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ON POSSIBLE VALUES OF THE INTERIOR ANGLE BETWEEN INTERMEDIATE SUBALGEBRAS

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Abstract

We show that all values in the interval $[0, \pi/2]$ can be attained as interior angles between intermediate subalgebras (as introduced by Bakshi and the first named author ['Lattice of intermediate subalgebras', *J. Lond. Math. Soc. (2)* **104**(2) (2021), 2082–2127]) of a certain inclusion of simple unital C^*-algebras. We also calculate the interior angles between intermediate crossed product subalgebras of any inclusion of crossed product algebras corresponding to any action of a countable discrete group and its subgroups on a unital C^*-algebra.

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1. Introduction

In any category, to classify its objects, the analysis of the relative positions of the subobjects of an object has proven to be a very rewarding approach. In the same vein, in the category of operator algebras, a great deal of work has been done by some eminent mathematicians—see, for instance, [1, 3, 6–8, 11] and the references therein. The theory of subfactors and, more generally, the theory of inclusions of (simple) C^*-algebras are two prominent aspects within this topic.

In this article, our focus lies only on unital C^*-algebras and their subalgebras. Over the years, various significant tools and theories have been developed to understand the relative positions of subalgebras of a given unital C^*-algebra. Among them, Watatani's notions of *finite-index conditional expectations* and C^*-basic constructions with respect to a finite-index conditional expectation [11] have proven to be fundamental in the development of the theory of inclusions of C^*-algebras—see [3, 6, 7, 9, 11]. Based on these two notions, and motivated by [1], very recently, Bakshi and the first named author, in [3], introduced the notions of *interior and exterior angles* between intermediate C^*-subalgebras of a given inclusion $B \subset A$ of unital C^*-algebras with a finite-index conditional expectation. As an application of the notion of interior angle, the authors in [3] were able to improve Longo's upper bound for the cardinality of the lattice of intermediate C^*-subalgebras of any irreducible inclusion of simple unital C^*-algebras.

Apart from the above mentioned quantitative application of the notion of interior angle, we expect some significant qualitative consequences too to be visible soon. In this direction, it is then quite natural to first ask whether one can make some concrete calculations of these angles and the possible values that they can attain. This article essentially answers these questions to a certain level of satisfaction. Being precise, through some elementary calculations, we are able to show that all values in the interval $[0, \pi/2]$ are attained as the interior angles between intermediate subalgebras of a certain inclusion of simple unital C^*-algebras. Further, motivated by [2], we also calculate the interior angle between intermediate crossed product subalgebras of any inclusion of crossed product algebras corresponding to any action of a countable discrete group and its subgroups on a given unital C^*-algebra.

The article is organized as follows.

After the introduction, we have a relatively longer section on preliminaries wherein we recall and derive some basic nuances related to finite-index conditional expectations and Watatani's C^*-basic construction related to inclusions of unital C^*-algebras. This discussion is fundamental to the formalism of the interior and exterior angles, which we briefly recall in Section 3; and, in the same section, we also derive some useful expressions related to them. Then, in Section 4, we prove that for any $t \in [0, \pi/2]$, there exists a 2×2 unitary matrix u such that the interior angle $\alpha(\Delta, u\Delta u^*) = t$ with respect to the canonical conditional expectation from $M_2(\mathbb{C})$ onto $\mathbb{C}$, where Δ denotes the diagonal subalgebra of $M_2(\mathbb{C})$; thereby, establishing that all values in the interval $[0, \pi/2]$ are attained as the interior angles between intermediate subalgebras. Finally, in Section 5, as an application of some expressions derived in Section 3, given any quadruple of countable discrete groups $H \subsetneq K, L \subsetneq G$ with $[G : H] < \infty$ and with an action α of G on a unital C^*-algebra P, we derive an expression for the interior angle between the (reduced as well as universal) intermediate crossed product subalgebras $P \rtimes K$ and $P \rtimes L$ of the inclusion $P \rtimes H \subset P \rtimes G$.

2. Preliminaries

2.1. Watatani's index and basic construction. In this subsection, we first recall Watatani's notions of finite-index conditional expectations and the C^*-basic constructions with respect to such conditional expectations; and then, we touch upon some generalities related to intermediate C^*-subalgebras.

2.1.1. Finite-index conditional expectations. Recall that, for an inclusion $B \subset A$ of unital C^*-algebras, a conditional expectation $E : A \to B$ is said to have *finite index* if there exists a finite set $\{\lambda_1, \ldots, \lambda_n\} \subset A$ such that

$$x = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^* = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x)$$

for every $x \in A$—see [6, 7, 11]. Such a set $\{\lambda_1, \ldots, \lambda_n\}$ is called a *quasibasis* for E and the Watatani index of E is defined as

$$\mathrm{Ind}(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*.$$

It is known that $\mathrm{Ind}(E)$ is a positive invertible element of $\mathcal{Z}(A)$ and is independent of the quasibasis $\{\lambda_i\}$—see [11, Section 2]. Also, E is faithful, $E(1_A) = 1_B$ and $\mathrm{Ind}(E) \geq 1$.

REMARK 2.1

(1) Suppose that $B \subset C \subset A$ are inclusions of unital C^*-algebras with $1_A \in B$, and $E : A \to B$, $F : A \to C$ and $G : C \to B$ are faithful conditional expectations satisfying $E = G \circ F$. Then E has finite index if and only if F and G have finite index—see [9, Proposition 3.5].

(2) For an inclusion $B \subset A$, in general, if $E, E' : A \to B$ are two conditional expectations, one may be of finite index and the other may fail to be so—see [11, Example 2.10.1].

Interestingly, if there exists a finite-index conditional expectation from A onto B, then all faithful conditional expectations from A onto B are of finite index if the centralizer of B in A (that is, $C_A(B) := \{x \in A : xb = bx \text{ for all } b \in B\}$) is finite-dimensional—see [11, Proposition 2.10.2].

Thus, when $C_A(B)$ is finite-dimensional, one can roughly say that the property of 'finite index' is an intrinsic property of the inclusion $B \subset A$ and not of a conditional expectation from A onto B.

(3) There exist finite-index conditional expectations even when the corresponding centralizers are not finite-dimensional. For instance, see [11, Example 2.6.7].

Let $A = C(X)$ and $B := A^\alpha$, where X is an infinite compact Hausdorff space and α is a free action of a finite group G on A. Define $E : A \to B$ by

$$E(f) = \frac{\sum_g \alpha_g(f)}{|G|}, \quad f \in A.$$

Then, E has finite index and $\mathrm{Ind}(E) = |G|$—see [11, Proposition 2.8.1]—whereas $C_A(B)$ is infinite dimensional as $A = C(X)$ is a commutative C^*-algebra.

2.1.2. Watatani's C^*-basic construction. Let $B \subset A$ be an inclusion of unital C^*-algebras with common unit and suppose $E : A \to B$ is a faithful conditional expectation. Let A_1 denote the Watatani C^*-basic construction of the inclusion $B \subset A$

with respect to the conditional expectation E, that is, in short, one essentially shows the following:

(1) A is a pre-Hilbert B-module with respect to the B-valued inner product given by

$$\langle a, a' \rangle_B := E(a^* a') \quad \text{for } a, a' \in A;$$

and, if $\mathfrak{A}$ denotes the Hilbert B-module completion of A, then

(2) the space of adjointable maps on $\mathfrak{A}$, denoted by $\mathcal{L}_B(\mathfrak{A})$, is a unital C^*-algebra (with the usual operator norm) and A embeds in it as a unital C^*-subalgebra (and, by a slight abuse of notation, we identify A with its image in $\mathcal{L}_B(\mathfrak{A})$);

(3) there exists a projection $e_B \in \mathcal{L}_B(\mathfrak{A})$ (called the *Jones projection* associated to E) such that $e_B a e_B = E(a) e_B$ for all $a \in A$ (it is standard to denote e_B by e_1 as well); and

(4) one considers $A_1 := \overline{\text{span}}\{x e_B y : x, y \in A\} \subseteq \mathcal{L}_B(\mathfrak{A})$, which turns out to be a C^*-algebra (not always unital) and is called the C^*-basic construction of the inclusion $B \subset A$.

The system (A, B, E, e_B, A_1) has the following natural universal property.

THEOREM 2.2 [11, Proposition 2.2.11]. *Let $B \subset A$ be an inclusion of unital C^*-algebras with a faithful conditional expectation $E : A \to B$. Suppose that A acts faithfully on some Hilbert space H and e is a projection on H satisfying $eae = E(a)e$ for all $a \in A$. If the linear map $B \ni b \mapsto be \in B(H)$ is injective, then there is a $*$-isomorphism $\theta : A_1 \to \overline{AeA} \subset B(H)$ such that $\theta(x e_B y) = xey$ for all $x, y \in A$.*

REMARK 2.3. If $E : A \to B$ has finite index with a quasibasis $\{\lambda_i\}$, then:

(1) the two norms $\|\cdot\|_A$ and $\|\cdot\|$ are equivalent on A (where $\|x\|_A := \|E_B(x^*x)\|^{1/2}$)—see [11] or the proof of [3, Lemma 2.11]; in particular, A itself is a Hilbert B-module;

(2) A_1 is unital and is equal to $C^*(A, e_B)$—see [11, Proposition 1.5];

(3) there exists a finite-index conditional expectation $E_1 : A_1 \to A$ (called the dual conditional expectation) with a quasibasis $\{\lambda_i e_B (\text{Ind}(E))^{1/2}\}$ that satisfies the equation

$$E_1(x e_B y) = \text{Ind}(E)^{-1} xy \tag{2-1}$$

for all $x, y \in A$ and $\text{Ind}(E_1) = \sum_i \lambda_i E(\text{Ind}(E)) e_B \lambda_i^*$; moreover, if $\text{Ind}(E) \in B$, then $\text{Ind}(E_1) = \text{Ind}(E)$—see [11, Propositions 2.3.2 and 2.3.4]; and

(4) if $F : A \to B$ is another finite index conditional expectation and $C^*(A, f_B)$ denotes the corresponding C^*-basic construction, then there exists a $*$-isomorphism $\theta : A_1 \to C^*(A, f_B)$ such that $\theta(e_B) = f_B$ and $\theta(a) = a$ for all $a \in A$—[11, Proposition 2.10.11]; and

(5) $A_1 = \text{span}\{x e_B y : x, y \in A\} =: A e_B A$—see [11, Lemma 2.2.2].

2.2. Intermediate C^*-subalgebras. Throughout this subsection, we let $B \subset A$ be an inclusion of unital C^*-algebras, $E : A \to B$ be a finite-index conditional expectation with a quasibasis $\{\lambda_i : 1 \leq i \leq n\}$, $A_1 := Ae_BA$ $(= C^*(A, e_B))$ denote the C^*-basic construction of $B \subset A$ with respect to E and $E_1 : A_1 \to A$ denote the dual conditional expectation.

As in [6], let $\mathrm{IMS}(B, A, E)$ denote the set of intermediate C^*-subalgebras C between B and A with a conditional expectation $F : A \to C$ satisfying the compatibility condition $E = E_{\restriction_C} \circ F$.

REMARK 2.4

(1) If $C \in \mathrm{IMS}(B, A, E)$ with respect to two compatible conditional expectations $F, F' : A \to C$, then $F = F'$—see [6, page 3].
(2) If $C \in \mathrm{IMS}(B, A, E)$ with respect to the compatible conditional expectation $F : A \to C$, then F is faithful (since E is so) and, therefore, by Remark 2.1(1), F has finite index.
(3) It must be mentioned here that it was presumed (without mention) in [3] that the compatible conditional expectation has finite index and was implicitly used while defining the notions of interior and exterior angles between intermediate subalgebras of an inclusion of unital C^*-algebras.
(4) For $C \in \mathrm{IMS}(B, A, E)$ with respect to the compatible conditional expectation $F : A \to C$, we observe that A is a Hilbert C-module (Remark 2.3(2)); we let e_C denote the corresponding Jones projection in $\mathcal{L}_C(A)$ and C_1 denote the Watatani basic construction of the inclusion $C \subset A$; and thus, $C_1 = C^*(A, e_C) \subseteq \mathcal{L}_C(A)$.

REMARK 2.5. In general, if $Q \subset P$ is an inclusion of unital C^*-algebras with a finite-index conditional expectation $G : P \to Q$, then not every intermediate C^*-subalgebra R of $Q \subset P$ belongs to $\mathrm{IMS}(Q, P, G)$—see [6, Example 2.5]. In fact, the example given in [6] illustrates that there need not exist even a single conditional expectation from P onto R.

Izumi observed that the intermediate subalgebras of an inclusion of simple C^*-algebras have certain specific structures.

PROPOSITION 2.6 [7]. *Let $B \subset A$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : A \to B$. If either A or B is simple, then every C in* $\mathrm{IMS}(B, A, E)$ *is a finite direct sum of simple closed two-sided ideals.*

PROOF. Let $C \in \mathrm{IMS}(\mathrm{B}, \mathrm{A}, \mathrm{E})$ with respect to the compatible conditional expectation $F : A \to C$. Then, by Remark 2.4(2) and [11, Proposition 2.1.5], F and $E_{\restriction_C}$ satisfy the Pimsner–Popa inequality. Further, since A or B is simple and unital, it then follows from [7, Theorem 3.3] that C is a finite direct sum of simple closed two-sided ideals. □

The following useful observations are needed ahead when we recall and derive some generalities related to the notions of interior and exterior angles.

PROPOSITION 2.7. *Let $B \subset A$ be an inclusion of unital C^*-algebras, $E : A \to B$ be a finite-index conditional expectation with a quasibasis $\{\lambda_i : 1 \leq i \leq n\}$, A_1 denote the*

C^*-basic construction of $B \subset A$ with respect to E, $E_1 : A_1 \to A$ denote the dual conditional expectation and $C \in \mathrm{IMS}(B, A, E)$ with respect to the compatible finite-index conditional expectation $F : A \to C$. Then:

(1) $\mathcal{L}_C(A) \subset \mathcal{L}_B(A)$;
(2) $C_1 \subset A_1$, *so that* $e_C \in A_1$;
(3) $e_C e_B = e_B = e_B e_C$;
(4) $E_{\restriction_C}$ *has finite index with a quasibasis* $\{F(\lambda_i)\}$ *and* $e_C = \sum \mu_j e_B \mu_j^*$ *for any quasibasis* $\{\mu_j\}$ *of the conditional expectation* $E_{\restriction_C}$;
(5) $E_1(e_B) = \mathrm{Ind}(E)^{-1} \in \mathcal{Z}(A)$;
(6) $E_1(e_C) = \mathrm{Ind}(E)^{-1}\mathrm{Ind}(E_{\restriction_C}) \in \mathcal{Z}(C)$; *and,*
(7) *in addition, if* $\mathrm{Ind}(E_{\restriction_C}) \in \mathcal{Z}(A)$, *then:*
   (a) $\mathrm{Ind}(E) = \mathrm{Ind}(F)\mathrm{Ind}(E_{\restriction_C})$;
   (b) $E_{1\restriction_{C_1}} = F_1$, *where* F_1 *denotes the dual conditional expectation of* F; *and*
   (c) $C_1 \in \mathrm{IMS}(A, A_1, E_1)$ *with respect to the conditional expectation* $G : A_1 \to C_1$ *satisfying* $G(xe_B y) = \mathrm{Ind}(E_{\restriction_C})^{-1} x e_C y$ *for all* $x, y \in A$ *and has a quasibasis* $\{\lambda_i e_B \mathrm{Ind}(E_{\restriction_C})^{1/2} : 1 \le i \le n\}$.

   In particular, we then have $E_1(e_C) = \mathrm{Ind}(F)^{-1}$.

PROOF. (1) Let $T \in \mathcal{L}_C(A)$ and T^* denote its adjoint in $\mathcal{L}_C(A)$. Then, we see that

$$\begin{aligned}\langle T(x), y\rangle_B &= E(T(x)^* y) = (E_{\restriction_C} \circ F)(T(x)^* y) = E_{\restriction_C}(\langle T(x), y\rangle_C) \\ &= E_{\restriction_C}(\langle x, T^*(y)\rangle_C) = (E_{\restriction_C} \circ F)(xT^*(y)) = \langle x, T^*(y)\rangle_B\end{aligned}$$

for all $x, y \in A$. Hence, $T \in \mathcal{L}_B(A)$.

Because of item (1), item (2) now follows on the lines of [3, Lemma 4.2].

(3) Clearly, $e_C e_B = e_B$ (as $B \subset C$). Next, we observe that

$$e_B e_C(a) = e_B(F(a)) = E(F(a)) = (E_{\restriction_C} \circ F)(a) = E(a) = e_B(a)$$

for all $a \in A$. Thus, $e_B e_C = e_B$.

(4) That $E_{\restriction_C}$ has finite index with quasibasis $\{F(\lambda_i)\}$ follows from [6, page 3] (also see [11, Proposition 1.7.2]). Further, for any quasibasis $\{\mu_j\}$ for $E_{\restriction_C}$, we have

$$\begin{aligned}\Big(\sum \mu_j e_B \mu_j^*\Big)(a) &= \sum \mu_j e_B \mu_j^*(a) \\ &= \sum \mu_j E(\mu_j^*(a)) \\ &= \sum \mu_j (E_{\restriction_C} \circ F)(\mu_j^*(a)) \\ &= \sum \mu_j E_{\restriction_C}(\mu_j^* F(a)) \\ &= F(a) \\ &= e_C(a)\end{aligned}$$

for all $a \in A$. Hence, $e_C = \sum \mu_j e_B \mu_j^*$.

(5) See [11, Proposition 2.3.2].

(6) For any quasibasis $\{\mu_j\}$ for the conditional expectation $E_{\restriction_C}$, we have $e_C = \sum \mu_j e_B \mu_j^*$. Hence,

$$\begin{aligned} E_1(e_C) &= E_1\Big(\sum \mu_j e_B \mu_j^*\Big) \\ &= \sum E_1(\mu_j e_B \mu_j^*) \\ &= \mathrm{Ind}(E)^{-1} \sum \mu_j \mu_j^* \\ &= \mathrm{Ind}(E)^{-1} \mathrm{Ind}(E_{\restriction_C}). \end{aligned}$$

(7a) Let $\{\mu_1, \mu_2, \ldots, \mu_n\}$ be a quasibasis for $E_{\restriction_C}$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a quasibasis for F. Then, it is (known and can be) easily seen that $\{\gamma_i \mu_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a quasibasis for E—see also [11, Proposition 1.7.1]. Thus,

$$\begin{aligned} \mathrm{Ind}(E) &= \sum_{i,j} (\gamma_i \mu_j)(\gamma_i \mu_j)^* \\ &= \sum_i \gamma_i \Big(\sum_j \mu_j \mu_j^*\Big) \gamma_i^* \\ &= \mathrm{Ind}(E_{\restriction_C}) \mathrm{Ind}(F). \end{aligned}$$

(7b) We have $C_1 = \mathrm{span}\{x e_C y : x, y \in A\}$. Fix a quasibasis $\{\mu_j\}$ for $E_{\restriction_C}$. Then, for every pair $x, y \in C$, we observe that

$$\begin{aligned} E_1(x e_C y) &= E_1\Big(x \sum_j \mu_j e_B \mu_j^* y\Big) \\ &= \mathrm{Ind}(E)^{-1} \sum_j x \mu_j \mu_j^* y \\ &= \mathrm{Ind}(E)^{-1} x \, \mathrm{Ind}(E_{\restriction_C}) y = \mathrm{Ind}(F)^{-1} xy \\ &= F_1(x e_C y), \end{aligned}$$

where the second last equality follows from item (7a). Hence, $(E_1)_{\restriction_{C_1}} = F_1$.

(7c) We have $A_1 = \mathrm{span}\{x e_B y : x, y \in A\}$. Consider the linear map $G : A_1 \to C_1$ given by

$$G\Big(\sum_i x_i e_B y_i\Big) = \mathrm{Ind}(E_{\restriction_C})^{-1} \sum_i x_i e_C y_i$$

for $x_i, y_i \in A$, $i = 1, \ldots, n$.

We first assert that G is a conditional expectation of finite-index.

Fix a quasibasis $\{\mu_j\}$ for $E_{\restriction_C}$. Then, for any $x, y \in A$, by item (4), we have

$$
\begin{aligned}
G(xe_C y) &= G\Big(x \sum_j \mu_j e_B \mu_j^* y\Big) \\
&= \mathrm{Ind}(E_{\restriction_C})^{-1} \sum_j x\mu_j e_C \mu_j^* y \\
&= \mathrm{Ind}(E_{\restriction_C})^{-1} \sum_j x\mu_j \mu_j^* e_C y \quad (\text{since } e_C \in C' \cap C_1) \\
&= \mathrm{Ind}(E_{\restriction_C})^{-1} x \mathrm{Ind}(E_{\restriction_C}) e_C y \\
&= xe_C y \quad (\text{since } \mathrm{Ind}(E_{\restriction_C}) \in \mathcal{Z}(A)).
\end{aligned}
$$

This implies that $G^2 = G$. Further, for any $\sum_i x_i e_B y_i \in A_1$,

$$
\begin{aligned}
G\Big(\Big(\sum_i x_i e_B y_i\Big)^* \Big(\sum_i x_i e_B y_i\Big)\Big) &= G\Big(\sum_{i,j} y_i^* E(x_i^* x_j) e_B y_j\Big) \\
&= \mathrm{Ind}(E_{\restriction_C})^{-1} \sum_{i,j} y_i^* e_C E(x_i^* x_j) e_C y_j.
\end{aligned}
$$

Then, taking $a_{i,j} := e_C E(x_i^* x_j) e_C \in C_1$, $i, j = 1, \ldots, n$,

$$
[a_{i,j}] = \mathrm{diag}(e_C, \ldots, e_C)[E(x_i^* x_j)]\mathrm{diag}(e_C, \ldots, e_C).
$$

By [10, Lemma 3.1], $[x_i^* x_j]$ is positive in $M_n(A)$ and since $E : A \to B$ is completely positive, it follows that $[E(x_i^* x_j)]$ is positive in $M_n(B)$. Hence, $[a_{i,j}]$ is positive in $M_n(C_1)$. Thus, by [10, Lemma 3.2], it follows that $\sum_{i,j} y_i^* e_C E(x_i^* x_j) e_C y_j \geq 0$ in C_1. Further, since $\mathrm{Ind}(E_{\restriction_C})^{-1} \in \mathcal{Z}(A) \cap \mathcal{Z}(C)$ and $e_C \in C' \cap C_1$, it follows that $\mathrm{Ind}(E_{\restriction_C})^{-1}$ commutes with $\sum_{i,j} y_i^* e_C E(x_i^* x_j) e_C y_j$ and hence

$$
G\Big(\Big(\sum_i x_i e_B y_i\Big)^* \Big(\sum_i x_i e_B y_i\Big)\Big) \geq 0.
$$

Thus, $G : A_1 \to C_1$ is positive and, therefore, it is a conditional expectation.

Further, $G : A_1 \to C_1$ has finite index with quasibasis $\{\lambda_i e_B \mathrm{Ind}(E_{\restriction_C})^{1/2} : 1 \leq i \leq n\}$ because, for any $x, y \in A$,

$$
\begin{aligned}
&\sum_i \lambda_i e_B \mathrm{Ind}(E_{\restriction_C})^{1/2} G((\mathrm{Ind}(E_{\restriction_C})^{1/2} e_B \lambda_i^* x e_B y) \\
&\quad = \sum_i \lambda_i e_B (\mathrm{Ind}(E_{\restriction_C})^{1/2}) G((\mathrm{Ind}(E_{\restriction_C})^{1/2}) E(\lambda_i^* x) e_B y) \\
&\quad = \sum_i \lambda_i e_B \mathrm{Ind}(E_{\restriction_C})^{1/2} \mathrm{Ind}(E_{\restriction_C})^{-1} \mathrm{Ind}(E_{\restriction_C})^{1/2} E(\lambda_i^* x) e_C y \\
&\quad = xe_B y \quad (\text{since } e_C \in C_1 \cap B', e_B e_C = e_B \text{ and } e_B \in A_1 \cap B').
\end{aligned}
$$

Finally, since $E_{1\restriction_{C_1}} = F_1$, we observe that

$$\begin{aligned}(E_{1\restriction_{C_1}} \circ G)(xe_By) &= E_1(\mathrm{Ind}(E_{\restriction_C})^{-1}xe_Cy)\\ &= F_1(\mathrm{Ind}(E_{\restriction_C})^{-1}xe_Cy)\\ &= \mathrm{Ind}(E_{\restriction_C})^{-1}\mathrm{Ind}(F)^{-1}xy\\ &= \mathrm{Ind}(E)^{-1}xy\\ &= E_1(xe_By)\end{aligned}$$

for all $x, y \in A$, where the second-last equality follows from item (7a). Hence, $(E_{1\restriction_{C_1}} \circ G) = E_1$.

These show that $C_1 \in \mathrm{IMS}(A, A_1, E_1)$ with respect to the finite-index conditional expectation $G : A_1 \to C_1$. □

Recall that for an inclusion $B \subset A$ of unital C^*-algebras, the normalizer of B in A is defined as

$$\mathcal{N}_A(B) := \{u \in \mathcal{U}(A) : uBu^* = B\},$$

where $\mathcal{U}(A)$ denotes the group of unitaries in A; and (as already mentioned above) the centralizer of B in A is defined as

$$C_A(B) := \{a \in A : ab = ba \text{ for all } b \in B\}.$$

Clearly, $\mathcal{U}(B)$ is a normal subgroup of $\mathcal{N}_A(B)$ and $C_A(B)$ is a unital C^*-subalgebra of A, which is also denoted by $B' \cap A$ and is called the *relative commutant* of B in A.

The following observation provides us with some easy examples of elements in $\mathrm{IMS}(B, A, E)$.

LEMMA 2.8. *Let $B \subset A$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : A \to B$. Let $C \in \mathrm{IMS}(B, A, E)$ with respect to the compatible finite-index conditional expectation $F : A \to C$ and $u \in \mathcal{U}(A)$. Then:*

(1) *$F_u : A \to uCu^*$ given by $F_u = \mathrm{Ad}_u \circ F \circ \mathrm{Ad}_{u^*}$, that is, $F_u(a) = uF(u^*au)u^*$ for $a \in A$, is a finite-index conditional expectation with a quasibasis $\{\eta_i u^* : 1 \leq i \leq n\}$ (as well as $\{u\eta_i u^* : 1 \leq i \leq n\}$), where $\{\eta_i : 1 \leq i \leq n\}$ is a quasibasis for F; $\mathrm{Ind}(F_u) = \mathrm{Ind}(F)$; and,*

(2) *in addition, if $u \in \mathcal{N}_A(B)$ and E satisfies the tracial property, that is, $E(xy) = E(yx)$ for all $x, y \in A$, then $uCu^* \in \mathrm{IMS}(B, A, E)$ with respect to F_u.*

PROOF. (1) is a straightforward verification.

(2) Let $D := uCu^*$. Since $u \in \mathcal{N}_A(B)$, $B = uBu^* \subset uCu^*$ and

$$\begin{aligned}E_{\restriction_D} \circ F_u(a) &= E_{\restriction_D}(uF(u^*au)u^*)\\ &= E(u^*uF(u^*au)) \quad \text{(by tracial property)}\\ &= E_{\restriction_C} \circ F(u^*au)\\ &= E(u^*au)\end{aligned}$$

$$\begin{aligned} &= E(uu^*a) \quad \text{(by tracial property again)} \\ &= E(a) \end{aligned}$$

for all $a \in A$. Hence, $uCu^* \in \mathrm{IMS}(B, A, E)$ with respect to F_u. □

REMARK 2.9. Note that ue_Cu^* is a projection in C_1 (as $u \in A \subset C_1$) and, for each $x \in A$,

$$(ue_Cu^*)x(ue_Cu^*) = uF(u^*xu)e_Cu^* = F_u(x)ue_Cu^*.$$

So, it is quite tempting to think that maybe the basic construction of $B \subset uCu^*$ is given by $(uCu^*)_1 = C_1$ (as the C^*-algebra) with Jones projection $e_{uCu^*} = ue_Cu^*$. However, this is not the case.

For instance, if we let $A, B, C, E : A \to B$ and $F : A \to C$ be the same as in Section 4, then taking

$$u = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

we observe that

$$ue_Cu^* = \begin{bmatrix} 1/2 & 0 & -i/2 & 0 \\ 0 & 1/2 & 0 & i/2 \\ i/2 & 0 & 1/2 & 0 \\ 0 & -i/2 & 0 & 1/2 \end{bmatrix}$$

whereas

$$e_{uCu^*} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

(using values of e_C from Lemma 4.2 and e_{uCu^*} from Lemma 4.3).

REMARK 2.10

(1) In general, the dual conditional expectation of a tracial conditional expectation need not be tracial.

For instance, consider the inclusion $B = \mathbb{C} \ni \lambda \hookrightarrow (\lambda, \lambda) \in A = \mathbb{C} \oplus \mathbb{C}$ with respect to the conditional expectation $E : A \to B$ given by $E((\lambda, \mu)) = (\lambda + \mu)/2$. Clearly, E is a finite-index tracial conditional expectation and we see that one can identify A_1 with $M_2(\mathbb{C})$ and then the dual conditional expectation $E_1 : A_1 \to A$ is given by $E_1([a_{ij}]) = (a_{11}, a_{22})$. Clearly, $E_1(AB) \neq E_1(BA)$ for $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(2) It is natural to wonder whether the traciality of E can be dropped or not while showing that uCu^* belongs to IMS(B, A, E) with respect to F_u. And, it turns out that it cannot always be dropped.

For instance, consider $A = M_2(\mathbb{C})$ and $B = \mathbb{C}I_2$ with the conditional expectation $E : A \to B$ given by $E([a_{ij}]) = a_{11}t + a_{22}(1-t)$ where t $\neq 1/2$ is fixed. Let $C = \{\mathrm{diag}(\lambda, \mu) : \lambda, \mu \in \mathbb{C}\}$ and $F : A \to C$ be the conditional expectation defined by $F([a_{ij}]) = \mathrm{diag}(a_{11}, a_{22})$. Clearly, E and F are finite-index conditional expectations with quasibases $\{\sqrt{t}e_{11}, \sqrt{1-t}e_{12}, \sqrt{t}e_{21}, \sqrt{1-t}e_{22}\}$ and $\{e_{ij} : 1 \leq i, j \leq 2\}$, and $E_{\restriction_C} \circ F = E$. If

$$u = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

then $u \in U(2)$ and

$$F_u([a_{ij}]) = \begin{bmatrix} (a_{11} + a_{22})/2 & (a_{12} - a_{21})/2 \\ (a_{21} - a_{12})/2 & (a_{11} + a_{22})/2 \end{bmatrix}$$

for all $[a_{ij}] \in A$. Thus, $E_{\restriction_{uCu^*}} \circ F_u([a_{ij}]) = (a_{11} + a_{22})/2$ which is not equal to $E([a_{ij}])$ (as $t \neq 1/2$).

3. Interior and exterior angles

Let $B \subset A$ be an inclusion of unital C^*-algebras with a conditional expectation $E : A \to B$. Then, for the B-valued inner product $\langle \cdot, \cdot \rangle_B$ on A given by $\langle x, y \rangle_B = E(x^*y)$, one has the following well-known analogue of the Cauchy–Schwarz inequality

$$\|\langle x, y \rangle_B\| \leq \|x\|_A \|y\|_A \quad \text{for all } x, y \in A, \tag{3-1}$$

where $\|x\|_A := \|E_B(x^*x)\|^{1/2}$. And, unlike for usual inner products, equality in (3-1) does not imply that $\{x, y\}$ is linearly dependent. For instance, consider the subalgebra $B = \{\mathrm{diag}(\lambda, \mu) : \lambda, \mu \in \mathbb{C}\}$ in $A = M_2(\mathbb{C})$ with the natural finite-index conditional expectation $E : A \to B$ given by $E([x_{ij}]) = \mathrm{diag}(x_{11}, x_{22})$. Then, for $x = \mathrm{diag}(1, 1)$ and $y = \mathrm{diag}(i, 1)$ in A, one easily verifies that $\|\langle x, y \rangle_B\| = \|x\|_A \|y\|_A$ whereas $\{x, y\}$ is linearly independent.

Employing (3-1), motivated by [1], Bakshi and the first named author introduced the following definitions of the interior and exterior angles between intermediate C^*-subalgebras.

Definition 3.1 [3]. Let $B \subset A$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : A \to B$. Then, for $C, D \in \mathrm{IMS}(B, A, E) \setminus \{B\}$, the interior angle between C and D (with respect to E), denoted as $\alpha(C, D)$, is given by the expression

$$\cos(\alpha(C, D)) = \frac{\|\langle e_C - e_B, e_D - e_B \rangle_A\|}{\|e_C - e_B\|_{A_1} \|e_D - e_B\|_{A_1}}; \tag{3-2}$$

and, for $C, D \in \mathrm{IMS}(B, A, E) \setminus \{A\}$ with $C_1, D_1 \in \mathrm{IMS}(A, A_1, E_1)$, the exterior angle between C and D is defined as

$$\beta(C, D) = \alpha(C_1, D_1), \tag{3-3}$$

where $\alpha(C_1, D_1)$ is defined with respect to the dual conditional expectation $E_1 : A_1 \to A$.

By definition, both angles are allowed to take values only in the interval $[0, \pi/2]$.

REMARK 3.2

(1) Note that if $\mathrm{Ind}(E_{\restriction_D}), \mathrm{Ind}(E_{\restriction_C}) \in \mathcal{Z}(A)$, then by Proposition 2.7(7), $C_1, D_1 \in \mathrm{IMS}(A, A_1, E_1)$. Thus, $\beta(C, D)$ is defined for such intermediate subalgebras.
(2) If $B \subset C, D \subset A$ is a quadruple of simple unital C^*-algebras, then $\beta(C, D)$ is always defined.

We now derive some useful formulae for the interior and exterior angles in terms of certain related quasibases.

PROPOSITION 3.3. *Let $B \subset A$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : A \to B$ with quasibasis $\{\lambda_i : 1 \leq i \leq p\}$. Let $C, D \in \mathrm{IMS}(B, A, E) \setminus \{B\}$ with respect to the conditional expectations $F : A \to C$ and $F' : A \to D$, respectively. Let $\{\mu_j : 1 \leq \mu_j \leq m\}$ and $\{\delta_k : 1 \leq \delta_k \leq n\}$ be quasibases for $E_{\restriction_C}$ and $E_{\restriction_D}$, respectively. Then, we have the following.*

(1) *The interior angle between C and D is given by*

$$\cos(\alpha(C, D)) = \frac{\|(\mathrm{Ind}(E))^{-1}(\sum_{j,k} \mu_j E(\mu_j^* \delta_k)\delta_k^* - 1)\|}{\|(\mathrm{Ind}(E))^{-1}(\mathrm{Ind}(E_{\restriction_C}) - 1)\|^{1/2} \|(\mathrm{Ind}(E))^{-1}(\mathrm{Ind}(E_{\restriction_D}) - 1)\|^{1/2}}.$$

In particular, if $\mathrm{Ind}(E)$ is a scalar, then

$$\cos(\alpha(C, D)) = \frac{\| \sum_{j,k} \mu_j E(\mu_j^* \delta_k)\delta_k^* - 1\|}{\|\mathrm{Ind}(E_{\restriction_C}) - 1\|^{1/2} \|\mathrm{Ind}(E_{\restriction_D}) - 1\|^{1/2}}. \tag{3-4}$$

(2) *Whenever $\mathrm{Ind}(E_{\restriction_C})$ and $\mathrm{Ind}(E_{\restriction_D})$ belong to $\mathcal{Z}(A)$, the exterior angle between C and D can be derived from (3-3) using the following expressions:*

$$\begin{aligned}\langle e_{C_1} - e_2, e_{D_1} - e_2\rangle_{A_1} &= (\mathrm{Ind}(E_1))^{-1}\Big[(\mathrm{Ind}(E_{\restriction_C}))^{-2}(\mathrm{Ind}(E_{\restriction_D}))^{-1} \sum_{i,i'} \lambda_i e_C \mathrm{Ind}(F') \\ &\quad \times \sum_{j,k} \mu_j E(\mu_j^* \lambda_i^* \lambda_{i'} \delta_k)\delta_k^*) e_D \lambda_{i'}^* - 1\Big],\end{aligned}$$

$$\|e_{C_1} - e_2\|_{A_2} = \left\|(\mathrm{Ind}(E_1))^{-1}\Big[(\mathrm{Ind}(E_{\restriction_C}))^{-1}\Big(\sum_i \lambda_i F(\mathrm{Ind}(F))e_C \lambda_i^*\Big) - 1\Big]\right\|^{1/2}$$

and a similar expression for $\|e_{D_1} - e_2\|_{A_2}$.

PROOF. (1) follows immediately by substituting the expressions for e_C, e_D, as obtained in Proposition 2.7(4),(5),(6), in the definition of interior angle (3-2).

(2) Note that the dual conditional expectation $E_1 : A_1 \to A$ is of finite index with a quasibasis $\{\lambda_i e_B(\mathrm{Ind}(E))^{1/2}\}$—see Remark 2.3(3). Further, from Proposition 2.7(4), a quasibasis for $E_1{\restriction_{C_1}}$ is given by $\{w_i := G(\lambda_i e_B(\mathrm{Ind}(E))^{1/2}) : 1 \le i \le n\}$, where $G : A_1 \to C_1$ is the conditional expectation as in the proof of Proposition 2.7(7). Thus, by Proposition 2.7(4), we see that

$$\begin{aligned} e_{C_1} &= \sum_i w_i e_2 w_i^* \\ &= \sum_i \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-1}(\mathrm{Ind}(E))^{1/2} e_2 (\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E{\restriction_C}))^{-1} e_C \lambda_i^*, \end{aligned}$$

since $\mathrm{Ind}(E{\restriction_C}) \in \mathcal{Z}(A) \cap \mathcal{Z}(C)$ and $e_C \in C' \cap C_1$. Thus,

$$\begin{aligned} &E_2(e_{C_1}) - E_2(e_2) \\ &\quad = (\mathrm{Ind}(E_1))^{-1}\Big[\Big(\sum_i \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-1}(\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E{\restriction_C}))^{-1} e_C\lambda_i^*\Big) - 1\Big] \\ &\quad = (\mathrm{Ind}(E_1))^{-1}\Big[\Big(\sum_i \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-2}(\mathrm{Ind}(E)) e_C\lambda_i^*\Big) - 1\Big] \\ &\quad = (\mathrm{Ind}(E_1))^{-1}\Big[\Big(\sum_i \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-1}(\mathrm{Ind}(F)) e_C\lambda_i^*\Big) - 1\Big] \quad \text{(by Proposition 2.7(7a))} \\ &\quad = (\mathrm{Ind}(E_1))^{-1}\Big[\Big((\mathrm{Ind}(E{\restriction_C}))^{-1} \sum_i \lambda_i F((\mathrm{Ind}(F)) e_C\lambda_i^*) - 1\Big], \end{aligned}$$

which shows that

$$\|e_{C_1} - e_2\|_{A_2} = \Big\|(\mathrm{Ind}(E_1))^{-1}\Big[(\mathrm{Ind}(E{\restriction_C}))^{-1}\Big(\sum_i \lambda_i F(\mathrm{Ind}(F)) e_C\lambda_i^*\Big) - 1\Big]\Big\|^{1/2}.$$

Further, as above,

$$e_{D_1} = \sum_i \lambda_i e_D(\mathrm{Ind}(E{\restriction_D}))^{-1}(\mathrm{Ind}(E))^{1/2} e_2 (\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E{\restriction_D}))^{-1} e_D \lambda_i^*;$$

so that

$$\begin{aligned} &e_{C_1} e_{D_1} \\ &\quad = \sum_{i,i'} \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-1}(\mathrm{Ind}(E))^{1/2} E_1[(\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E{\restriction_C}))^{-1} e_C \lambda_i^* \lambda_{i'} e_D \\ &\qquad \times (\mathrm{Ind}(E{\restriction_D}))^{-1}(\mathrm{Ind}(E))^{1/2}] e_2 (\mathrm{Ind}(E))^{1/2}(\mathrm{Ind}(E{\restriction_D}))^{-1} e_D \lambda_{i'}^* \\ &\quad = \sum_{i,i'} \lambda_i e_C(\mathrm{Ind}(E{\restriction_C}))^{-1}\, \mathrm{Ind}(F)\, E_1(e_C\lambda_i^*\lambda_{i'} e_D)(\mathrm{Ind}(E{\restriction_D}))^{-1} \mathrm{Ind}(F') e_2 e_D \lambda_{i'}^*, \end{aligned}$$

where the last equality holds because of Proposition 2.7(7a). Then,

$$
\begin{aligned}
&E_2(e_{C_1}e_{D_1}) - E_2(e_2)\\
&\quad= (\mathrm{Ind}(E_1))^{-1}\sum_{i,i'}\lambda_i e_C(\mathrm{Ind}(E_{\restriction_C}))^{-1}(\mathrm{Ind}(F))E_1(e_C\lambda_i^*\lambda_{i'}e_D)(\mathrm{Ind}(E_{\restriction_D}))^{-1}\\
&\qquad\times \mathrm{Ind}(F')e_D\lambda_{i'}^* - (\mathrm{Ind}(E_1))^{-1}\\
&\quad= (\mathrm{Ind}(E_1))^{-1}\Big[(\mathrm{Ind}(E_{\restriction_C}))^{-2}(\mathrm{Ind}(E_{\restriction_D}))^{-1}\sum_{i,i'}\lambda_i e_C\mathrm{Ind}(F')\\
&\qquad\times\sum_{j,k}\mu_j E(\mu_j^*\lambda_i^*\lambda_{i'}\delta_k)\delta_k^*)e_D\lambda_{i'}^* - 1\Big].
\end{aligned}
$$

Since $\langle e_{C_1} - e_2, e_{D_1} - e_2\rangle_{A_1} = E_2(e_{C_1}e_{D_1}) - E_2(e_2)$, we are done. □

REMARK 3.4. *A priori*, for $C, D \in \mathrm{IMS}(B, A, E) \setminus \{B\}$, it is not clear whether $\alpha(C, D) = 0$ implies $C = D$ or not. However, when $B \subset A$ is an irreducible inclusion of simple unital C^*-algebras, then it is known to be true—see [3, Proposition 5.10]. Also, this phenomenon holds for a certain collection of intermediate subalgebras even in some nonirreducible setup, as we see in Corollary 4.5.

4. Possible values of the interior angle

Throughout this section, we let $A = M_2(\mathbb{C})$, $B = \mathbb{C}I_2$, $\Delta = \{\mathrm{diag}(\lambda,\mu) : \lambda,\mu \in \mathbb{C}\}$, $E : A \to B$ denote the canonical (tracial) conditional expectation given by

$$E([a_{ij}]) = \frac{(a_{11} + a_{22})}{2} I_2 \text{ for } [a_{ij}] \in A$$

and $F : A \to \Delta$ denote the conditional expectation given by $F([a_{ij}]) = \mathrm{diag}(a_{11}, a_{22})$.

The following useful observations are standard—see [11, Example 2.4.5].

LEMMA 4.1. *With running notation, the following hold:*

(1) *E is a finite-index conditional expectation with a quasibasis*

$$\{\sqrt{2}e_{ij} : 1 \le i, j \le 2\},$$

where $\{e_{ij} : 1 \le i, j \le 2\}$ denotes the set of standard matrix units of $M_2(\mathbb{C})$;

(2) $\mathrm{Ind}(E) = 4$ *and E is the (unique) minimal conditional expectation from A onto B;*

(3) *the C^*-basic construction A_1 for $B \subset A$, with respect to the conditional expectation E, can be identified with $M_4(\mathbb{C})$ and the Jones projection e_1 corresponding to the conditional expectation E is given by*

$$e_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix};$$

(4) *identifying $M_4(\mathbb{C})$ with $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, the dual conditional expectation $E_1 : A_1 \to A$ is given by $E_1 = \mathrm{id}_{\mathrm{M}_2} \otimes E$; thus,*

$$E_1\,(X) = \begin{bmatrix} E(X_{(1,1)}) & E(X_{(1,2)}) \\ E(X_{(2,1)}) & E(X_{(2,2)}) \end{bmatrix}, \quad X \in M_4(\mathbb{C}),$$

where $X_{(i,j)}$ denotes the (i, j) th 2×2 block of a matrix $X \in M_4(\mathbb{C})$.

LEMMA 4.2

(1) *F has finite index with a quasibasis $\{e_{ij} : 1 \leq i, j \leq 2\}$ and scalar index equal to 2.*
(2) *$\Delta \in \mathrm{IMS}(B, A, E)$ with respect to the conditional expectation F and the corresponding Jones projection in Δ_1 ($\subset A_1 = M_4(\mathbb{C})$) is given by*

$$e_\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

LEMMA 4.3. *For every unitary u in A:*

(1) *the map $F_u : A \to u\Delta u^*$ given by $F_u = \mathrm{Ad}_u \circ F \circ \mathrm{Ad}_{u^*}$ is a finite-index conditional expectation with a quasibasis $\{e_{ij}u^* : 1 \leq i, j \leq 2\}$ and $\mathrm{Ind}(F_u) = 2$;*
(2) *$D := u\Delta u^* \in \mathrm{IMS}(B, A, E)$ with respect to the conditional expectation F_u; and,*
(3) *if $u = [\lambda_{ij}]$, then the corresponding Jones projection in D_1 ($\subset A_1 = M_4(\mathbb{C})$) is given by $e_D = [x_{ij}]$, where*

$$x_{11} = |\lambda_{11}|^4 + |\lambda_{12}|^4,\ x_{12} = \lambda_{21}\bar{\lambda}_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2),\ x_{14} = 2|\lambda_{11}|^2|\lambda_{12}|^2,$$

$$x_{22} = 2|\lambda_{11}|^2|\lambda_{21}|^2,\ x_{23} = 2\bar{\lambda}_{21}^2\lambda_{11}^2$$

and the remaining entries are given by $x_{12} = \bar{x}_{13} = \bar{x}_{21} = x_{31} = -\bar{x}_{24} = -x_{42} = -x_{34} = -\bar{x}_{43}$, $x_{41} = x_{14}$, $x_{33} = x_{22}$, $x_{32} = \bar{x}_{23}$ and $x_{44} = x_{11}$.

PROOF. (1) Clearly, the map $F_u : A \to D$ is a conditional expectation and we can easily verify that

$$x = \sum_{i,j} e_{ij}u^* F_u(ue_{ij}^* x) = \sum_{i,j} F_u(xe_{ij}u^*)ue_{ij}^*$$

for all $x \in A$. Thus, $\{e_{ij}u^* : 1 \leq i, j \leq 2\}$ is a quasibasis for F_u and $\mathrm{Ind}(F_u) = 2 = \mathrm{Ind}(F)$.

Since E satisfies the tracial property and $\mathcal{N}_A(B) = \mathcal{A}$, item (2) follows from Lemma 2.8.

(3) After some routine calculation, for any $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in A$, we obtain

$$F_u\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} x|\lambda_{11}|^2 + y|\lambda_{12}|^2 & x\bar{\lambda}_{21}\lambda_{11} + y\bar{\lambda}_{22}\lambda_{12} \\ x\lambda_{21}\bar{\lambda}_{11} + y\lambda_{22}\bar{\lambda}_{12} & x|\lambda_{12}|^2 + y|\lambda_{11}|^2 \end{bmatrix},$$

where $x = a|\lambda_{11}|^2 + d|\lambda_{12}|^2 + b\lambda_{21}\bar{\lambda}_{11} + c\bar{\lambda}_{21}\lambda_{11}$ and $y = a|\lambda_{12}|^2 + d|\lambda_{11}|^2 + b\lambda_{22}\bar{\lambda}_{12} + c\bar{\lambda}_{22}\lambda_{12}$. Since u is a unitary, we have $\bar{\lambda}_{12}\lambda_{22} = -(\bar{\lambda}_{11}\lambda_{21})$, $\lambda_{12}\bar{\lambda}_{22} = -(\lambda_{11}\bar{\lambda}_{21})$, $|\lambda_{11}|^2 = |\lambda_{22}|^2$ and $|\lambda_{12}|^2 = |\lambda_{21}|^2$; thus, we further deduce that

$$\begin{aligned} x|\lambda_{11}|^2 + y|\lambda_{12}|^2 &= a(|\lambda_{11}|^4 + |\lambda_{12}|^4) + 2d|\lambda_{11}|^2|\lambda_{12}|^2 + c\bar{\lambda}_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) \\ &\quad + b\lambda_{21}\bar{\lambda}_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2); \text{ and} \\ x|\lambda_{12}|^2 + y|\lambda_{11}|^2 &= 2a|\lambda_{11}|^2|\lambda_{12}|^2 + d(|\lambda_{11}|^4 + |\lambda_{12}|^4) + b\lambda_{21}\bar{\lambda}_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) \\ &\quad + c\bar{\lambda}_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2). \end{aligned}$$

Then, using the above expression for $F_u([a_{ij}])$ for $[a_{ij}] \in A$ and the matrix e_D, as given in the statement, we can easily verify that:

(1) $e_D x e_D = F_u(x) e_D$ and
(2) $e_D(x) = F_u(x)$

for all $x \in A$. This completes the proof. □

We are now all set to derive a concrete expression for the interior angle between Δ and its conjugate $u\Delta u^*$, in terms of the entries of u.

THEOREM 4.4. *If $u = [\lambda_{ij}] \in U(2)$, then*

$$\cos(\alpha(\Delta, u\Delta u^*)) = \sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4}.$$

PROOF. Let $D = u\Delta u^*$. From the matrix expressions of e_1, e_Δ and e_D obtained above, we easily see that $E_1(e_1) = \frac{1}{4}I_2$, $E_1(e_\Delta) = \frac{1}{2}I_2$, $E_1(e_D) = \frac{1}{2}I_2$ and

$$E_1(e_\Delta e_D) = \begin{bmatrix} (|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 & \bar{\lambda}_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2)/2 \\ \lambda_{21}\bar{\lambda}_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2)/2 & (|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 \end{bmatrix}.$$

Thus,

$$\|e_\Delta - e_1\|_{A_1} = \sqrt{\|E_1(e_\Delta - e_1)\|} = \tfrac{1}{2} = \sqrt{\|E_1(e_D - e_1)\|} = \|e_D - e_1\|_{A_1}.$$

Next, we calculate $\|E_1(e_\Delta e_D - e_1)\|$. Let

$$T = E_1(e_\Delta e_D - e_1) = \begin{bmatrix} (|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 - 1/4 & \bar{\lambda}_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2)/2 \\ \lambda_{21}\bar{\lambda}_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2)/2 & (|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 - 1/4 \end{bmatrix}.$$

Note that, T^*T turns out to be a scalar matrix with eigenvalue λ, where

$$\begin{aligned} \lambda &= \left(\frac{|\lambda_{11}|^4 + |\lambda_{12}|^4}{2} - \frac{1}{4}\right)^2 + \frac{|\lambda_{11}|^2|\lambda_{21}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4} \\ &= \frac{1}{16}(2(|\lambda_{11}|^4 + |\lambda_{12}|^4) - 1)^2 + \frac{|\lambda_{11}|^2|\lambda_{21}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4} \\ &= \frac{1}{16}(2(|\lambda_{11}|^4 + |\lambda_{12}|^4) - (|\lambda_{11}|^2 + |\lambda_{12}|^2)^2)^2 + \frac{|\lambda_{11}|^2|\lambda_{12}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4} \\ &\qquad \text{(since } |\lambda_{11}|^2 + |\lambda_{12}|^2 = 1 \text{ and } |\lambda_{21}| = |\lambda_{12}|) \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{|\lambda_{11}|^2 - |\lambda_{12}|^2}{4}\Big)^2 + \frac{|\lambda_{11}|^2|\lambda_{12}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4} \\
&= \Big(\frac{(|\lambda_{11}|^2 - |\lambda_{12}|^2)}{4}\Big)^2(1 + 4|\lambda_{11}|^2|\lambda_{12}|^2) \\
&= \frac{1}{16}((|\lambda_{11}|^2 + |\lambda_{12}|^2)^2 - 4|\lambda_{11}|^2|\lambda_{12}|^2)(1 + 4|\lambda_{11}|^2|\lambda_{12}|^2) \\
&= \frac{1}{16}(1 - (2|\lambda_{11}||\lambda_{12}|)^4).
\end{aligned}
$$

Thus,

$$\|\langle e_\Delta - e_1,\ e_D - e_1\rangle_A\| = \|E_1(e_\Delta e_D - e_1)\| = \|T\| = \Big(\sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4}\Big)/4.$$

Finally, substituting the values of $\|e_\Delta - e_1\|_{A_1}$, $\|e_D - e_1\|_{A_1}$ and $\|\langle e_\Delta - e_1, e_D - e_1\rangle_A\|$ above into (3-2), we obtain

$$\cos(\alpha(\Delta, u\Delta u^*)) = \sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4}. \qquad \square$$

Recall that a unitary matrix whose entries all have the same modulus is called a *Hadamard matrix*. Also, if (B, C, D, A) is a quadruple of finite von Neumann algebras (that is, $B \subset C, D \subset A$) with a faithful normal tracial state $\tau : A \to \mathbb{C}$, then (B, C, D, A) is said to be a commuting square if $E^A_C E^A_D = E^A_B = E^A_D E^A_C$, where $E^A_X : A \to X$ denotes the unique τ-preserving conditional expectation from A onto any von Neumann subalgebra X of A.

COROLLARY 4.5. *Let $u \in U(2)$. Then:*

(1) $\alpha(\Delta, u\Delta u^*) = \pi/2$ *if and only if u is a Hadamard matrix if and only if $(B, \Delta, u^*\Delta u, A)$ is a commuting square; and,*
(2) *if $u = [\lambda_{ij}]$, then $\alpha(\Delta, u\Delta u^*) = 0$ if and only if either $u \in \Delta$ or $\lambda_{11} = 0 = \lambda_{22}$.*
In particular, $\alpha(\Delta, u\Delta u^*) = 0$ if and only if $\Delta = u\Delta u^*$.

PROOF. (1) From Theorem 4.4, we observe that

$$
\begin{aligned}
\cos(\alpha(\Delta, u\Delta u^*)) = 0 &\Leftrightarrow \sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4} = 0 \\
&\Leftrightarrow |\lambda_{11}||\lambda_{12}| = \tfrac{1}{2} \\
&\Leftrightarrow |\lambda_{11}| = |\lambda_{12}| \quad (\text{since } |\lambda_{11}|^2 + |\lambda_{12}|^2 = 1) \\
&\Leftrightarrow |\lambda_{11}| = |\lambda_{12}| = |\lambda_{21}| = |\lambda_{22}| \\
&\Leftrightarrow u \text{ is a Hadamard matrix.}
\end{aligned}
$$

Note that $F : M_2 \to \Delta$ and $F_u : M_2 \to u\Delta u^*$ are the unique trace-preserving conditional expectations. Additionally, it is a well-known fact—see, for instance, [8, Section 5.2.2]—that $(B, \Delta, u^*\Delta u, A)$ is a commuting square if an only if u is a Hadamard matrix.

(2) Again, from Theorem 4.4,

$$\begin{aligned}\cos(\alpha(\Delta, u\Delta u^*)) = 1 &\Leftrightarrow |\lambda_{11}||\lambda_{12}| = 0\\ &\Leftrightarrow |\lambda_{11}| = 0 \quad \text{or} \quad |\lambda_{12}| = 0\\ &\Leftrightarrow |\lambda_{11}| = 0 = |\lambda_{22}| \quad \text{or} \quad u \text{ is diagonal} \quad (\text{as } u \text{ is unitary}). \qquad \square\end{aligned}$$

We can now deduce our assertion that the interior angle attains all values in $[0, \pi/2]$.

COROLLARY 4.6

$$\{\alpha(\Delta, u\Delta u^*) : u \in U(2)\} = \left[0, \frac{\pi}{2}\right].$$

PROOF. Note that, for each $u = [\lambda_{ij}] \in U(2)$, $0 \le (2|\lambda_{11}||\lambda_{12}|)^4 \le 1$. Thus, we can define a map $\varphi : U(2) \to [0, 1]$ given by

$$\varphi([\lambda_{ij}]) = \sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4}.$$

Clearly, φ is a continuous function. Since $U(2)$ is connected, it follows that $\varphi(U(2))$ is also connected. Note that, from Corollary 4.5, we have $\varphi(u) = 0$ for any complex Hadamard matrix $u \in U(2)$ and $\varphi(I_2) = 1$. Hence, $\varphi(U(2)) = [0, 1]$. In particular, in view of Theorem 4.4,

$$\{\alpha(\Delta, u\Delta u^*) : u \in U(2)\} = \left[0, \frac{\pi}{2}\right],$$

as desired. $\square$

COROLLARY 4.7. *There exist $C, D \in \mathrm{IMS}(B, A, E)$ such that $e_C e_D \neq e_D e_C$.*

PROOF. Fix a $u = [\lambda_{ij}] \in U(2)$ and let $C = \Delta$ and $D = u\Delta u^*$. Then, both $C, D \in \mathrm{IMS}(A, B, E)$, and using the values of e_C and e_D from Lemmas 4.2 and 4.3,

$$e_C e_D = \begin{bmatrix} |\lambda_{11}|^4 + |\lambda_{12}|^4 & \lambda_{21}\bar{\lambda}_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) & \bar{\lambda}_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) & 2|\lambda_{11}|^2|\lambda_{12}|^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2|\lambda_{11}|^2|\lambda_{12}|^2 & \lambda_{21}\bar{\lambda}_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) & \bar{\lambda}_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) & |\lambda_{11}|^4 + |\lambda_{12}|^4 \end{bmatrix}$$

and

$$e_D e_C = \begin{bmatrix} |\lambda_{11}|^4 + |\lambda_{12}|^4 & 0 & 0 & 2|\lambda_{11}|^2|\lambda_{12}|^2 \\ \bar{\lambda}_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) & 0 & 0 & \bar{\lambda}_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) \\ \lambda_{21}\bar{\lambda}_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) & 0 & 0 & \lambda_{21}\bar{\lambda}_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) \\ 2|\lambda_{11}|^2|\lambda_{12}|^2 & 0 & 0 & |\lambda_{11}|^4 + |\lambda_{12}|^4 \end{bmatrix}.$$

Thus, if u is neither a diagonal matrix nor a Hadamard matrix nor $\lambda_{11} = 0 = \lambda_{22}$, then we see that $C \neq D$ and $e_C e_D \neq e_D e_C$. $\square$

5. Angles between intermediate crossed product subalgebras of crossed product inclusions

Recall that if a countable discrete group G acts on a unital C^*-algebra P via a map $\alpha : G \to \mathrm{Aut}(P)$, then the space $C_c(G, P)$ consisting of compactly supported P-valued functions on G can be identified with the space $\{\sum_{\text{finite}} a_g g : a_g \in P, g \in G\}$ of formal finite sums and is a unital $*$-algebra with respect to (the so-called twisted) multiplication given by the convolution operation

$$\Big(\sum_{s\in I} a_s s\Big)\Big(\sum_{t\in J} b_t t\Big) = \sum_{s\in I, t\in J} a_s \alpha_s(b_t) st$$

and involution given by

$$\Big(\sum_{s\in I} a_s s\Big)^* = \sum_{s\in I} \alpha_{s^{-1}}(a_s^*) s^{-1}$$

for any two finite sets I and J in G. Further, the reduced crossed product $P \rtimes_{\alpha,r} G$ and the universal crossed product $P \rtimes_\alpha G$ are defined, respectively, as the completions of $C_c(G, P)$ with respect to the reduced norm

$$\Big\| \sum_{\text{finite}} a_g g \Big\|_r := \Big\| \sum_{\text{finite}} \pi(a_g)(1 \otimes \lambda_g) \Big\|_{B(H \otimes \ell^2(G))},$$

where $P \subset B(H)$ is a (equivalently, any) fixed faithful representation of P, $\lambda : G \to B(\ell^2(G))$ is the left regular representation and $\pi : P \to B(H \otimes \ell^2(G))$ is the representation satisfying $\pi(a)(\xi \otimes \delta_g) = \alpha_{g^{-1}}(a)(\xi) \otimes \delta_g$ for all $\xi \in H$ and $g \in G$; and the universal norm

$$\|x\|_u := \sup_\pi \|\pi(x)\| \quad \text{for } x \in C_c(G, P),$$

where the supremum runs over all (cyclic) $*$-homomorphisms $\pi : C_c(G, P) \to B(H)$. We suggest the reader refer to [4, 12] for more on crossed products.

When G is a finite group, then it is well known that the reduced and universal norms coincide on $C_c(G, P)$, and $C_c(G, P)$ is complete with respect to the common norm; thus, $P \rtimes_{\alpha,r} G = C(G, P) = P \rtimes_\alpha G$ (as $*$-algebras).

In this section, analogous to [2, Proposition 2.7], we derive a concrete value for the interior angle between intermediate crossed product subalgebras of an inclusion of crossed product algebras.

The following important observations are well known.

PROPOSITION 5.1 [5, 9]. *Let G be a countable discrete group and H be its subgroup. Let P be a unital C^*-algebra such that G acts on P via a map $\alpha : G \to$* Aut(P). *Let*

$A := P \rtimes_{\alpha,r} G$ (respectively, $A := P \rtimes G$) and $B := P \rtimes_{\alpha,r} H$ (respectively, $B := P \rtimes H$). Then:

(1) *the canonical injective $*$-homomorphism*

$$C_c(H,P) \ni \sum_{\text{finite}} a_h h \mapsto \sum_{\text{finite}} a_h h \in C_c(G,P)$$

extends to an injective $*$-homomorphism from B into A; and

(2) *the natural map*

$$C_c(G,P) \ni \sum_{\text{finite}} a_g g \mapsto \sum a_h h \in C_c(H,P)$$

extends to a conditional expectation $E : A \to B$.

Moreover, E has finite index if and only if $[G : H] < \infty$ and in that case, a quasibasis for E is given by $\{g_i : 1 \le i \le [G : H]\}$ for any set $\{g_i\}$ of left coset representatives of H in G and E has scalar index equal to $[G : H]$.

PROOF. (1) follows from [5] (also see [9, Remark 3.2]) and the first part of the proof of [9, Proposition 3.1].

(2) Consider the canonical $C_c(H,P)$-bilinear projection $E_0 : C_c(G,P) \to C_c(H,P)$ given by

$$E_0\Big(\sum_{\text{finite}} a_g g\Big) = \sum a_h h.$$

Then, from [5] (also see [9, Remark 3.2]) and [9, Proposition 3.1, Remark 3.2], it follows that E_0 extends to a conditional expectation from A onto B. Also, from [9, Theorem 3.4], it follows that E has finite index (with a quasibasis as in the statement) if and only $[G : H] < \infty$. □

PROPOSITION 5.2. *Let G, H, P, α, A, B and E be as in Proposition 5.1 with $[G : H] < \infty$ and $\{g_i : 1 \le i \le [G : H]\}$ be a set of left coset representatives of H in G. Let K and L be proper intermediate subgroups of $H \subset G$ and let $C := P \rtimes_{\alpha,r} K$ (respectively, $P \rtimes_\alpha K$) and $D := P \rtimes_{\alpha,r} L$ (respectively, $P \rtimes_\alpha L$). Then, $C, D \in \mathrm{IMS}(B, A, E) \setminus \{A, B\}$ and the interior angle between them is given by*

$$\cos(\alpha(C,D)) = \frac{[K \cap L : H] - 1}{\sqrt{[K : H] - 1}\,\sqrt{[L : H] - 1}}. \tag{5-1}$$

PROOF. Note that $B \subset C, D \subset A$, by Proposition 5.1. Also, $[G : K]$ and $[G : L]$ are both finite as $[G : H]$ is finite. So, $C, D \in \mathrm{IMS(B, A, E)}$ with respect to the natural finite-index conditional expectations guaranteed by Proposition 5.1.

Fix left coset representatives $\{k_r : 1 \le r \le [K : H]\}$ and $\{l_s : 1 \le s \le [L : H]\}$ of H in K and L, respectively. Then, it is readily seen that $E_{\restriction_C} : C \to B$ and $E_{\restriction_D} : D \to B$ have quasibases $\{k_r : 1 \le r \le [K : H]\}$ and $\{l_s : 1 \le s \le [L : H]\}$, respectively. Then, from (3-4), we obtain

$$\begin{aligned}\cos(\alpha(C,D)) &= \frac{\|(\sum_{r,s} k_r E(k_r^* l_s) l_s^*) - 1\|}{\| \sqrt{[K:H]-1} \| \| \sqrt{[L:H]-1} \|} \\ &= \frac{\|(\sum_{\{r,s:(k_rH)\cap(l_sH)\neq\phi\}} k_r k_r^* l_s l_s^*) - 1\|}{\sqrt{[K:H]-1}\sqrt{[L:H]-1}} \\ &= \frac{[(K\cap L):H]-1}{\sqrt{[K:H]-1}\sqrt{[L:H]-1}},\end{aligned}$$

where the last equality holds because the map

$$\{(r,s) : k_rH \cap l_sH \neq \emptyset\} \ni (r,s) \mapsto k_rH = l_sH \in (K\cap L)/H$$

is a bijection. □

COROLLARY 5.3. *Let the notation be as in Proposition 5.2. Then:*

(1) $\alpha(C,D) = \pi/2$ *if and only if* $K \cap L = H$; *and*
(2) $\alpha(C,D) = 0$ *if and only if* $K = L$.

In particular, if $C_g := P \rtimes_\alpha (g^{-1}Kg)$ *(respectively,* $P \rtimes_{\alpha,r} (g^{-1}Kg)$*) then* $\alpha(C,C_g) = 0$ *for all* $g \in G$ *if and only if* K *is normal in* G.

PROOF. (1) is straight forward and, for item (2), we just need to show the necessity.

Note that $\alpha(C,D) = 0$ implies that $([(K\cap L):H]-1)/(\sqrt{[K:H]-1}\sqrt{[L:H]-1}) = 1$, which then implies that

$$\left(\sqrt{\frac{[(K\cap L):H]-1}{[K:H]-1}}\right)\left(\sqrt{\frac{[(K\cap L):H]-1}{[L:H]-1}}\right) = 1$$

and that $[K\cap L : H] \neq 1$. Note that

$$0 < \frac{[(K\cap L):H]-1}{[K:H]-1}, \frac{[(K\cap L):H]-1}{[L:H]-1} \le 1;$$

so, it follows that

$$\frac{[(K\cap L):H]-1}{[K:H]-1} = 1 = \frac{[(K\cap L):H]-1}{[L:H]-1}.$$

Hence, $K = K\cap L = L$. □

Recall that for a subgroup H of a group G, its normalizer is given by

$$\mathcal{N}_G(H) = \{g \in G : g^{-1}Hg = H\}.$$

COROLLARY 5.4. *Let* G, H *and* K *be as in Proposition 5.2. If* $g \in \mathcal{N}_G(H)$*, then* $\alpha(C,C_g) = 0$ *if and only if* $g \in \mathcal{N}_G(K)$*, where* C_g *is the same as in Corollary 5.3*

PROOF. Let $L := g^{-1}Kg$. Since $[K : H] = [L : H]$, from (5-1), we obtain

$$\cos(\alpha(C, C_g)) = \frac{[(K \cap L) : H] - 1}{[K : H] - 1}.$$

Thus, $\alpha(C, C_g)) = 0$ if and only if $K \cap (g^{-1}Kg) = K$ if and only if $g \in \mathcal{N}_G(K)$. □

Note that if $P = \mathbb{C}$ and $\alpha : G \to \mathrm{Aut}(\mathbb{C})$ is the trivial representation, then we know that $C^*_r(G) = \mathbb{C} \rtimes_{\alpha,r} G$ and $C^*(G) = \mathbb{C} \rtimes_\alpha G$. Thus, we readily deduce the following.

COROLLARY 5.5. *Let G be a countable discrete group with proper subgroups H, K and L such that $H \subseteq K \cap L$, $H \neq K, L$ and $[G : H] < \infty$. Let $A := C^*_r(G)$ (respectively, $C^*(G)$), $B := C^*_r(H)$ (respectively, $C^*(H)$), $C := C^*_r(K)$ (respectively, $C^*(K)$) and $D := C^*_r(L)$ (respectively, $C^*(L)$). Then, $C, D \in \mathrm{IMS}(B, A, E) \setminus \{A, B\}$ and*

$$\cos(\alpha(C, D)) = \frac{[K \cap L : H] - 1}{\sqrt{[K : H] - 1}\,\sqrt{[L : H] - 1}},$$

where $E : A \to B$ is the conditional expectation as in Proposition 5.1 with $P = \mathbb{C}$.

EXAMPLE 5.6. Let $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$. Consider its subgroups $K = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus (0)$, $L = (0) \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus (0)$ and $H = (0) \oplus (0) \oplus \mathbb{Z}_5 \oplus (0)$. Then,

$$\cos(\alpha(\mathbb{C}[K], \mathbb{C}[L])) = \tfrac{1}{2}.$$

Thus, $\alpha(\mathbb{C}[K], \mathbb{C}[L]) = \pi/3$.

In particular, this illustrates that if $B \neq C \subsetneq D \subsetneq A$, then $\alpha(C, D)$ need not be 0.

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