

MICROSCOPIC ASYMPTOTICS FOR SOLUTIONS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

TAKAYOSHI OGAWA AND TAKASHI SUZUKI

Dedicated to the memory of Professor Jongsik Kim

1. Introduction

In our previous work [8], we picked up the elliptic equation

$$(1) \quad \begin{cases} -\Delta u = \lambda f(u) e^{u^\alpha} & \text{in } B \equiv \{|x| < 1\} \subset \mathbf{R}^2, \\ u = 0 & \text{on } \partial B \end{cases}$$

with the nonlinearity $f(u) \geq 0$ in C^1 . We studied the asymptotics of the family $\{(\lambda, u(x))\}$ of classical solutions satisfying

$$(2) \quad \lambda \downarrow 0 \quad \text{and} \quad \|u\|_{L^\infty} \rightarrow +\infty.$$

Taking the result by Gidas-Ni-Nirenberg [5] into account, we may assume that the solution is radially symmetric and decreasing in $r = |x|$, i.e.,

$$u = u(|x|) \geq 0, \quad u_r < 0 \quad (0 < r = |x| \leq 1).$$

Furthermore, the coefficient nonlinear term $f(u)$ is supposed to have the polynomial growth. More precisely,

$$\begin{aligned} f'(u) &\geq 0 \quad (u \gg 1), \\ \lim_{u \rightarrow +\infty} (\log f)'(u) &= 0, \end{aligned}$$

and for some $k \in \mathbf{R}$,

$$(3) \quad 0 < \liminf_{u \rightarrow +\infty} f(u) u^{-k+\alpha-1} \leq \limsup_{u \rightarrow +\infty} f(u) u^{-k+\alpha-1} < +\infty.$$

First, the *global asymptotics* is stated as follows.

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PROPOSITION 1 ([8]). *Let (u, λ) be a family of solutions of (1) with (2).*

1. *If $0 < \alpha < 1$, then for any $x \in B$, $u(x) \rightarrow +\infty$ as $\lambda \rightarrow 0$.*
2. *If $\alpha > 1$, then $u(x) \rightarrow 0$ for any $x \in B \setminus \{0\}$ as $\lambda \rightarrow 0$.*

It is well-known that the solutions are expressed explicitly if $f(u) \equiv 1$ and $\alpha = 1$. In this case the singular limit is explicitly determined as

$$u(x) \rightarrow 4 \log \frac{1}{|x|} \text{ as } \lambda \downarrow 0.$$

Thus the exponent $\alpha = 1$ is the borderline of the global asymptotics.

Incidentally, the number of solutions for $f(u) = 1$ and $\alpha = 1$ is 0, 1, and 2 according to $\lambda > 2$, $\lambda = 2$, and $0 < \lambda < 2$, respectively. The unique solution for

$$\begin{cases} -\Delta u = 2e^u & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

is given as

$$u(x) = 2 \log \frac{2}{1 + |x|^2}.$$

This function plays an important role in *microscopic asymptotics* in the following. Henceforth, we suppose that $\alpha > 1$.

PROPOSITION 2 ([8]). *Passing to a subsequence, it holds that*

$$(4) \quad u^\alpha(e^{-\tau/2}y) = u^\alpha(e^{-\tau/2}) + 2 \log \frac{2}{1 + |y|^2} + o(1)$$

locally uniformly in $y \in \mathbf{R}^2 \setminus \{0\}$, where $\tau \rightarrow +\infty$ is taken appropriately.

The purpose of the present paper is to study the uniformity of (4). When the exponent is in $1 < \alpha < 2$, the following fact has proven in [8] with the aid of o.d.e. approach by Atkinson-Peletier [2].

PROPOSITION 3 ([8]). *In the case of $f(u) \equiv 1$ and $1 < \alpha < 2$, the uniform convergence in (4);*

$$\sup_{|y| \leq e^{\tau/2}} \left| u^\alpha(e^{-\tau/2}y) - u^\alpha(e^{-\tau/2}) - 2 \log \frac{2}{1 + |y|^2} \right| \rightarrow 0$$

never holds for any $\{\tau\}$.

In case of $\alpha > 2$, it is not known whether classical solutions for (1) with (2) exist or not ([1], [2]). The exponent $\alpha = 2$ is considered to be a borderline for the existence. What we want to claim here is that it is also the borderline from the microscopic asymptotic point of view. We shall give a uniform convergence result for this borderline case. The following theorem is the main result of the present paper.

THEOREM 4. *If $\alpha = 2$ and*

$$(5) \quad E_0 \equiv \limsup_{\lambda \rightarrow 0} \int_B |\nabla u|^2 dx < 6\pi,$$

then the convergence (4) is locally uniform in $y \in \mathbf{R}^2$. In other words, the uniform asymptotics near $y = 0$ is exactly expressed as in (4).

Concerning the existence of such a family, we have the following theorem.

THEOREM 5. *If $k > 2$ in (3), there exists a family $\{(\lambda, u(x))\}$ of classical solutions of (1) with $\alpha = 2$, satisfying (2) and*

$$(6) \quad E \equiv \int_B |\nabla u|^2 dx \rightarrow 4\pi.$$

In case of $\alpha < 2$, $E_0 < \infty$ implies that

$$\lambda f(u) e^{u^\alpha} \in L^{1+\varepsilon}(\Omega)$$

for some $\varepsilon > 0$ because of the Trudinger–Moser inequality ([11], [7]) i.e.

$$(7) \quad \sup_v \left\{ \int_\Omega e^{v^2} dx \mid \|\nabla v\|_2^2 \leq 4\pi \right\} \leq C |\Omega|.$$

Consequently the blow-up (2) does not occur by the standard elliptic estimates. In this sense, Theorems 4 and 5 are peculiar to the case $\alpha = 2$.

A similar observation also yields for the case $\alpha = 2$, that

$$\liminf_{\lambda \rightarrow 0} \int_B |\nabla u|^2 dx \geq 4\pi,$$

for the solution of (1) with (2). We shall give a more specified estimate (Lemma 8) for the Dirichlet integral for the solution by using the scaling parameter which will be a key estimate to show Theorem 4.

The special case

$$-\Delta u = \lambda u e^{u^2}, \quad u > 0 \text{ in } B = \{|x| < 1\} \subset \mathbf{R}^2$$

with

$$u = 0 \text{ on } \partial B$$

is closely related to the Trudinger–Moser inequality and also Carleson–Chang’s theorem ([4], see also [6], [10]). However, this case of $k = 2$ is not treated in Theorem 5. We shall pick up such a kind of equations in a forthcoming paper.

For the proof of Theorem 4, we invoke the following uniform estimate by Brezis–Merle [3]. Assume that $\Omega \subset \mathbf{R}^2$ is a bounded domain. Consider a family of solutions to

$$(8) \quad \begin{cases} -\Delta u_n = V_n e^{u_n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\{V_n\}$ is a given family of functions on Ω .

LEMMA 6 ([3]). *Let $\{V_n\}$ be given functions with*

$$\|V_n\|_{L^{p'}(\Omega)} \leq \beta$$

for some $1 < p \leq \infty$ and u_n be a solution of (8) in the sense of distribution. Suppose that

$$(9) \quad \int_{\Omega} |V_n| e^{v_n} dx \leq \gamma < \frac{4\pi}{p'}, \quad p' = p/p - 1$$

then the solution u_n is bounded independent of n , i.e.,

$$\|v_n\|_{L^\infty} \leq C(\beta, \gamma, \Omega, p).$$

The smallness assumption (5) in Theorem 4 comes from the assumption (9).

The proof of Theorem 4 is based on that of Proposition 2. In §2 we shall review the latter to perform the former in §3. The proof of Theorem 5 is independent and shall be given in §4.

2. Summary of the Proof of Proposition 2

We take the case $\alpha = 2$ for simplicity. Namely, we consider the smooth solution u of

$$(10) \quad \begin{cases} -\Delta u = \lambda f(u) e^{u^2} & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $f(u) \geq 0$ is a C^1 function satisfying (3).

The solution u becomes radially symmetric and has the property that

$$u_r < 0 \quad (0 < r = |x| \leq 1).$$

We put the scaling solution $v(r)$ as

$$(11) \quad v(r) = u^2(\gamma r) - u^2(\gamma)$$

for some scaling constant $\gamma \rightarrow 0$. This function v is subject to

$$(12) \quad \begin{cases} -\Delta v = k(r)e^v - \rho(r) & \text{in } B, \\ v > 0 & \text{in } B, \\ v = 0 & \text{on } \partial B, \\ v < 0 & \text{on } B_{\gamma^{-1}} \setminus B, \end{cases}$$

where we set

$$(13) \quad \begin{cases} k(r) = 2\lambda u(\gamma r) f(u(\gamma r)) e^{u(\gamma r)} \gamma^2, \\ \rho(r) = 2\gamma^2 |\nabla u(\gamma r)|^2 \end{cases}$$

and $B_{\gamma^{-1}} = \{x \in \mathbf{R}^2, |x| < \gamma^{-1}\}$.

Writing both equations (10) and (12) into the ODE form, we introduce the transformation $r = e^{-t/2}$, $U(t) = u(r)$ and $V(t) = v(r)$ to get

$$(14) \quad \begin{cases} \ddot{U} + \frac{\lambda}{4} f(U) e^{U^2-t} = 0, \\ U > 0 \quad (t > 0), \\ \dot{U} > 0 \quad (t > 0), \\ \dot{U} e^{t/2} \rightarrow 0 \quad (t \rightarrow +\infty) \end{cases}$$

and

$$(15) \quad \begin{cases} \ddot{V} + \frac{1}{4} K(t) e^{V(t)-t} - 2\dot{U}_\tau^2(t) = 0, \\ V > 0 \quad (t > 0), \\ \dot{V} > 0 \quad (t > 0), \\ \dot{V} e^{t/2} \rightarrow 0 \quad (t \rightarrow +\infty), \end{cases}$$

where $\tau = -2 \log \gamma$, $U_\tau(t) = U(t + \tau)$ and $K(t) \equiv 2\lambda U_\tau f(U_\tau) e^{U(\tau)^2-\tau} = k(r)$.

The equation (15) has a representation of the integral equation as

$$(16) \quad V(t) - V(+\infty) + \int_t^\infty (s-t) \frac{K(s)}{4} e^{V(s)-s} ds = 2 \int_t^\infty (s-t) U_\tau^2(s) ds.$$

In the proof of Proposition 1 (cf. [8]), the asymptotics

$$(17) \quad \eta \equiv \max_{0 \leq r \leq 1} |ru_r| \rightarrow 0$$

is proven. Since

$$\dot{U}_\tau(t) = -\frac{1}{2} ru_r \Big|_{r=\exp(-\frac{t+\tau}{2})},$$

it holds that

$$(18) \quad \|\dot{U}_\tau\|_{L^\infty(-\tau, \infty)} \rightarrow 0.$$

This relation deduces that

$$(19) \quad K(t) \rightarrow \text{constant, locally uniformly in } t \in \mathbf{R}.$$

Two cases should be distinguished for the parameter $\tau \rightarrow +\infty$ to be specified. Let

$$(20) \quad \begin{aligned} m &= \max_{0 \leq r \leq 1} 2\lambda u(r) f(u(r)) e^{u^2(r)} r^2 \\ &= \sup_{t \in \mathbf{R}} 2\lambda U(t) f(U(t)) e^{U^2(t)-t}. \end{aligned}$$

Case 1: $m \rightarrow +\infty$

In this case we can take $\tau \rightarrow +\infty$ as

$$(21) \quad K(0) = 2.$$

The asymptotics (18) and (19) imply that

$$(22) \quad \rho \rightarrow 0, \quad k \rightarrow 2 \text{ locally uniformly in } \mathbf{R}^2 \setminus \{0\}.$$

The relations (12) is reduced to

$$\|-\Delta v\|_{L^\infty(1 < |y| < R)} = O(1)$$

with

$$v = 0 \text{ on } |y| = 1$$

for any $R > 1$. Hence $\{v\}$ never blows-up on $\{|y| \geq 1\}$.

On the other hand, by (22), the boundedness of the equation (12) near ∂B follows and this implies

$$\|v_r\|_{L^\infty(\partial B)} \leq C.$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} \int_{B/B_{\varepsilon/2}} \{k(r)e^v - \rho(r)\} dx \\ = \int_{B/B_{\varepsilon/2}} (-\Delta v) dx \\ \leq -\omega_2 v_r(1) \leq C. \end{aligned}$$

While by (22),

$$\begin{aligned} \int_{B/B_{\varepsilon/2}} \{k(r)e^v - \rho(r)\} dx \\ \geq \int_{B/B_{\varepsilon/2}} \left\{ \frac{1}{2} e^v - \frac{1}{4} \right\} dx \\ \geq \pi \int_{\varepsilon/2}^{\varepsilon} v(r) r dr \\ \leq C\varepsilon^2 v(\varepsilon). \end{aligned}$$

Hence we obtain an a priori estimate for v on $\mathbf{R}^2 \setminus \{0\}$. Together with the equation (12), we may obtain the limit function v_0 as

$$v(r) \rightarrow v_0(r) \text{ locally uniformly on } \mathbf{R}^2 \setminus \{0\}$$

by the Ascoli-Arzelà theorem.

Finally, the singular limit $v_0(y) = 2 \log \frac{2}{1 + |y|^2}$ is specified through

$$-\Delta v_0 = 2e^{v_0} \text{ in } \mathbf{R}^2 \setminus \{0\}$$

and

$$v_0 \geq 0 \text{ on } |y| \leq 1.$$

Case 2: $m = O(1)$

In this case, we choose $r \rightarrow +\infty$ by

$$(23) \quad U^2(+\infty) = U^2(\tau) + 2 \log 2.$$

The condition $m = O(1)$ implies that

$$(24) \quad \|K\|_{L^\infty(-\tau, \infty)} = O(1)$$

and hence by passing to a subsequence,

$$K(0) \rightarrow 2\mu$$

for some $\mu \geq 0$.

From (19), it follows the convergence

$$K(t) \rightarrow 2\mu \text{ locally uniformly in } t \in (-\infty, +\infty),$$

while

$$\|v\|_{L^\infty(|y| < e^{\tau/2})} = 2 \log 2$$

holds by (23). Utilizing the elliptic estimate, we see that a subsequence of $\{v\}$ converges locally uniformly in $\mathbf{R}^2 \setminus \{0\}$. The limiting function $v_0(y)$ satisfies

$$\begin{aligned} -\Delta v_0 &= 2\mu e^{v_0} \text{ in } \mathbf{R}^2, \\ v_0 &= 0 \text{ on } |y| = 1 \end{aligned}$$

and

$$\|v_0\|_{L^\infty} = v_0(0) \leq 2 \log 2.$$

The conclusion $v_0(y) = 2 \log \frac{2}{1 + |y|^2}$ follows from $\mu = 1$ or equivalently $v_0(0) = 2 \log 2$. However, the right-hand side of (16) is non-negative and $V(+\infty) = 2 \log 2$. Therefore, the dominated convergence theorem implies that

$$\begin{aligned} 0 &\leq V_0(t) + \int_t^\infty (s-t) \frac{\mu}{2} e^{V_0(s)-s} ds - 2 \log 2 \\ &= V_0(+\infty) - 2 \log 2 \end{aligned}$$

for $V_0(t) = v_0(r)$, or equivalently,

$$v_0(0) \geq 2 \log 2.$$

This completes the proof. □

We note that the relation

$$\lambda u(e^{-\tau/2}) f(u(e^{-\tau/2})) e^{u^2(e^{-\tau/2})-\tau} = 1 + o(1)$$

follows from the proof.

3. Proof of Theorem 4

We have to prepare a few lemmas.

LEMMA 7. *The function $k(|y|)$ defined by (13) satisfies that*

$$(25) \quad \|k\|_{L^p(|y|<1)} = O(1) \text{ for } 1 < p < \infty.$$

Proof. In the case of $m = O(1)$, the uniform estimate (24) holds. Therefore, we have only to consider the case $m \rightarrow +\infty$.

Then, $\tau \rightarrow +\infty$ is determined through (21), i.e.,

$$2 = K(0) = 2\lambda U(\tau) f(U(\tau)) e^{U^2(\tau) - \tau}.$$

Hence

$$(26) \quad \begin{aligned} K(t) &= 2\lambda U_\tau(t) f(U_\tau(t)) e^{U^2(t) - \tau} \\ &= 2 \frac{U_\tau(t) f(U_\tau(t))}{U(\tau) f(U(\tau))} \approx \left(\frac{U_\tau(t)}{U(\tau)} \right)^k \end{aligned}$$

by (3).

Writing

$$\frac{U_\tau(t)}{U(\tau)} = 1 + \frac{1}{U(\tau)} \int_0^t \dot{U}_\tau(s) ds,$$

we reach

$$(27) \quad 0 \leq \frac{U_\tau(t)}{U(\tau)} \leq C(1+t) \quad (t \geq 0)$$

by (18). The conclusion (25) follows from (26), (27), and

$$k(r) = K(t) \text{ for } r = e^{-t/2}.$$

□

LEMMA 8. *For any fixed $R > 0$, we have*

$$(28) \quad 4\pi \leq \liminf_{\lambda \rightarrow 0} \int_{\{Re^{-\tau/2} < |x| < 1\}} |\nabla u|^2 dx.$$

Proof. As is described in the previous section,

$$V(t) \rightarrow V_0(t) \text{ locally uniformly in } t \in (-\infty, +\infty)$$

for

$$V_0(t) = 2 \log \frac{2}{1 + e^{-t}}.$$

Making use of the elliptic estimate in (15), this implies that

$$\dot{V}(t) \rightarrow \dot{V}_0(t) \text{ locally uniformly in } t \in (-\infty, +\infty).$$

Here,

$$\dot{V}(t) = 2U(t + \tau)\dot{U}(t + \tau)$$

and

$$\dot{V}_0(t) = \frac{2e^{-t}}{1 + e^{-t}}.$$

Therefore, writing $R = e^{-t/2}$, we obtain

$$(29) \quad -Re^{-\tau/2}u(Re^{-\tau/2})u_r(Re^{-\tau/2}) \rightarrow \frac{2R^2}{1 + R^2}.$$

The equation (10) deduces that

$$(30) \quad \int_{\{Re^{-\tau/2} < |x| < 1\}} |\nabla u|^2 dx = -2\pi Re^{-\tau/2}u(Re^{-\tau/2})u_r(Re^{-\tau/2}) \\ + \int_{Re^{-\tau/2} < |x| < 1} \lambda u f(u) e^2 dx.$$

Therefore, combining (29) and (30), we see for $\tilde{R} > R$

$$(31) \quad \frac{4\pi\tilde{R}^2}{1 + \tilde{R}^2} \leq \liminf_{\lambda \rightarrow 0} \int_{\{\tilde{R}e^{-\tau/2} < |x| < 1\}} (|\nabla u|^2 - \lambda u f(u) e^{u^2}) dx \\ \leq \liminf_{\lambda \rightarrow 0} \int_{\{\tilde{R}e^{-\tau/2} < |x| < 1\}} |\nabla u|^2 dx \\ \leq \liminf_{\lambda \rightarrow 0} \int_{\{Re^{-\tau/2} < |x| < 1\}} |\nabla u|^2 dx.$$

By taking \tilde{R} arbitrarily large, we obtain (28). □

LEMMA 9. *Under the assumption of*

$$(32) \quad \limsup_{\lambda \rightarrow 0} \int_B |\nabla u|^2 dx < 6\pi,$$

we have

$$(33) \quad \|v\|_{L^\infty(|y| < R)} = O(1)$$

for any $R > 0$. Here, the function $v(|y|)$ is defined by (11).

Proof. We prove this lemma by the aid of Lemma 6 in Section 1. The estimates (28) and (32) imply that

$$\limsup_{\lambda \rightarrow 0} \int_{\{|x| < R e^{-\tau/2}\}} |\nabla u|^2 dx < 2\pi \quad \text{for any } R > 0.$$

Hence the function $\rho(|y|)$ introduced in (13) satisfies that

$$(34) \quad \limsup_{\lambda \rightarrow 0} \|\rho\|_{L^1(|y| < 1)} = 4\pi \int_0^{r^{-1}} |\nabla u(r)|^2 dr < 4\pi.$$

We may suppose that $0 < R \ll 1$ in showing (33). Let us take the functions h_1 and h_2 as

$$-\Delta h_1 = 0 \quad \text{in } |y| < R, \quad h_1 = v \quad \text{on } |y| = R$$

and

$$-\Delta h_2 = -\rho \quad \text{in } |y| < R, \quad h_2 = 0 \quad \text{on } |y| = R.$$

We have already proven that

$$\|v\|_{L^\infty(|y|=R)} = O(1)$$

so that

$$(35) \quad \|h_1\|_{L^\infty(|y| < 1)} = O(1)$$

holds by the maximum principle. On the other hand, $\rho \geq 0$ and hence

$$(36) \quad h_2 \leq 0 \quad \text{in } |y| < R.$$

This implies the estimate

$$(37) \quad \|e^h\|_{L^\infty(|y| < R)} = O(1)$$

for

$$h = h_1 + h_2.$$

Because of (12), the function $w = v - h$ solves that

$$(38) \quad \begin{aligned} -\Delta w &= -\Delta v + \Delta h = -\Delta v + \rho \\ &= k e^v = k e^h e^w \quad \text{in } |y| < R \end{aligned}$$

and

$$(39) \quad w = 0 \quad \text{on } |y| = R.$$

Here, Lemma 7 and (37) are utilized to deduce

$$\|ke^h\|_{L^p(|y|<R)} = O(1) \text{ for } 1 < p < \infty.$$

On the other hand we have

$$(40) \quad \begin{aligned} \|ke^h e^w\|_{L^1(|y|<R)} &= \|ke^v\|_{L^1(|y|<R)} \\ &= \|\rho\|_{L^1(|y|<R)} + \int_{\{|y|<R\}} (-\Delta v) dx \end{aligned}$$

by (12) and $k, \rho \geq 0$. By Proposition 2, we have

$$v_r(R) = v_{0r}(R) + o(1).$$

Therefore, from (40) and (29) we obtain

$$\begin{aligned} \|ke^h e^w\|_{L^1(|y|<R)} &= \|\rho\|_{L^1(|y|<R)} - 2\pi R v_r(R) \\ &\leq \|\rho\|_{L^1(|y|<1)} - 2\pi R v_{0r}(R) + o(1) \\ &\leq \|\rho\|_{L^1(|y|<1)} + \frac{4\pi R^2}{1+R^2} + o(1) \end{aligned}$$

for $0 < R < 1$.

Here, we take $R > 0$ sufficiently small to deduce that

$$\limsup_{\lambda \rightarrow 0} \|ke^h e^w\|_{L^1(|y|<R)} < 4\pi$$

by (34).

Now, we can apply Lemma 6 for (38) with (29). Then it follows that

$$\|w\|_{L^\infty(|y|<R)} = O(1).$$

However, we have from (36),

$$(41) \quad 0 \leq v = w + h_1 + h_2 \leq w + h_1 \text{ in } |y| < R.$$

Consequently (33) follows from (35) and (41). □

We are in position to complete the proof of Theorem 4.

Proof of Theorem 4. As we have shown,

$$v \rightarrow v_0 \text{ locally uniformly in } \mathbf{R}^2 \setminus \{0\}$$

so that

$$(42) \quad V_0(+\infty) = 2 \log 2 \leq \liminf_{\lambda \rightarrow 0} V(+\infty).$$

Furthermore,

$$(43) \quad K(t) \rightarrow 2, \quad V(t) \rightarrow V_0(t) \text{ locally uniformly in } (-\infty, +\infty)$$

and also

$$(44) \quad \|V\|_{L^\infty(t_1, \infty)} = O(1) \text{ for any } t_1 \in \mathbf{R}$$

by Lemma 9. Finally, we have

$$(45) \quad |K(t)| \leq C(1+t)^m \text{ for } t \gg 1$$

from (26) and (27).

Here, the dominated convergence theorem is utilized to take the limit in (16). We obtain

$$\begin{aligned} 0 &\leq \liminf_{\lambda \rightarrow 0} \int_t^\infty 2(s-t) \dot{U}_\tau^2(s) ds \leq \limsup_{\lambda \rightarrow 0} \int_t^\infty 2(s-t) \dot{U}_\tau^2(s) ds \\ &\leq V_0(t) + \int_t^\infty \frac{1}{2} (s-t) e^{V_0(s)-s} ds - \liminf_{\lambda \rightarrow 0} V(+\infty) \\ &= V_0(+\infty) - \liminf_{\lambda \rightarrow 0} V(+\infty) \leq 0 \end{aligned}$$

by (42). Therefore,

$$(46) \quad \int_t^\infty 2(s-t) \dot{U}_\tau^2(s) ds \rightarrow 0 \quad (t \in \mathcal{R}).$$

Furthermore,

$$\begin{aligned} 0 &= \lim_{\lambda \rightarrow 0} \int_t^\infty 2(s-t) \dot{U}_\tau^2(s) ds \\ &= V_0(t) + \int_t^\infty \frac{1}{2} (s-t) e^{V_0(s)-s} ds - \lim_{\lambda \rightarrow 0} V(+\infty) \\ &= V_0(+\infty) - \lim_{\lambda \rightarrow 0} V(+\infty). \end{aligned}$$

Hence

$$(47) \quad V(+\infty) \rightarrow V_0(+\infty).$$

Going back to (16), we have

$$|V(t) - V_0(t)| \leq \int_t^\infty (s-t) 2 \dot{U}_\tau^2(s) ds + |V(+\infty) - V_0(+\infty)|$$

$$+ \int_t^\infty (s - t) \left| \frac{K(s)}{4} e^{V(s)} - \frac{1}{2} e^{V_0(s)} \right| e^{-s} ds$$

so that

$$\begin{aligned} \sup_{t \geq t_1} |V(t) - V_0(t)| &\leq \int_{t_1}^\infty (s - t_1) 2\dot{U}_\tau^2(s) ds + |V(+\infty) - V_0(+\infty)| \\ &\quad + \int_{t_1}^\infty (s - t_1) \left| \frac{K(s)}{4} e^{V(s)} - \frac{1}{2} e^{V_0(s)} \right| e^{-s} ds, \end{aligned}$$

where $t_1 \in \mathcal{R}$.

The first two terms converges to zero by (46) and (47). For the last term, we utilize (43)-(45) and the dominated convergence theorem. Thus we obtain

$$\|V - V_0\|_{L^\infty(t_1, \infty)} \rightarrow 0,$$

which means that

$$v \rightarrow v_0 \text{ locally uniformly in } \mathbf{R}^2,$$

the desired convergence. □

4. Proof of Theorem 5

The Trudinger-Moser inequality mentioned in Section 1 is expressed as

$$(48) \quad \sup_v \left\{ \int_B e^{v^2} dx \mid \|\nabla v\|_2^2 \leq 4\pi \right\} \leq C |\Omega|.$$

The constant 4π in (48) is shown to be best possibly by [7]. The following proposition is a slight refinement.

PROPOSITION 10. *For any continuous function $k(u) \geq 0$ with*

$$\lim_{u \rightarrow +\infty} k(u) = +\infty,$$

there exists a family $\{w\} \subset H_0^1(B)$ satisfying

$$w \geq 0, \int_B |\nabla w|^2 dx < 4\pi$$

and

$$\int_B k(w) e^{w^2} dx \rightarrow +\infty.$$

This fact is combined with the following lemma proven by Shaw via the Lagrange multiplier principle.

LEMMA 11 ([9]). *Suppose the existence of a non-negative function $w \in H_0^1(B)$ such that*

$$\int_B |\nabla w|^2 dx = \gamma < 4\pi.$$

Then, there exists a solution $(\lambda, u(x))$ for (10) such that

$$\int_B G(u) dx = \int_B G(w) dx$$

and

$$\int_B |\nabla u|^2 dx \leq \gamma,$$

where

$$G(u) = \int_0^u f(u) e^{u^2} du.$$

The condition (3) with $k > 2$ implies that

$$\lim_{u \rightarrow +\infty} f(u)/u = +\infty.$$

If

$$G(u) = k(u) e^{u^2},$$

this means that

$$\lim_{u \rightarrow +\infty} k(u) = +\infty.$$

Hence Proposition 10 and Lemma 11 are applicable.

We get a family $\{(\lambda, u(x))\}$ of solutions for (10) satisfying

$$(49) \quad \limsup_{\lambda \rightarrow 0} \int_B |\nabla u|^2 dx \leq 4\pi$$

and

$$(50) \quad \int_B G(u) dx \rightarrow +\infty.$$

The asymptotics (50) holds only when

$$\|u\|_{L^\infty} \rightarrow +\infty.$$

Furthermore, $f'(u) > 0$ for $u \gg 1$ so that there exists a constant $C > 0$ such that

$$G(u) = \int_0^u f(u) e^{u^2} du \leq C u f(u) e^{u^2} \quad (u \geq 0).$$

Therefore,

$$\begin{aligned} \lambda \int_B G(u) dx &\leq C \int_B \lambda u f(u) e^{u^2} dx \\ &= C \int_B |\nabla u|^2 dx = O(1) \end{aligned}$$

by (49) and hence

$$\lambda \downarrow 0$$

by (50).

In this way Theorem 5 has been reduced to Proposition 10. For the sake of completeness we show the proof, although it is quite similar to [7].

Proof of Proposition 10. The family is constructed from $W(t) = w(r)$ for $r = e^{-t/2}$. We have

$$\int_0^\infty \dot{W}^2 dt = \frac{1}{4\pi} \int_B |\nabla w|^2 dx$$

and

$$\int_0^\infty k(W) e^{W^2-t} dt = \frac{1}{\pi} \int_B k(w) e^{w^2} dx.$$

Therefore, the desired relations are reduced to

$$(51) \quad \{W\} \subset AC[0, \infty), \quad W(0) = 0, \quad W \geq 0,$$

$$(52) \quad \int_0^\infty \dot{W}^2 dt < 1$$

and

$$\int_0^\infty k(W) e^{W^2-t} dt \rightarrow +\infty,$$

where AC denotes the set of absolutely continuous functions.

Taking ε sufficiently small, we put

$$\eta_\varepsilon(s) = \min(s, 1 - \varepsilon)$$

and

$$W(t) = \varepsilon^{-1/2} \eta_\varepsilon(\varepsilon t).$$

For this function, the requirement (51) is obvious. The inequality (52) is examined as

$$\int_0^\infty \dot{W}^2 dt = \int_0^\infty \eta'_\varepsilon(s)^2 ds = (1 - \varepsilon)^2 < 1.$$

Finally, we conclude that

$$\begin{aligned} \int_0^\infty k(W) e^{W^2-t} dt &= \int_0^\infty k(\varepsilon^{-1/2} \eta_\varepsilon(s)) e^{\varepsilon^{-1} \eta_\varepsilon - \eta_\varepsilon^{-1} s} \eta^{-1} ds \\ &\geq k(\eta^{-1/2}(1 - \varepsilon)) \int_{1-\varepsilon}^\infty e^{(1-\varepsilon-s)\varepsilon^{-1}} \varepsilon^{-1} ds \\ &= k(\varepsilon^{-1/2}(1 - \varepsilon)) \rightarrow +\infty \text{ as } \varepsilon \downarrow 0. \end{aligned}$$

Thus the proof has been completed. \square

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Takayoshi Ogawa
Graduate School of Polymathematics
Nagoya University,
Furôchô, Chikusa-ku, Nagoya, 464-01, Japan

Takashi Suzuki
Department of Mathematics
Osaka University
Machikaneyamachô, Toyonaka, Osaka 560, Japan