminate the generalized acceleration yields the condition

$$b^\rho(a_{\rho;\beta} - a_{\beta;\rho})\dot{q}^\beta - b^\rho a_{;\rho} = \tfrac{1}{2}(b_{\alpha;\beta} + b_{\beta;\alpha})\dot{q}^\alpha\dot{q}^\beta. \qquad (3)$$

Now assume that (3) is *not* an identity but is satisfied by virtue of (2). In fact (2) can be used to express (3) as a homogeneous quadratic form in the generalized

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velocities.   Comparing coefficients in this quadratic form leads to the condition

$$b^\rho(a_{\rho;\,\beta}-a_{\beta;\,\rho})b_\alpha+b^\rho(a_{\rho;\,\alpha}-a_{\alpha;\,\rho})b_\beta-2b^\rho a_{;\,\rho}b_\alpha b_\beta = b_{\alpha;\,\beta}+b_{\beta;\,\alpha} \qquad (4)$$

for the existence of the linear integral (2). Kilmister's conditions are contained in (4). It is shown later that (4) is a necessary and sufficient condition for the existence of the linear integral (2).

Suppose now that (4) admits a solution b_α. Consider $u = b_\alpha \dot{q}^\alpha$. Then a simple calculation, using (4), shows that

$$\dot{u} = (u-1)[b^\rho(a_{\rho;\,\beta}-a_{\beta;\,\rho})\dot{q}^\beta-(1+u)b^\rho a_{;\,\rho}].$$

Hence $u = 1$ is a *singular* solution unless

$$b^\rho(a_{\rho;\,b}-a_{\beta;\,\rho}) = b^\rho a_{;\,\rho} = 0 \qquad (5)$$

Substituting (5) into (4) yields

$$b_{\alpha;\,\beta}+b_{\beta;\,\alpha} = 0$$

and this is the condition for the existence of an ignorable coordinate.

As Kilmister states, the general situation, when more than one linear integral exists, is far more complicated.

3. Derivation of the General Lagrangian Admitting a Linear Integral

It is instructive to split (4) into components along b_α and perpendicular to b_α. The latter components are found by contracting indices with the perpendicular projection operator $h^\alpha_\nu = \delta^\alpha_\nu - b^{-2}b^\alpha b_\nu$, which satisfies $h^\alpha_\nu b_\alpha = 0$. The condition (4) can then be replaced by the following three conditions

$$b_{(\alpha;\,\beta)}h^\alpha_\nu h^\beta_\mu = 0, \qquad (6)$$

$$b_{(\alpha;\,\beta)}h^\alpha_\nu b^\beta = b^\rho a_{[\rho;\,\alpha]}b^2 h^\alpha_\nu \qquad (7)$$

and

$$b_{(\alpha;\,\beta)}b^\alpha b^\beta = -b^\rho a_{;\,\rho}b^4, \qquad (8)$$

where ( ) and [ ] denote the symmetric and antisymmetric parts respectively.

Suppose now that these equations admit a solution. Then the coordinates can be chosen to make $b^\alpha = \delta^\alpha_1$ so that $b_\alpha = a_{\alpha 1}$ and $b_{(\alpha;\,\beta)} = \frac{1}{2}a_{\alpha\beta,\,1}$. Equation (6) becomes

$$a_{\nu\mu,\,1}-a_{\nu 1,\,1}a_{\mu 1}a_{11}^{-1}-a_{1\mu,\,1}a_{\nu 1}a_{11}^{-1}+a_{11,\,1}a_{\nu 1}a_{\mu 1}a_{11}^{-2} = 0,$$

or

$$[a_{\nu\mu}-a_{\nu 1}a_{\mu 1}a_{11}^{-1}]_{,\,1} = 0.$$

Hence

$$a_{\nu\mu} = a_{\nu 1}a_{\mu 1}a_{11}^{-1}+A_{\nu\mu} \qquad (9)$$

where $A_{\nu\mu} = A_{\mu\nu}$ are functions independent of q^1 with $A_{\nu1} = 0$. Equation (7) becomes

$$a_{\nu1,1} - a_{11,1} a_{\nu1} a_{11}^{-1} = (a_{1,\nu} - a_{\nu,1}) a_{11}^{-1},$$

or

$$(a_{\nu1} a_{11}^{-1} + a_\nu)_{,1} = (a_{11} a_{11}^{-1} + a_1)_{,\nu}.$$

Hence

$$a_{\nu1} a_{11}^{-1} + a_\nu = \phi_{,\nu} + A_\nu$$

where A_ν are functions independent of q^1 with $A_1 \equiv 0$ and ϕ is an arbitrary function. In fact terms involving ϕ do not appear in the equations of motion and so, without loss of generality, it can be assumed that

$$a_\nu = A_\nu - a_{\nu1} a_{11}^{-1}. \tag{10}$$

Finally equation (8) yields

$$a = \tfrac{1}{2} a_{11}^{-1} + A \tag{11}$$

where A is a function independent of q^1. Combining (9), (10) and (11) gives an expression for the most general Lagrangian admitting a linear integral. The Lagrangian can best be written by introducing indices $i, j, \ldots$ which range and sum from 2 to n. Then

$$L = \tfrac{1}{2} a_{11}^{-1} (a_{\nu1} \dot{q}^\nu - 1)^2 + \tfrac{1}{2} A_{ij} \dot{q}^i \dot{q}^j + A_i \dot{q}^i + A,$$

where A_{ij}, A_j and A are all independent of q^1. The subcase discussed by Kilmister and Boyer is equivalent to $a_{\nu1} a_{11}^{-1}$ being independent of q^1.

4. Proof that the Condition is Necessary and Sufficient

Return now to equation (3). Suppose a linear integral exists and choose $b^\alpha = \delta_1^\alpha$. Then the integral can be written as

$$a_{\alpha1} \dot{q}^\alpha = 1.$$

Hence, since $a_{11} \neq 0$,

$$\dot{q}^1 = a_{11}^{-1} - a_{i1} a_{11}^{-1} \dot{q}^i.$$

Using this to eliminate $\dot{q}^1$ from (3) yields:

$$(a_{1;i} - a_{i;1}) \dot{q}^i - a_{,1} = \tfrac{1}{2} a_{ij,1} \dot{q}^i \dot{q}^j + a_{1j,1} \dot{q}^j (a_{11}^{-1} - a_{i1} a_{11}^{-1} \dot{q}^i)$$

$$+ \tfrac{1}{2} a_{11,1} (a_{11}^{-1} - a_{i1} a_{11}^{-1} \dot{q}^i)(a_{11}^{-1} - a_{j1} a_{11}^{-1} \dot{q}^j).$$

Since this is to be true for all $\dot{q}^i$ three equations are obtained. These are necessary and sufficient conditions for the existence of a linear integral and are the same equations as were obtained in the last section. It follows that the covariant condition (4) is both a necessary and sufficient condition for the existence of a linear integral.

5. Conditional Linear Integrals

Birkhoff discusses linear integrals for which (3) is not an identity nor is satisfied by virtue of (2) but is satisfied by virtue of the energy equation. Multiplying the equation of motion (1) by $\dot{q}_\alpha$ yields

$$\dot{q}^\beta \dot{q}_\alpha \dot{q}^\alpha{}_{;\,\beta} - \dot{q}^\rho a_{;\,\rho} = 0$$

so that

$$\frac{d}{dt}\left[\tfrac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - a\right] = 0.$$

Hence

$$\tfrac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - a = E \tag{12}$$

where the constant E is, of course, the total energy. Now (12) can be used to write the condition (3) for a linear integral in the form

$$b^\rho(a_{\rho;\,\beta} - a_{\beta;\,\rho})\dot{q}^\beta - \frac{\tfrac{1}{2} b^\rho a_{;\,\rho}}{E+a}\, a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = \tfrac{1}{2}(b_{\alpha;\,\beta} + b_{\beta;\,\alpha})\dot{q}^\alpha \dot{q}^\beta$$

which is satisfied provided that

$$b^\rho(a_{\rho;\,\beta} - a_{\beta;\,\rho}) = 0$$

and

$$\tfrac{1}{2}(b_{\alpha;\,\beta} + b_{\beta;\,\alpha}) = -\frac{b^\rho a_{;\,\rho}}{E+a}\, a_{\alpha\beta}.$$

The last equation is familiar in Riemannian geometry (4) and expresses the fact that the vector field b_α describes a conformal motion of the metric $a_{\alpha\beta}$. The most general Lagrangian admitting a conditional linear integral can be written in the form (i.e. choosing $b^\alpha = \delta^\alpha_1$)

$$L = \tfrac{1}{2} A_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta (E+a)^{-1} + A_\alpha \dot{q}^\alpha + a$$

where $A_{\alpha\beta}$ and A_α are independent of the coordinate q^1.

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