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KK-THEORY OF CIRCLE ACTIONS WITH THE ROKHLIN PROPERTY

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ABSTRACT. We investigate the structure of circle actions with the Rokhlin property, particularly in relation to equivariant KK-theory. Our main results are $\mathbb{T}$-equivariant versions of celebrated results of Kirchberg: any Rokhlin action on a separable, nuclear C*-algebra is $KK^{\mathbb{T}}$-equivalent to a Rokhlin action on a Kirchberg algebra; and two circle actions with the Rokhlin property on a Kirchberg algebra are conjugate if and only if they are $KK^{\mathbb{T}}$-equivalent.

In the presence of the UCT, $KK^{\mathbb{T}}$-equivalence for Rokhlin actions reduces to isomorphism of a K-theoretical invariant, namely of a canonical pure extension naturally associated to any Rokhlin action, and we provide a complete description of the extensions that arise from actions on nuclear C^*-algebras. In contrast with the non-equivariant setting, we exhibit an example showing that an isomorphism between the $K^{\mathbb{T}}$-theories of Rokhlin actions on Kirchberg algebras does not necessarily lift to a $KK^{\mathbb{T}}$-equivalence; this is the first example of its kind, even in the absence of the Rokhlin property.

1. INTRODUCTION

Crossed products have provided some of the most relevant examples in the theory of C^*-algebras, and the study of their structure and classification is a very active field of research. Moreover, the classification of group actions has a long history within the theory of operator algebras. For example, Connes' classification of automorphisms of the hyperfinite II_1-factor $\mathcal{R}$ was instrumental in his award-winning classification of amenable factors [4]. Connes' success motivated significant efforts towards classifying amenable group actions on hyperfinite factors, which culminated almost two decades later.

By comparison to the von Neumann algebra setting, the classification of C*-dynamical systems is a far less developed field of research. A major difficulty is the complicated behavior that finite order automorphisms exhibit at the level of K-theory. Even in the absence of torsion in the acting group, the induced action on the trace space may be wild. These difficulties resulted in a somewhat scattered collection of results, and a lack of a systematic approach. A notable exception is the fruitful analysis of the Rokhlin property in C^*-algebras, as is seen in the works of Kishimoto [22, 23], Matui [26], Nakamura [30], Izumi [16, 17], Sato [40], Nawata [31], and the author and Santiago [14], to mention a few.

The advances in Elliott's classification programme (which is by now essentially complete; see [6, 44]) suggest that group actions on purely infinite C^*-algebras may be more accessible, and there already exist very encouraging results in this direction. In [43], Szabo proved versions of Kirchberg's absorption results $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes A \cong A$ (with suitable A) for outer actions of amenable groups, using actions on $\mathcal{O}_2$ and $\mathcal{O}_\infty$ which have an appropriate version of the Rokhlin property. (An equivariant version of the absorption result $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ was proved for *exact* groups by Suzuki [42].) More recently, Meyer [27] began exploring the classification of actions of torsion-free amenable groups using KK^G-theory, particularly in what refers to lifting an isomorphism between

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K-theoretical data to a KK^G-equivalence. A recurrent issue in this setting is that satisfactory results can only be expected if either the action has some variation of the Rokhlin property, or the group is torsion-free.

In this work, we use equivariant KK-theory to study circle actions with the Rokhlin property, and obtain $\mathbb{T}$-equivariant versions of celebrated results of Kirchberg concerning simple, purely infinite, separable, nuclear C^*-algebras (also known as Kirchberg algebras); see Theorems C and F below. By comparison to the *continuous* Rokhlin property (studied in [1] and [12]), Rokhlin actions are a much richer class with less rigid behavior. Accordingly, more involved arguments are needed in this setting.

The main reason to focus on circle actions is that the combination of one-dimensionality with the fact that $\mathbb{T}$ is a Lie group produces phenomena that cannot be expected beyond this setting. An example of this, which is crucial to our work and already fails for $\mathbb{T}^2$, is the existence of a predual automorphism:

Theorem A. (See Theorem 2.3) Let A be a unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there exists $\breve{\alpha} \in \mathrm{Aut}(A^\alpha)$ such that α is conjugate to the dual action $\widehat{\breve{\alpha}}$.

We completely characterize the automorphisms that arise as preduals of circle actions with the Rokhlin property; see Proposition 2.8. Using this, we show that every circle action with the Rokhlin property has a naturally associated PExt-class.

Theorem B. (See Theorem 3.3) Let A be a unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a Rokhlin action. Then the natural map $K_0(A^\alpha) \hookrightarrow K_0(A)$ is an injective order-embedding, and there is a canonical *pure* extension $\mathrm{Ext}_*(\alpha)$ given by

$$0 \longrightarrow K_*(A^\alpha) \longrightarrow K_*(A) \longrightarrow K_*(SA^\alpha) \longrightarrow 0\ .$$

The fact that the above extension is pure is far from obvious, and it ultimately depends on the existence of a sequence of ucp maps $A \to A^\alpha$ which are asymptotically multiplicative and asymptotically the identity on A^α.

Our most interesting results are related to equivariant $KK^{\mathbb{T}}$-theory in the setting of Kirchberg algebras. When A is a Kirchberg algebra, we show that so is A^α and that $\breve{\alpha}$ is aperiodic (Proposition 4.2), which gives us access to Nakamura's work [30]. We use this to obtain a $\mathbb{T}$-equivariant version of the Kirchberg-Phillips classification theorem for actions with the Rokhlin property:

Theorem C. (See Theorem 4.6) Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be actions on unital Kirchberg algebras with the Rokhlin property. Then α and β are conjugate if and only if they are unitally $KK^{\mathbb{T}}$-equivalent. In the presence of the UCT, this is in turn equivalent to the existence of a graded isomorphism $\mathrm{Ext}_*(\alpha) \cong \mathrm{Ext}_*(\beta)$, which preserves unit classes and is compatible with suspension shifts (see Definition 4.5).

We also obtain a range result in the context of above theorem, showing that the only K-theoretic obstructions are the ones obtained in Theorem B.

Theorem D. (See Theorem 4.8) Let K_0 and K_1 be abelian groups, let $k_0 \in K_0$, and let $\mathcal{E}_0 \in \mathrm{Ext}(K_0, K_1)$ and $\mathcal{E}_1 \in \mathrm{Ext}(K_1, K_0)$ be extensions. The following are equivalent:

(a) There is a Rokhlin action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ on a unital UCT Kirchberg algebra A with $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) \cong (\mathcal{E}_0, \mathcal{E}_1, k_0)$;

(b) $\mathcal{E}_0$ and $\mathcal{E}_1$ are pure.

Unlike in Kirchberg-Phillips' classification, in the presence of the UCT it does not suffice to assume that both actions have isomorphic $K^{\mathbb{T}}$-theory (this is a big difference with the case of the continuous Rokhlin property [1, 12]). It also does not suffice for the actions to have isomorphic Meyer's L-invariant $L_*^{\mathbb{T}}(A,\alpha) = K_*^{\mathbb{T}}(A,\alpha) \oplus K_*(A)$ ([27]):

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Example E. (See Example 4.7.) There exist a UCT Kirchberg algebra A and Rokhlin actions $\alpha, \beta\colon \mathbb{T} \to \mathrm{Aut}(A)$ such that $K_*^{\mathbb{T}}(A,\alpha) \cong K_*^{\mathbb{T}}(A,\beta)$ as $R(\mathbb{T})$-modules, although α and β are not $KK^{\mathbb{T}}$-equivalent. One can even construct the actions so that $A^\alpha \cong A^\beta$ and $A \rtimes_\alpha \mathbb{T} \cong A \rtimes_\beta \mathbb{T}$, all satisfying the UCT.

The above example shows an interesting phenomenon, which we put into perspective. In Example 10.6 in [39], Rosenberg and Schochet construct two circle actions on commutative C^*-algebras with isomorphic $K^{\mathbb{T}}$-theory, which are not $KK^{\mathbb{T}}$-equivalent. In their example, the underlying algebras are not even KK-equivalent, so the actions cannot be $KK^{\mathbb{T}}$-equivalent. As communicated to us by Claude Schochet, Example E is the first construction of two circle actions on *the same* C^*-algebra, satisfying the UCT, with isomorphic fixed point algebras and crossed products, which all satisfy the UCT, and isomorphic $K^{\mathbb{T}}$-theory, that are not $KK^{\mathbb{T}}$-equivalent.

With a classification of Rokhlin actions on Kirchberg algebras in terms of $KK^{\mathbb{T}}$-theory at our disposal, it is natural to ask which Rokhlin actions are $KK^{\mathbb{T}}$-equivalent to a Rokhlin action on a Kirchberg algebra. As it turns out, Rokhlin actions on Kirchberg algebras represent all separable, nuclear $KK^{\mathbb{T}}$-classes of Rokhlin actions:

Theorem F. (See Theorem 4.15.) Let A be a separable, nuclear, unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ have the Rokhlin property. Then there exist a unique unital Kirchberg algebra D and a unique circle action $\delta\colon \mathbb{T} \to \mathrm{Aut}(D)$ with the Rokhlin property such that $(A,\alpha) \sim_{KK^{\mathbb{T}}} (D,\delta)$ unitally.

The theorem above cannot be extended to actions α that do not necessarily have the Rokhlin property, since there are obstructions to being $KK^{\mathbb{T}}$-equivalent to a Rokhlin action (for example as in Theorem B). Theorem F should be compared to Theorem 2.1 of [27], where Meyer shows that every circle action on a separable, nuclear C^*-algebra is $KK^{\mathbb{T}}$-equivalent to an outer action on a Kirchberg algebra. It is not clear from Meyer's construction that the resulting action on the Kirchberg algebra has the Rokhlin property if the original one does. We therefore could not adapt his argument to our context, and instead use older ideas of Kirchberg, applied at the level of the predual $\breve{\alpha}$.

The results here presented are an expanded version of Chapter IX of my PhD thesis [10]. Since the first preprint version of this work appeared on the arxiv, Arano and Kubota generalized the first part of Theorem C to compact groups other than $\mathbb{T}$; see Proposition 4.8 in [1]. The methods are quite different, since we take full advantage of the existence of a predual automorphism. Moreover, for circle actions (as opposed to general compact group actions), $KK^{\mathbb{T}}$-equivalence can be detected via a K-theoretical invariant in the presence of the UCT, and the range of this invariant can be completely described. This makes the classification of circle actions with the Rokhlin property comparatively more accessible than that of general compact groups. Even more recently, Gabe and Szabo obtained in [8] an equivariant version of the Kirchberg-Phillips classification for amenable actions that are "isometrically shift-absorbing". These developments subsume[1] the part of Theorem C which does not assume the UCT, using significantly heavier machinery, but do not have overlap with the other results here stated. In this sense, our proof Theorem C should be regarded as a shorter and simpler proof of [8, Theorem F] in the case of Rokhlin actions of the circle.

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[1]This is because Rokhlin actions of compact groups are automatically isometrically shift-absorbing. This follows, for example, using an argument similar to the first part of the proof of [8, Corollary 6.15].

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2. Duality for circle actions with the Rokhlin property

In this section, we study the Rokhlin property for circle actions in connection to duality. There are two main results in this section. First, it is shown that every circle action with the Rokhlin property is a dual action, that is, there is an automorphism of the fixed point algebra whose dual action is conjugate to the given one; see Theorem 2.3. Such an automorphism is essentially unique, and is called the *predual* automorphism. Second, we characterize those automorphisms that are predual to a circle action with the Rokhlin property; see Proposition 2.8.

We begin by recalling the definition of the Rokhlin property for a circle action (Definition 3.2 in [15]) in a way that is useful for our purposes.

Definition 2.1. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T}\to \mathrm{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for every $\varepsilon>0$ and every compact subset $F\subseteq A$, there exists a unitary $u\in\mathcal{U}(A)$ such that

(a) $\|\alpha_z(u)-zu\|<\varepsilon$ for all $z\in\mathbb{T}$.
(b) $\|ua-au\|<\varepsilon$ for all $a\in F$.

Next, we show that we can replace the unitary u in the above definition by a nearby unitary which satisfies condition (a) exactly. In the terminology of [37], the following shows that the action of $\mathbb{T}$ on $C(\mathbb{T})$ is equivariantly semiprojective.

Proposition 2.2. For every $\varepsilon>0$ there exists $\delta>0$ with the following property: whenever $\alpha\colon\mathbb{T}\to\mathrm{Aut}(A)$ is an action on a unital C^*-algebra A and $u\in\mathcal{U}(A)$ is a unitary satisfying $\|\alpha_z(u)-zu\|<\delta$ for all $z\in\mathbb{T}$, then there exists a unitary $v\in\mathcal{U}(A)$ with $\|u-v\|<\varepsilon$ and $\alpha_z(v)=zv$ for all $z\in\mathbb{T}$.

Proof. Given $\varepsilon>0$, choose $\delta<\frac{1}{3}$ small enough so that

$$\frac{2\delta}{\sqrt{1-2\delta}}+\delta<\varepsilon. \tag{2.1}$$

Let $\alpha\colon\mathbb{T}\to\mathrm{Aut}(A)$ and $u\in\mathcal{U}(A)$ be as in the statement. Set $x=\int_{\mathbb{T}}\overline{z}\alpha_z(u)\,dz\in A$. Then $\|x\|\leq 1$ and $\|x-u\|\leq\delta$. One checks that $\|x^*x-1\|\leq 2\delta<1$, so x^*x is invertible. Moreover,

$$\left\|(x^*x)^{-1}\right\|\leq\frac{1}{1-\|1-x^*x\|}\leq\frac{1}{1-2\delta}. \tag{2.2}$$

Set $u=x(x^*x)^{-\frac{1}{2}}$. Then u is a unitary in A. Using that $\|x\|\leq 1$ at the first step, and that $0\leq 1-(x^*x)^{\frac{1}{2}}\leq 1-x^*x$ at the second step, we get

$$\begin{aligned}\|u-x\|&\leq\left\|(x^*x)^{-\frac{1}{2}}-1\right\|\leq\left\|(x^*x)^{-\frac{1}{2}}\right\|\left\|1-(x^*x)^{\frac{1}{2}}\right\|\\&\overset{(2.2)}{\leq}\frac{1}{\sqrt{1-2\delta}}\|1-x^*x\|\leq\frac{2\delta}{\sqrt{1-2\delta}}.\end{aligned}$$

Thus

$$\|u-v\|\leq\|u-x\|+\|x-v\|\leq\frac{2\delta}{\sqrt{1-2\delta}}+\delta\overset{(2.1)}{<}\varepsilon.$$

For $z\in\mathbb{T}$, we have

$$\alpha_z(x)=\int_{\mathbb{T}}\overline{\omega}\alpha_{z\omega}(u)d\omega=\int_{\mathbb{T}}z\overline{\omega}\alpha_\omega(u)d\omega=zx.$$

It follows that $\alpha_z(x^*x)=x^*x$ and hence $\alpha_z(u)=\alpha_z\big(x(x^*x)^{-\frac{1}{2}}\big)=zu$, for all $z\in\mathbb{T}$, so u satisfies the condition in the statement. ☐

It follows from Proposition 2.2 that condition (1) in Definition 2.1 can be replaced with $\alpha_z(u)=zu$ for all $z\in\mathbb{T}$.

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Theorem 2.3. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

(1) There exists an automorphism $\theta \in \mathrm{Aut}(A^\alpha)$ such that $(A^\alpha \rtimes_\theta \mathbb{Z}, \widehat{\theta})$ is conjugate to (A, α).

(2) If $\theta' \in \mathrm{Aut}(A^\alpha)$ is another automorphism for which $(A^\alpha \rtimes_{\theta'} \mathbb{Z}, \widehat{\theta'})$ is conjugate to (A, α), then there is a unitary $w \in A^\alpha$ such that $\theta = \mathrm{Ad}(w) \circ \theta'$.

Proof. (1). Using Proposition 2.2, let $u \in \mathcal{U}(A)$ be a unitary satisfying $\alpha_z(u) = zu$ for all $z \in \mathbb{T}$. For $a \in A^\alpha$, we have $\alpha_z(uau^*) = uau^*$ for all $z \in \mathbb{T}$, and thus conjugation by u determines an automorphism θ of A^α. Let $v \in A^\alpha \rtimes_\theta \mathbb{Z}$ denote the canonical unitary implementing θ. Since the pair $(\mathrm{id}_{A^\alpha}, u)$ is a covariant representation of (A^α, θ) on A, there is a unique homomorphism $\varphi\colon A^\alpha \rtimes_\theta \mathbb{Z} \to A$ satisfying $\varphi(a) = a$ for all $a \in A^\alpha$ and $\varphi(v) = u$.

We claim that φ is an equivariant isomorphism. Equivariance of φ is clear, since for all $z \in \mathbb{T}$ we have $\widehat{\theta}_z(a) = a$ for all $a \in A^\alpha$ and $\widehat{\theta}_z(v) = zv$. Injectivity of φ follows from the fact that id_{A^α} is injective (and that $\mathbb{Z}$ is amenable). Surjectivity can be deduced using spectral subspaces, as follows. Given $n \in \mathbb{Z}$, we set

$$A_n = \{a \in A\colon \alpha_z(a) = z^n a \text{ for all } z \in \mathbb{T}\},$$

which is a closed subspace of A. It is well-known that $\sum_{n\in\mathbb{Z}} A_n$ is dense in A; see, for example, part (ix) of Theorem 8.1.4 in [35]. Note that $A_0 = A^\alpha$ and that u belongs to A_1. Moreover, using that u is a unitary, it is easy to see that $A_n = u^n A_0$ for all $n \in \mathbb{Z}$. In particular, A is generated as a C^*-algebra by A_0 and u. Since $A_0 \cup \{u\}$ is contained in the image of φ, we conclude that φ is surjective.

(2). Let θ' be as in the statement, and let $\varphi\colon (A^\alpha \rtimes_\theta \mathbb{Z}, \widehat{\theta}) \to (A, \alpha)$ and $\varphi'\colon (A^\alpha \rtimes_{\theta'} \mathbb{Z}, \widehat{\theta'}) \to (A, \alpha)$ be equivariant isomorphisms. Let v be the canonical unitary in $A^\alpha \rtimes_\theta \mathbb{Z}$ that implements θ, and let v' be the canonical unitary in $A^\alpha \rtimes_{\theta'} \mathbb{Z}$ that implements θ'. Set $w = \varphi(v)\varphi'(v')^*$, which is a unitary in A. We claim that w is fixed by α. For $z \in \mathbb{T}$, we use equivariance of φ and φ' to get

$$\alpha_z(w) = \varphi(\widehat{\theta}_z(v))\varphi'(\widehat{\theta'}_z(v'))^*) = z\varphi(v)\overline{z}\varphi'(v') = w.$$

Finally, given $a \in A^\alpha$, we have

$$\begin{aligned}(\mathrm{Ad}(w) \circ \theta)(a) &= (\varphi(v)\varphi'(v')^*)(\varphi'(v')a\varphi'(v')^*)(\varphi'(v')\varphi(v)^*)\\ &= \varphi(v)a\varphi(v)^* = \theta'(a).\end{aligned}$$

□

Definition 2.4. Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. In view of Theorem 2.3, we denote by $\check{\alpha} \in \mathrm{Aut}(A^\alpha)$ the unique automorphism for which $\widehat{\check{\alpha}}$ is conjugate to α. We call $\check{\alpha}$ the *predual automorphism* of α.

Corollary 2.5. Let A be a C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there is a natural isomorphism $A \rtimes_\alpha \mathbb{T} \cong A^\alpha \otimes \mathcal{K}(L^2(\mathbb{T}))$.

Proof. This is an immediate from Theorem 2.3 and Takai duality. □

Obtaining a characterization of those automorphisms that arise as preduals of Rokhlin actions as in Definition 2.4 will be a critical tool in the rest of this work. Such a characterization is obtained in Proposition 2.8, using the following notion:

Definition 2.6. Let B be a C^*-algebra and let β be an automorphism of B. Then β is said to be *approximately representable* if for every finite subset $F \subseteq B$ and every $\varepsilon > 0$, there exists a contraction $v \in B$ satisfying

(a) $\|v^*v - vv^*\| < \varepsilon$;
(b) $\|v^*vb - b\| < \varepsilon$ for all $b \in F$;
(c) $\|\beta(v) - v\| < \varepsilon$; and

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(d) $\|\beta(b) - vbv^*\| < \varepsilon$ for all $b \in F$.

Using functional calculus, it is clear that the contraction v in the above definition can be chosen to be a unitary whenever B is unital. In particular, approximately representable automorphisms of unital C^*-algebras are approximately inner.

Remark 2.7. We endow $\mathbb{T}$ with its Haar probability measure. For an action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ on a unital C^*-algebra A, we endow $L^1(\mathbb{T}, A)$ with the usual L^1-norm $\|\cdot\|_1$ and the operations of twisted convolution and involution

$$(\xi * \eta)(z) = \int_{\mathbb{T}} \xi(\omega)\alpha_\omega(\eta(\omega^{-1}z))\, d\omega \quad \text{and} \quad \xi^*(z) = \alpha_z(\xi(\overline{z})^*)$$

for all $\xi, \eta \in L^1(\mathbb{T}, A)$ and all $z \in \mathbb{T}$. Then $L^1(\mathbb{T}, A)$ is a dense $*$-subalgebra of $A \rtimes_\alpha \mathbb{T}$, and the canonical inclusion is contractive with respect to the L^1-norm on $L^1(\mathbb{T}, A)$ and the C^*-norm on $A \rtimes_\alpha \mathbb{T}$. Recall that there is a canonical nondegenerate inclusion $C^*(\mathbb{T}) \subseteq A \rtimes_\alpha \mathbb{T}$; in particular, any (contractive) approximate identity for $C^*(\mathbb{T})$ is also a (contractive) approximate identity for $A \rtimes_\alpha \mathbb{T}$. Recall that the dual automorphism $\widehat{\alpha} \in \mathrm{Aut}(A \rtimes_\alpha \mathbb{T})$ is given by $\widehat{\alpha}(fa)(z) = zf(z)a$ for all $z \in \mathbb{T}$.

Next, we show that approximate representability is dual to the Rokhlin property.

Proposition 2.8. Let A be a unital C^*-algebra, let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action, and let $\beta \in \mathrm{Aut}(A)$ be an automorphism.

(1) The action α has the Rokhlin property if and only if $\widehat{\alpha} \in \mathrm{Aut}(A \rtimes_\alpha \mathbb{T})$ is approximately representable.

(2) The automorphism β is approximately representable if and only if $\widehat{\beta}\colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_\beta \mathbb{Z})$ has the Rokhlin property.

Proof. (1). Assume that α has the Rokhlin property. Let $F \subseteq A \rtimes_\alpha \mathbb{T}$ be a finite subset and let $\varepsilon > 0$. For $\xi \in L^1(\mathbb{T})$ and $a \in A$, write ξa for the for the function given by $(\xi a)(z) = \xi(z)a$ for all $z \in \mathbb{T}$. Since the linear span of the elements of this form is dense in $L^1(\mathbb{T}, A)$, and hence also in $A \rtimes_\alpha \mathbb{T}$, we may assume that there exist finite subsets $F_A \subseteq A$ and $F_{\mathbb{T}} \subseteq L^1(\mathbb{T})$ such that every element of F has the form ξa for $a \in F_A$ and $\xi \in F_{\mathbb{T}}$. Without loss of generality, we may assume that the sets F_A and $F_{\mathbb{T}}$ contain only self-adjoint contractions.

Let $f\colon \mathbb{T} \to \mathbb{C}$ be a positive, continuous function whose support is a small enough neighborhood of $1 \in \mathbb{T}$ so that the following conditions are satisfied:

(i) $\|(f * f)b - b\| < \varepsilon$ for all $b \in F \cup \widehat{\alpha}(F)$;

(ii) $\|f\|_1 = 1 = \|f * f\|_1$;

(iii) $f(z) = f(\overline{z})$ for all $z \in \mathbb{T}$;

(iv) with $\widetilde{f}(z) = zf(z)$ for all $z \in \mathbb{T}$, we have

$$\|f - \widetilde{f}\|_1 < \varepsilon \quad \text{and} \quad \|f * f - \widetilde{f} * \widetilde{f}\|_1 < \varepsilon;$$

(v) given $\xi \in F_{\mathbb{T}}$ and $a \in F_A$, if $z, \sigma, \omega \in \mathbb{T}$ satisfy $f(\omega)f(\overline{z}\sigma\omega) \neq 0$, then

$$\|\alpha_\omega(a) - a\| < \frac{\varepsilon}{2} \quad \text{and} \quad |\xi(\sigma) - \xi(z)| < \frac{\varepsilon}{2}.$$

Using Proposition 2.2, find a unitary $u \in A$ satisfying

(vi) $\alpha_\zeta(u) = \zeta u$ for all $\zeta \in \mathbb{T}$, and

(vii) $\|ua - au\| < \varepsilon/2$ for all $a \in \bigcup_{\omega \in \mathbb{T}} \alpha_\omega(F_A)$.

We regard u as a unitary in the multiplier algebra of $A \rtimes_\alpha \mathbb{T}$ via the canonical unital embedding $A \hookrightarrow M(A \rtimes_\alpha \mathbb{T})$, and set $v = fu^*$. Then v is a contraction in $L^1(\mathbb{T}, A)$, and hence also in $A \rtimes_\alpha \mathbb{T}$. We proceed to check the conditions in Definition 2.6. Let $z \in \mathbb{T}$.

Then

$$(v^* * v)(z) = \int_{\mathbb{T}} v^*(\omega)\alpha_\omega(v(\overline{\omega}z))\ d\omega = \int_{\mathbb{T}} \omega f(\omega)u\alpha_\omega(f(\overline{\omega}z)u^*)\ d\omega$$
$$\overset{\text{(vi)}}{=} \int_{\mathbb{T}} \omega f(\omega)uf(\overline{\omega}z)\overline{\omega}u^*\ d\omega = (f * f)(z).$$

Thus, $v^* * v = f * f$ and hence condition (a) in Definition 2.6 follows from condition (ii) above. In order to check (b), we compute as follows for $z \in \mathbb{T}$:

$$(v * v^*)(z) = \int_{\mathbb{T}} v(\omega)\alpha_\omega(v^*(\overline{\omega}z))\ d\omega = \int_{\mathbb{T}} f(\omega)u^*\alpha_\omega(\overline{\omega}zf(\overline{\omega}z)u)\ d\omega$$
$$\overset{\text{(vi)}}{=} \int_{\mathbb{T}} f(\omega)u^*\overline{\omega}zf(\overline{\omega}z)\omega u\ d\omega = \int_{\mathbb{T}} \omega f(\omega)\overline{\omega}zf(\overline{\omega}z)\ d\omega = (\widetilde{f} * \widetilde{f})(z).$$

Thus $v * v^* = \widetilde{f} * \widetilde{f}$. We deduce that

$$\|v^*v - vv^*\| \leq \|v^* * v - v * v^*\|_1 = \|f * f - \widetilde{f} * \widetilde{f}\|_1 \overset{\text{(iv)}}{<} \varepsilon,$$

as desired. In order to check (c), let $\zeta \in \mathbb{T}$. Then

$$\widehat{\alpha}(v)(z) = zf(z)u^* = \widetilde{f}(z)u^*,$$

so $\widehat{\alpha}(v) = \widetilde{f}u^*$. Using this at the second step, we get

$$\|\widehat{\alpha}(v) - v\| \leq \|\widehat{\alpha}(v) - v\|_1 = \|f - \widetilde{f}\|_1 \overset{\text{(iv)}}{<} \varepsilon,$$

as desired. Finally, in order to check (d), it suffices to take $\xi \in F_{\mathbb{T}}$ and $a \in F_A$, and show the desired inequality for $b = \xi a$. Given $z \in \mathbb{T}$, we have

$$\begin{aligned}
(v * b * v^*)(z) &= \int_{\mathbb{T}} v(\omega)\alpha_\omega((b * v^*)(\overline{\omega}z))\ d\omega \\
&= \int_{\mathbb{T}} f(\omega)u^*\alpha_\omega\Big(\int_{\mathbb{T}} \xi(\sigma)a\alpha_\sigma(v^*(\overline{\sigma}\overline{\omega}z))\ d\sigma\Big)d\omega \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} f(\omega)u^*\xi(\sigma)\alpha_\omega(a)\alpha_{\omega\sigma}\big(\overline{\sigma}\overline{\omega}zf(\overline{\sigma}\overline{\omega}z)u\big)d\sigma d\omega \\
&\overset{\text{(vi)}}{=} \int_{\mathbb{T}}\int_{\mathbb{T}} f(\omega)u^*\xi(\sigma)\alpha_\omega(a)\overline{\sigma}\overline{\omega}zf(\overline{\sigma}\overline{\omega}z)\omega\sigma u\ d\sigma d\omega \\
&= z\int_{\mathbb{T}}\int_{\mathbb{T}} \xi(\sigma)u^*\alpha_\omega(a)uf(\omega)f(\overline{\sigma}\overline{\omega}z)\ d\sigma d\omega \\
&\overset{\text{(vii)}}{\approx_{\frac{\varepsilon}{2}}} z\int_{\mathbb{T}}\int_{\mathbb{T}} \xi(\sigma)\alpha_\omega(a)f(\omega)f(\overline{\sigma}\overline{\omega}z)\ d\sigma d\omega.
\end{aligned}$$

By (iv), if in the above expression we replace $\xi(\sigma)\alpha_\omega(a)$ by $\xi(z)a$, we obtain an element in A whose distance to $(v * b * v^*)(z)$ is at most $\varepsilon/2$. Hence,

$$\begin{aligned}
(v * b * v^*)(z) &\approx_\varepsilon z\xi(z)a\int_{\mathbb{T}}\int_{\mathbb{T}} f(\omega)f(\overline{\sigma}\overline{\omega}z)\ d\sigma d\omega \\
&= zb(z)\int_{\mathbb{T}} (f * f)(\overline{\sigma}z)\ d\sigma = zb(z)\|f * f\|_1 \overset{\text{(ii)}}{=} \widehat{\alpha}(b)(z).
\end{aligned}$$

We conclude that

$$\|vbv^* - \widehat{\alpha}(b)\| \leq \|v * b * v^* - \widehat{\alpha}(b)\|_1 < \varepsilon,$$

as desired. This shows that $\widehat{\alpha}$ is approximately representable.

Conversely, assume that $\widehat{\alpha}$ is approximately representable. Denote the left regular representation of G by $\lambda\colon \mathbb{T} \to \mathcal{U}(L^2(\mathbb{T}))$. By Takai duality, there is a canonical equivariant identification

$$(A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z}, \widehat{\widehat{\alpha}}) \cong (A \otimes \mathcal{K}(L^2(\mathbb{T})), \alpha \otimes \mathrm{Ad}(\lambda)). \tag{2.3}$$

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Let $p \in \mathcal{K}(L^2(\mathbb{T}))$ be the projection onto the constant functions, and let $e \in M(A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z})$ be the projection corresponding to $1_A \otimes p$ under the identification in (2.3). Then e and p are $\mathbb{T}$-invariant, and there is a canonical equivariant isomorphism

$$\left(e(A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z})e, \widehat{\widehat{\alpha}}\right) \cong (A, \alpha). \tag{2.4}$$

Let $u \in M(A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z})$ be the canonical unitary implementing $\widehat{\alpha}$. Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Let $\delta > 0$ such that whenever $x \in A$ satisfies $\|x^*x - 1\| < \delta$ and $\|xx^* - 1\| < \delta$, then there is $w \in \mathcal{U}(A)$ with $\|w - x\| < \varepsilon/2$. Set

$$F'' = \{e\} \cup \{a \otimes p \colon a \in F\} \subseteq A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z}.$$

Let $F' \subseteq A \rtimes_\alpha \mathbb{T}$ be a finite subset and let $n \in \mathbb{N}$ such that any element in F'' is within $\varepsilon/2$ of the span of $\{bu^k \colon b \in F', -n \leq k \leq n\}$. Using approximate representability of $\widehat{\alpha}$, let $v \in A \rtimes_\alpha \mathbb{T}$ be a contraction satisfying conditions (a), (b), (c) and (d) in Definition 2.6 for $\varepsilon_0 = \min\left\{\frac{\varepsilon}{26n^2|F'|}, \delta\right\}$ and F'. Set $y = v^*u$, which is a contraction in $A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z}$. Then

$$\widehat{\widehat{\alpha}}_z(y) = v^*\widehat{\widehat{\alpha}}_z(u) = zv^*u = zy \tag{2.5}$$

for all $z \in \mathbb{T}$. Moreover, given $b \in F'$ and $k \in \mathbb{Z}$ with $|k| \leq n$, we have

$$\begin{aligned} ybu^k = v^*ubu^k = v^*\widehat{\alpha}(b)u^{k+1} &\overset{\text{(b)}}{\approx}_{\varepsilon_0} v^*\widehat{\alpha}(b)v^*vu^{k+1} \\ &\overset{\text{(a)}}{\approx}_{\varepsilon_0} v^*\widehat{\alpha}(b)vv^*u^{k+1} \overset{\text{(d)}}{\approx}_{\varepsilon_0} v^*vbv^*vv^*u^{k+1} \\ &\overset{\text{(b)}}{\approx}_{2\varepsilon_0} bv^*u^{k+1} \overset{\text{(c)}}{\approx}_{k\varepsilon_0} bu^kv^*u = bu^ky. \end{aligned}$$

It follows from the choice of F', n and ε_0 that

$$\|yc - cy\| < \frac{\varepsilon}{2} \tag{2.6}$$

for every $c \in F''$. Set $x = eye$, which we regard as an element in A. By (2.5) we have $\alpha_z(x) = zx$ for all $z \in \mathbb{T}$, since e is $\mathbb{T}$-invariant. For $a \in F$ we have

$$\|xax^* - a\| = \|eye(a \otimes p)ey^*e - (a \otimes p)\| \leq \|y(a \otimes p)y^* - (a \otimes p)\| \overset{(2.6)}{<} \frac{\varepsilon}{2},$$

since $a \otimes p$ belongs to F''. Moreover, $\|x^*x - 1\| = \|ey^*eye - e\| < \varepsilon_0 \leq \delta$, and similarly $\|xx^* - 1\| < \delta$. By the choice of δ, there exists a unitary $w \in A$ such that $\|w - x\| < \varepsilon/2$. It is then straightforward to check that $\|wa - aw\| < \varepsilon$ for all $a \in F$ and $\max_{z \in \mathbb{T}} \|\alpha_z(w) - zw\| < \varepsilon/2$. This shows that α has the Rokhlin property.

(2). Assume that β is approximately representable. Let F be a finite subset of $A \rtimes_\beta \mathbb{Z}$ and let $\varepsilon > 0$. Denote by u the canonical unitary in the crossed product. Since A and u generate $A \rtimes_\beta \mathbb{Z}$, one can assume that $F = F' \cup \{u\}$, where F' is a finite subset of A. Using approximate representability for β, find $v \in \mathcal{U}(A)$ with

- $\|\beta(b) - vbv^*\| < \varepsilon$ for all $b \in \beta(F')$, and
- $\|\beta(v) - v\| < \varepsilon$.

Set $w = v^*u$, which is a unitary in $A \rtimes_\beta \mathbb{Z}$. For $b \in F'$ we have

$$wb = v^*ub = v^*\beta(b)u \approx_\varepsilon bv^*u = bw.$$

Moreover,

$$wu = v^*uu = u\beta^{-1}(v)u \approx_\varepsilon uw.$$

It follows that $\|wa - aw\| < \varepsilon$ for all $a \in F = F' \cup \{u\}$. On the other hand,

$$\widehat{\beta}_z(w) = \widehat{\beta}_z(v^*u) = v^*(zu) = zw$$

for all $z \in \mathbb{T}$. Thus, w is the desired unitary, and $\widehat{\beta}$ has the Rokhlin property.

Conversely, assume that $\widehat{\beta}$ has the Rokhlin property. We continue to denote by u the canonical unitary in $A \rtimes_\beta \mathbb{Z}$ that implements β. Let $F \subseteq A$ be a finite subset and set

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$F' = F \cup \{u\} \subseteq A \rtimes_\beta \mathbb{Z}$. Use Proposition 2.2 to choose a unitary $w \in \mathcal{U}(A \rtimes_\beta \mathbb{Z})$ such that

- $\widehat{\beta}_z(w) = zw$ for all $z \in \mathbb{T}$;
- $\|wb - bw\| = 0$ for all $b \in F'$.

Set $v = uw^* \in A \rtimes_\beta \mathbb{Z}$. Then the first condition above implies that $\widehat{\beta}_z(v) = v$ for all $z \in \mathbb{T}$, and hence the unitary v belongs to the fixed point algebra $(A \rtimes_\beta \mathbb{Z})^{\widehat{\beta}}$, which equals A by Proposition 7.8.9 in [35]. For $a \in F$, we have

$$\|vav^* - \beta(a)\| = \|uw^*awu^* - uau^*\| = \|w^*aw - a\| < \varepsilon.$$

Moreover, $\|\beta(v) - v\| = \|u(uw^*)u^* - uw^*\| < \varepsilon$, since $u \in F'$. It follows that v satisfies the conditions of Definition 2.6, so β is approximately representable. □

Using Proposition 2.8, we show that for actions with the Rokhlin property, every ideal in the crossed product is induced by an invariant ideal of the algebra.

Proposition 2.9. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ have the Rokhlin property. Then every ideal in A^α has the form $I \cap A^\alpha$, for some $\mathbb{T}$-invariant ideal I in A, and every ideal in $A \rtimes_\alpha \mathbb{T}$ has the form $I \rtimes_\alpha \mathbb{T}$ for some $\mathbb{T}$-invariant ideal I in A. In particular, if A is simple then so are A^α and $A \rtimes_\alpha \mathbb{T}$.

Proof. Since $A^\alpha \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \cong A \rtimes_\alpha \mathbb{T}$ by Corollary 2.5, it is enough to show the statement for A^α. We identify A with $A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z}$, where $\check{\alpha}$ is the predual of α; see Definition 2.4. Let J be an ideal in A^α. Since $\check{\alpha}$ is approximately inner by part (2) of Proposition 2.8, it follows that $\check{\alpha}(J) = J$. Hence $I = J \rtimes_{\check{\alpha}} \mathbb{Z}$ is canonically an ideal in A satisfying $I \cap A^\alpha = J$, as desired. □

3. K-theoretic obstructions to the Rokhlin property

In this section, we study the K-theory of C^*-algebras that admit circle actions with the Rokhlin property. First, we show that the canonical inclusion $A^\alpha \to A$ induces an injective order-embedding $K_*(\iota)\colon K_*(A^\alpha) \to K_*(A)$; see Proposition 3.1. Moreover, the quotient of $K_*(A)$ by $K_*(A^\alpha)$ can be canonically identified with $K_*(SA^\alpha)$, and the induced extension

$$0 \longrightarrow K_*(A^\alpha) \longrightarrow K_*(A) \longrightarrow K_*(SA^\alpha) \longrightarrow 0$$

is pure; see Definition 3.2 and Theorem 3.3.

In the following proposition, we will use the fact that if $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the Rokhlin property, then so does its n-amplification $\alpha \otimes \mathrm{id}_{M_n}\colon \mathbb{T} \to \mathrm{Aut}(M_n(A))$ for every $n \in \mathbb{N}$, and that $M_n(A)^{\alpha\otimes \mathrm{id}_{M_n}} = M_n(A^\alpha)$.

Proposition 3.1. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, and let $\iota\colon A^\alpha \hookrightarrow A$ denote the canonical inclusion. Then $K_0(\iota)$ and $K_1(\iota)$ are injective. Moreover, for $x, y \in K_0(A^\alpha)$ we have $K_0(\iota)(x) \leq K_0(\iota)(y)$ in $K_0(A)$ if and only if $x \leq y$ in $K_0(A^\alpha)$.

Proof. Consider the Pimsner-Voiculescu exact sequence associated to the predual automorphism $\check{\alpha} \in \mathrm{Aut}(A^\alpha)$ of α (Definition 2.4):

$$\begin{array}{ccccc}
K_0(A^\alpha) & \xrightarrow{1-K_0(\check{\alpha})} & K_0(A^\alpha) & \xrightarrow{K_0(\iota)} & K_0(A) \\
\uparrow & & & & \downarrow \\
K_1(A) & \xleftarrow[K_1(\iota)]{} & K_1(A^\alpha) & \xleftarrow[1-K_1(\check{\alpha})]{} & K_1(A^\alpha).
\end{array}$$

Since $\check{\alpha}$ is approximately inner by part (2) of Proposition 2.8, we get $K_*(\check{\alpha}) = 1$ and thus $K_*(\iota)$ is injective.

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We turn to the last part of the statement. Let $x, y \in K_0(A^\alpha)$, and assume that $K_0(\iota)(x) \leq K_0(\iota)(y)$. Set $z = y - x$, so that $K_0(\iota)(z) \geq 0$ in $K_0(A)$. Find projections $p, q \in \bigcup_{m\in\mathbb{N}} M_m(A^\alpha)$ with $z = [p] - [q]$ in $K_0(A^\alpha)$, and let $e \in \bigcup_{m\in\mathbb{N}} M_m(A)$ satisfy $K_0(\iota)(z) = [e]$ in $K_0(A)$. Then $[p] = [e] + [q]$ in $K_0(A)$. By increasing the matrix sizes, we may assume that there exist $k, n \in \mathbb{N}$ such that:

- p and q belong to $M_n(A^\alpha)$;
- e belongs to $M_n(A)$;
- e is orthogonal to $q \oplus 1_k$;
- $p \oplus 1_k$ is Murray-von Neumann equivalent to $(e + q) \oplus 1_k$ in $M_{n+k}(A)$.

Note that $z = [p \oplus 1_k] - [q \oplus 1_k]$. Without loss of generality, upon replacing p and q with $p \oplus 1_k$ and $q \oplus 1_k$, respectively, and n with $n + k$, we may moreover assume that:

- e is orthogonal to q;
- p is Murray-von Neumann equivalent to $e + q$ in $M_n(A)$.

Set $\alpha^{(n)} = \alpha \otimes \mathrm{id}_{M_n}$, which has the Rokhlin property. Let $s \in M_n(A)$ be a partial isometry satisfying $s^*s = p$ and $ss^* = e + q$. For $\varepsilon = 1/12$, let $\delta > 0$ such that whenever B is a C^*-algebra and $a \in B_{\mathrm{sa}}$ satisfies $\|a^2 - a\| < \delta$, then there exists a projection $r \in B$ with $\|r - a\| < \varepsilon$. Set $\varepsilon_0 = \min\{\varepsilon, \delta/5\}$, and let $\sigma\colon M_n(A) \to M_n(A^\alpha)$ be a unital completely positive map as in the conclusion of Theorem 2.11 in [13] for ε_0, $F_1 = \{p, q, e, s, s^*\}$ and $F_2 = \{p, q\}$. Set $t = \sigma(s)$ and $f = \sigma(e)$. Then

$$t^*t \approx_{\varepsilon_0} \sigma(s^*s) = \sigma(p) \approx_{\varepsilon_0} p,$$

so $\|t^*t - p\| < 2\varepsilon_0$. In particular, $(1 - q)f \approx_{2\varepsilon_0} f$ and hence $\|qf\| < 2\varepsilon_0$. Similarly, $\|fq\| < 2\varepsilon_0$.

Set $a = (1 - q)f(1 - q)$. Then $a = a^*$ and $\|a - f\| \leq 4\varepsilon_0$. Moreover,

$$a^2 = (1 - q)f(1 - q)f(1 - q) \approx_{2\varepsilon_0} (1 - q)f^2(1 - q) \approx_{2\varepsilon_0} (1 - q)f(1 - q) = a,$$

so $\|a^2 - a\| \leq 4\varepsilon_0 < \delta$. By the choice of δ applied to $B = (1 - q)M_n(A^\alpha)(1 - q)$, there exists a projection $r \in M_n(A^\alpha)$ with $\|r - a\| < \varepsilon$ and $rq = 0$.

Set $x = (q + r)tp \in M_n(A^\alpha)$. Then

$$\begin{aligned} x^*x = pt^*(q + r)tp &\approx_{\varepsilon} pt^*(q + a)tp \\ &\approx_{4\varepsilon_0} pt^*(q + f)tp \approx_{2\varepsilon_0} pt^*tt^*tp \\ &\approx_{2\varepsilon_0} p^4 = p, \end{aligned}$$

so $\|x^*x - p\| \leq 8\varepsilon_0 + \varepsilon < 1$. Similarly, we have

$$\begin{aligned} xx^* &= (q + r)tpt^*(q + r) \approx_{2\varepsilon_0} (q + r)tt^*tt^*(q + r) \\ &\approx_{4\varepsilon_0} (q + r)(q + f)(q + f)(q + r) \approx_{4\varepsilon_0+\varepsilon} (q + r)^4 = q + r, \end{aligned}$$

so $\|xx^* - (q + r)\| \leq 10\varepsilon_0 + \varepsilon < 1$. Since $x = (q + r)xp$, it follows from Lemma 2.5.3 in [25] that there exists a partial isometry $v \in M_n(A^\alpha)$ such that $v^*v = p$ and $vv^* = q + r$. It follows that $[p] = [q] + [r]$ in $K_0(A^\alpha)$, and thus $z = [p] - [q]$ is positive in $K_0(A^\alpha)$, as desired. □

Next, we recall the definition of a pure subgroup and a pure extension.

Definition 3.2. Let G be an abelian group and let H be a subgroup. We say that H is *pure* if $nH = nG \cap H$ for all $n \in \mathbb{N}$. In other words, for every $h \in H$ and $n \in \mathbb{N}$, if there exists $g \in G$ with $ng = h$ then there exists $h' \in H$ with $nh' = h$.

An extension $0 \to H \to G \to Q \to 0$ is said to be *pure* if H is pure in G.

The notion of a pure subgroup generalizes that of a direct summand. For example, the torsion subgroup of any abelian group is always a pure subgroup, although it is not always a direct summand. If G is finitely generated, then any pure subgroup is automatically a direct summand. On the other hand, there exist pure subgroups which are finitely generated, yet not a direct summand (despite being direct summands in every finitely generated subgroup that contains them).

In the following theorem, note that part (3) does not follow from part (2) since in (3) we only make assumptions about *one* of the K-groups of A, and not both. Also, if we assume that A satisfies the UCT, then part (1) admits an easier proof: by the universal multicoefficient theorem, the 6-term Pimsner-Voiculescu exact sequence decomposes into two pure extensions if and only if its boundary map vanishes once tensored with $\mathcal{O}_n$ for $n = 2, 3, \ldots, \infty$. That said maps vanish follows from Proposition 3.1, since the dynamical system $(A \otimes \mathcal{O}_n, \alpha \otimes \mathrm{id}_{\mathcal{O}_n})$ has the Rokhlin property for all $n = 2, 3, \ldots, \infty$. We thank the anonymous referee for providing us with this argument.

Theorem 3.3. Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action on a unital C^*-algebra A with the Rokhlin property. Denote by $\iota\colon A^\alpha \to A$ the canonical inclusion.

(1) There is a canonical class $\mathrm{Ext}_*(\alpha) = (\mathrm{Ext}_0(\alpha), \mathrm{Ext}_1(\alpha))$, where $\mathrm{Ext}_j(\alpha) \in \mathrm{Ext}(K_j(SA^\alpha), K_j(A^\alpha))$ is the pure extension

$$0 \longrightarrow K_j(A^\alpha) \xrightarrow{K_j(\iota)} K_j(A) \longrightarrow K_j(SA^\alpha) \longrightarrow 0\ .$$

(2) If both $K_0(A)$ and $K_1(A)$ are (possibly infinite) direct sums of cyclic groups, then there are isomorphisms

$$K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$$

such that $[1_A] \in K_0(A)$ is sent to $([1_{A^\alpha}], 0) \in K_0(A^\alpha) \oplus K_1(A^\alpha)$.

(3) If at least one of $K_0(A)$ or $K_1(A)$ is finitely generated, then there are isomorphisms as in (2).

Proof. (1). Arguing as in the beginning of the proof of Proposition 3.1, we deduce that the Pimsner-Voiculescu exact sequence associated to $\breve{\alpha}$ splits into the short exact sequence

$$0 \to K_*(A^\alpha) \to K_*(A) \to K_*(SA^\alpha) \to 0.$$

We claim that the extension is pure. By taking considering the tensor product of A with any unital C^*-algebra which is KK-equivalent to $C_0(\mathbb{R})$, endowed with the trivial action of $\mathbb{T}$, it follows that it suffices to prove that $K_0(A^\alpha)$ is a pure subgroup of $K_0(A)$. Let $x \in K_0(A^\alpha)$, let $k \in \mathbb{N}$ and let $y \in K_0(A)$, and suppose that $ky = K_0(\iota)(x)$. Find projections $p_x, q_x \in \bigcup_{m\in\mathbb{N}} M_m(A^\alpha)$ and $p_y, q_y \in \bigcup_{m\in\mathbb{N}} M_m(A)$ such that $x = [p_x] - [q_x]$ and $y = [p_y] - [q_y]$. It follows that $k[p_y] + [q_x] = [p_x] + k[q_y]$ in $K_0(A)$. Without loss of generality, we may assume that there is $n \in \mathbb{N}$ such that

- p_x, q_x to $M_n(A)^\alpha$ and p_y, q_y belong to $M_n(A)$;
- p_x is orthogonal to q_y and p_y is orthogonal to q_x;
- there exists $s \in M_{nk}(A)$ with

$$s^*s = \begin{pmatrix} q_x + p_y & & & \\ & p_y & & \\ & & \ddots & \\ & & & p_y \end{pmatrix} \quad \text{and} \quad ss^* = \begin{pmatrix} p_x + q_y & & & \\ & q_y & & \\ & & \ddots & \\ & & & q_y \end{pmatrix}.$$

Note that $\alpha^{(nk)}\colon \mathbb{T} \to \mathrm{Aut}(M_{nk}(A))$ has the Rokhlin property. For $\varepsilon = 1/12$, let $\delta > 0$ such that whenever B is a C^*-algebra and $a \in B$ is a self-adjoint element satisfying $\|a^2 - a\| < \delta$, then there exists a projection $r \in B$ with $\|r - a\| < \varepsilon$. Set $\varepsilon_0 = \min\{\varepsilon, \delta/10\}$, and let $\sigma\colon M_{nk}(A) \to M_{nk}(A^\alpha)$ be a unital completely positive map as in the conclusion of Theorem 2.11 in [13] for ε_0, $F_1 = \{p_x, q_x, p_y, q_y, s, s^*\}$ and $F_2 = \{p_x, q_x\}$. Then $\|\sigma(p_x) - p_x\| < \varepsilon$ and $\|\sigma(q_x) - q_x\| < \varepsilon$.

Set $e_y = \sigma(p_y)$ and $f_y = \sigma(q_y)$, which are self-adjoint contractions in $M_{nk}(A^\alpha)$ satisfying $\|e_y^2 - e_y\| < 2\varepsilon_0$ and $\|f_y^2 - f_y\| < 2\varepsilon_0$. Set $t = \sigma(s)$. Then

$$t^*t \approx_{2\varepsilon_0} \mathrm{diag}(q_x + e_y, e_y, \ldots, e_y) \quad \text{and} \quad tt^* \approx_{2\varepsilon_0} \mathrm{diag}(p_x + f_y, f_y, \ldots, f_y).$$

Set $a = (1-p_x)f_y(1-p_x)$ and $b = (1-q_x)e_y(1-q_x)$, which are self-adjoint elements in the corners of $M_{nk}(A^\alpha)$ by $1-p_x$ and $1-q_x$, respectively. Moreover, $\|a - f_y\| \leq 4\varepsilon_0$ and $\|b - e_y\| \leq 4\varepsilon_0$. On the other hand,

$$\begin{aligned} a^2 &= (1-p_x)f_y(1-p_x)f_y(1-p_x) \\ &\approx_{2\varepsilon_0} (1-p_x)f_y^2(1-p_x) \\ &\approx_{2\varepsilon_0} (1-p_x)f_y(1-p_x) = a, \end{aligned}$$

so $\|a^2 - a\| < 4\varepsilon_0 < \delta$. Similarly, we have $\|b^2 - b\| < \delta$. Using the definition of δ with $B = (1-p_x)M_{nk}(A^\alpha)(1-p_x)$, there exits a projection $\widetilde{q}_y \in M_{nk}(A^\alpha)$ satisfying $\|\widetilde{q}_y - a\| < \varepsilon$ and $p_x\widetilde{q}_y = 0$. Similarly, there exits a projection $\widetilde{p}_y \in M_{nk}(A^\alpha)$ satisfying $\|\widetilde{p}_y - b\| < \varepsilon$ and $\widetilde{p}_y q_x = 0$. Set

$$r = \mathrm{diag}(p_x + \widetilde{q}_y, \widetilde{q}_y, \ldots, \widetilde{q}_y) t \mathrm{diag}(q_x + \widetilde{p}_y, \widetilde{p}_y, \ldots, \widetilde{p}_y),$$

which belongs to $M_{nk}(A^\alpha)$. One checks that $\|r^*r - \mathrm{diag}(q_x + \widetilde{p}_y, \widetilde{p}_y, \ldots, \widetilde{p}_y)\| < 1$, and that $\|r^*r - \mathrm{diag}(p_x + \widetilde{q}_y, \widetilde{q}_y, \ldots, \widetilde{q}_y)\| < 1$. By Lemma 2.5.3 in [25], there exists a partial isometry $w \in M_{nk}(A^\alpha)$ such that

$$w^*w = \begin{pmatrix} q_x + \widetilde{p}_y & & & \\ & \widetilde{p}_y & & \\ & & \ddots & \\ & & & \widetilde{p}_y \end{pmatrix} \quad \text{and} \quad ww^* = \begin{pmatrix} p_y + \widetilde{q}_y & & & \\ & \widetilde{q}_y & & \\ & & \ddots & \\ & & & \widetilde{q}_y \end{pmatrix}.$$

It follows that $[p_x] + k[\widetilde{q}_y] = k[\widetilde{p}_y] + [p_y]$ in $K_0(A^\alpha)$. With $z = [\widetilde{q}_x] - [\widetilde{q}_y] \in K_0(A^\alpha)$, we have $kz = x$, as desired.

(2). The displayed isomorphisms follow from Proposition 5.4 in [41]. Since parts of the proposition are left as an exercise, and since we need to show that the isomorphism is compatible with the classes of the unit, we give a brief argument here for the existence of an isomorphism $K_0(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$ sending $[1_A]$ to $([1_{A^\alpha}], 0)$. (The argument for $K_1(A)$ is identical because the assumptions are symmetric.) Abbreviate $K_j(A)$ to K_j, and $K_j(A^\alpha)$ to K_j^α, for $j = 0, 1$. It is a standard result in group theory, usually attributed to Kulikov, that subgroups of direct sums of cyclic groups are again direct sums of cyclic groups; see Theorem 18.1 in [7]. Since K_1 is a direct sum of cyclic groups, we deduce that the same is true for K_1^α.

By part (1) of this theorem, there is a canonical quotient map $\pi\colon K_0 \to K_1^\alpha$ whose kernel is K_0^α. Choose a presentation $K_1^\alpha \cong \bigoplus_{s\in S} C_s$, where each C_s is a cyclic group with generator x_s. In particular, $\{x_s : s \in S\}$ generates K_1^α. Let $s \in S$. If x_s has infinite order in K_1^α, we let $y_s \in K_0$ be any group element (necessarily of infinite order) satisfying $\pi(y_s) = x_s$. If x_s has order $n < \infty$, let $z_s \in K_0$ be any lift of x_s, and note that nz_s belongs to K_0^α, which is a pure subgroup of K_0. Hence there exists $k_s \in K_0^\alpha$ with $nk_s = nz_s$, and we set $y_s = z_s - k_s$, which also lifts x_s.

Let L be the subgroup of K_0 generated by $\{y_s : s \in S\}$, which is mapped isomorphically onto K_1^α via π. In particular, $K_0^\alpha \cap L = \{0\}$ and $K_0^\alpha + L = K_0$. (Equivalently, L defines a splitting for the quotient map π.) We deduce that the extension $0 \to K_0^\alpha \to K_0 \to K_1^\alpha \to 0$ splits, and thus $K_0 \cong K_0^\alpha \oplus K_1^\alpha$. The isomorphism can be clearly chosen to send $[1_A] \in K_0$ to $[1_{A^\alpha}] \in K_0^\alpha$.

(3). Assume that $K_1(A)$ is finitely generated (and in particular a direct sum of cyclic groups). Hence $K_0(A^\alpha)$ and $K_1(A^\alpha)$ are both finitely generated, being a quotient and a subgroup of $K_1(A)$, respectively. The argument given in the proof of part (2) above shows that there is an isomorphism $K_0(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$. Thus $K_0(A)$ is also finitely generated, and repeating the same argument again, exchanging the roles of K_0 and K_1, shows that $K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$. □

In reference to part (2) of Theorem 3.3, we mention that it is not in general true that a pure subgroup of a direct sum of cyclic groups is automatically a direct summand. For example, set $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{2^n}$, with canonical generators $x_n \in \mathbb{Z}_{2^n}$ for $n \in \mathbb{N}$, and let H be the subgroup generated by $\{x_n - 2x_{n+1} : n \in \mathbb{N}\}$. Then H is pure in G but not a direct summand.

4. Circle actions on Kirchberg algebras

This section contains our main results concerning $KK^{\mathbb{T}}$-theory for Rokhlin actions. This includes Theorems C and F from the introduction. In the presence of the UCT, any isomorphism between the pure extensions from Theorem 3.3 lifts to a $KK^{\mathbb{T}}$-equivalence. We show by means of an example that an isomorphism of the K- and $K^{\mathbb{T}}$-theories does not necessarily lift to a $KK^{\mathbb{T}}$-equivalence; see Example 4.7. Finally, we also describe the extensions that arise as $\mathrm{Ext}_*(\alpha)$ for a Rokhlin action α on a Kirchberg algebra satisfying the UCT; see Theorem 4.8.

Definition 4.1. Let A be a simple unital C^*-algebra. Then A is said to be:

(1) *purely infinite*, if for every $a \in A \setminus \{0\}$ there are $x, y \in A$ with $xay = 1$.
(2) a *Kirchberg algebra*, if it is purely infinite, separable and nuclear.

Recall that an automorphism φ of a C^*-algebra is said to be *aperiodic* if φ^n is not inner for all $n \geq 1$.

Proposition 4.2. Let A be a unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

(1) A is simple if and only if A^α is simple and $\breve{\alpha}$ is aperiodic.
(2) A is purely infinite simple if and only if A^α is purely infinite simple and $\breve{\alpha}$ is aperiodic.
(3) A is a Kirchberg algebra if and only if A^α is a Kirchberg algebra and $\breve{\alpha}$ is aperiodic.
(4) A satisfies the UCT if and only if A^α satisfies the UCT.

Proof. (1). If A is simple, then A^α is simple by Corollary 2.5. We show that $\breve{\alpha}$ is aperiodic. Arguing by contradiction, suppose that there exist $n \geq 1$ and a unitary $v \in A^\alpha$ such that $\breve{\alpha}^n = \mathrm{Ad}(v)$. Set

$$w = v\breve{\alpha}(v)\cdots\breve{\alpha}^{n-1}(v).$$

Then $\breve{\alpha}^{n^2} = \mathrm{Ad}(w)$. Using that $\breve{\alpha}^n(v) = v$ at the second step, that $vx = \breve{\alpha}^n(x)v$ for all $x \in A^\alpha$ at the third, and that $\breve{\alpha}^{-n}(v) = v$ at he fifth, we get

$$\begin{aligned}\breve{\alpha}(w) &= \breve{\alpha}(v)\breve{\alpha}^2(v)\cdots\breve{\alpha}^{n-1}(v)\breve{\alpha}^n(v) = \breve{\alpha}(v)\breve{\alpha}^2(v)\cdots\breve{\alpha}^{n-1}(v)v\\ &= v\breve{\alpha}^{-n}\big(\breve{\alpha}(v)\breve{\alpha}^2(v)\cdots\breve{\alpha}^{n-1}(v)\big) = v\breve{\alpha}\big(\breve{\alpha}^{-n}(v)\big)\breve{\alpha}^2\big(\breve{\alpha}^{-n}(v)\big)\cdots\breve{\alpha}^{n-1}\big(\breve{\alpha}^{-n}(v)\big)\\ &= v\breve{\alpha}(v)\cdots\breve{\alpha}^{n-1}(v) = w.\end{aligned}$$

It follows that w is $\breve{\alpha}$-invariant. With u denoting the canonical unitary in $A^\alpha \rtimes_{\breve{\alpha}} \mathbb{Z}$ that implements $\breve{\alpha}$, we therefore have $uwu^* = w$. Set $z = u^{n^2}w^*$. It is clear that z commutes with u, and for $a \in A^\alpha$ we have

$$zaz^* = u^{n^2}w^*aw\big(u^{n^2}\big)^* = u^{n^2}\breve{\alpha}^{-n^2}(a)\big(u^{n^2}\big)^* = a,$$

so z belongs to the center of $A^\alpha \rtimes_{\breve{\alpha}} \mathbb{Z} \cong A$. Since A is simple, its center is trivial and thus there is $\lambda \in \mathbb{C}$ with $u^{n^2} = \lambda w$. In particular, u^{n^2} belongs to A^α, which is a contradiction. This shows that $\breve{\alpha}$ is aperiodic.

Conversely, if $\check{\alpha}$ is aperiodic and A^α is simple, it follows from Theorem 3.1 in [21] that the crossed product $A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z} \cong A$ is simple.

(2). Assume that A is purely infinite simple. Then $\check{\alpha}$ is aperiodic by part (1). Let $a \in A^\alpha$ be nonzero and let $\varepsilon > 0$ small enough so that $\varepsilon^3 + 3\varepsilon < 1$. Without loss of generality, we assume that $\|a\| = 1$. Find $x, y \in A$ such that $xay = 1$. By Lemma 4.1.7 in [38], we may assume that $\|x\| < 1 + \varepsilon$ and $\|y\| < 1 + \varepsilon$. Let $\sigma\colon A \to A^\alpha$ a completely positive unital map as in the conclusion of Theorem 2.11 in [13] for $\varepsilon > 0$, $F_2 = \{x, y, xa, a\}$ and $F_1 = \{a\}$. Then

$$\sigma(x)a\sigma(y) \approx_{(1+\varepsilon)^2\varepsilon} \sigma(x)\sigma(a)\sigma(y) \approx_\varepsilon \sigma(xa)\sigma(y) \approx_\varepsilon \sigma(xay) = 1.$$

Hence $\|\sigma(x)a\sigma(y) - 1\| \leq \varepsilon^3 + 3\varepsilon < 1$. It follows that $\sigma(x)a\sigma(y)$ is invertible. With $z \in A^\alpha$ denoting its inverse, we have $\sigma(x)a\sigma(y)z = 1$, as desired.

Conversely, assume that $\check{\alpha}$ is aperiodic and A^α is purely infinite simple. Then $A \cong A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z}$ is purely infinite simple by Corollary 4.6 in [18].

(3). This follows from (1) and (2), since A is nuclear (respectively, separable) if and only if so is A^α.

(4). If A satisfies the UCT, then so does A^α by Theorem 3.13 in [11]. The converse follows from the fact that the UCT is preserved by $\mathbb{Z}$-crossed products. □

Remark 4.3. Let the notation be as in Proposition 4.2.

- Simplicity of A is really needed in (1) to conclude that $\check{\alpha}$ is aperiodic, even if A^α is simple. Consider for example the trivial automorphism of $\mathbb{C}$, whose dual action is the left translation action $\mathtt{Lt}\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T}))$.
- Outerness of α is not enough in (2) to deduce pure infiniteness of A^α from pure infiniteness of A (unlike for finite groups). For example, the fixed point algebra of $\mathcal{O}_\infty$ by its gauge action is AF (and not even simple).

We will need some terminology.

Definition 4.4. Let G be a second countable, locally compact group (in this work either $\mathbb{T}$, $\mathbb{Z}$, or the trivial group), let A and B be separable, unital C^*-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions. We say that (A, α) and (B, β) are *unitally KK^G-equivalent*, if there is an invertible class $\eta \in KK^G(A, B)$ with $[1_A] \times \eta = [1_B]$. In this situation, we say that η is a *unital KK^G-equivalence*.

A unital $KK^{\mathbb{Z}}$-equivalence between two $\mathbb{Z}$-actions (A, σ) and (B, θ) naturally induces an isomorphism between the Pimsner-Voiculescu 6-term exact sequences of σ and θ which moreover preserves the classes of the units; this can be found in [29], and a special case will be proved in Theorem 4.6. If the automorphisms are moreover approximately inner, then each of these 6-term exact sequences splits into two short exact sequences. The resulting equivalence relation for sums of short exact sequences (with distinguished classes) is the following:

Definition 4.5. For $j = 0, 1$, let $K_j^{\mathcal{E}}, G_j^{\mathcal{E}}, K_j^{\mathcal{F}}$, and $G_j^{\mathcal{F}}$ be countable abelian groups, let $k_0^{\mathcal{E}} \in K_0^{\mathcal{E}}$ and $k_0^{\mathcal{F}} \in K_0^{\mathcal{F}}$, and let

$$(\mathcal{E}_j)\ \ 0 \to K_j^{\mathcal{E}} \to G_j^{\mathcal{E}} \to K_{1-j}^{\mathcal{E}} \to 0 \quad \text{and} \quad (\mathcal{F}_j)\ \ 0 \to K_j^{\mathcal{F}} \to G_j^{\mathcal{F}} \to K_{1-j}^{\mathcal{F}} \to 0$$

be short exact sequences. We say that $(\mathcal{E}_0, \mathcal{E}_1, k_0^{\mathcal{E}})$ is *isomorphic* to $(\mathcal{F}_0, \mathcal{F}_1, k_0^{\mathcal{F}})$ if there exist group isomorphisms $\varphi_j\colon K_j^{\mathcal{E}} \to K_j^{\mathcal{F}}$ and $\psi_j\colon G_j^{\mathcal{E}} \to G_j^{\mathcal{F}}$ with $\varphi_0(k_0^{\mathcal{E}}) = k_0^{\mathcal{F}}$ making the following diagram commute:

$$\begin{array}{ccccc} K_j^{\mathcal{E}} & \longrightarrow & G_j^{\mathcal{E}} & \longrightarrow & K_{1-j}^{\mathcal{E}} \\ \downarrow{\scriptstyle \psi_j} & & \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \psi_{1-j}} \\ K_j^{\mathcal{F}} & \longrightarrow & G_j^{\mathcal{F}} & \longrightarrow & K_{1-j}^{\mathcal{F}}. \end{array} \tag{4.1}$$

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This is different from having two isomorphisms $(\mathcal{E}_0, k_0^{\mathcal{E}}) \cong (\mathcal{F}_0, k_0^{\mathcal{F}})$ and $\mathcal{E}_1 \cong \mathcal{F}_1$, since, for example, we require the isomorphism $K_0^{\mathcal{E}} \cong K_0^{\mathcal{F}}$ to be the same both in the isomorphism $\mathcal{E}_0 \cong \mathcal{F}_0$ and in $\mathcal{E}_1 \cong \mathcal{F}_1$.

We are now ready to prove that Rokhlin actions of the circle on unital Kirchberg algebras are conjugate if they are unitally $KK^{\mathbb{T}}$-equivalent.

Theorem 4.6. Let A and B be unital Kirchberg algebras, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be actions with the Rokhlin property. Then (A, α) and (B, β) are conjugate if and only if they are unitally $KK^{\mathbb{T}}$-equivalent.

When A and B satisfy the UCT, these conditions are equivalent to the existence of an isomorphism $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) \cong (\mathrm{Ext}_*(\beta), [1_{B^\beta}])$ in the sense of Definition 4.5.

Proof. Assume that α and β are unitally $KK^{\mathbb{T}}$-equivalent, and fix a unital $KK^{\mathbb{T}}$-equivalence $\xi \in KK^{\mathbb{T}}\big((A, \alpha), (B, \beta)\big)$. Denote by $\xi \rtimes \mathbb{T}$ the $KK^{\mathbb{Z}}$-equivalence between $(A \rtimes_\alpha \mathbb{T}, \widehat{\alpha})$ and $(B \rtimes_\beta \mathbb{T}, \widehat{\beta})$ that ξ induces. Combining Takai duality with Theorem 2.3, it follows that $\xi \rtimes \mathbb{T}$ induces a $KK^{\mathbb{Z}}$-equivalence η between $(A^\alpha, \check{\alpha})$ and $(B^\beta, \check{\beta})$. Since ξ is unital, we can choose η to be unital as well.

Since A^α and B^β are Kirchberg algebras by part (3) of Proposition 4.2, it follows from Theorem 4.2.1 in [36] that there exists an isomorphism $\phi\colon A^\alpha \to B^\beta$ such that $KK(\phi) = \eta$. Since η is equivariant, it follows that $\phi \circ \check{\alpha} \circ \phi^{-1}$ and $\check{\beta}$ determine the same class in $KK(B^\beta, B^\beta)$. Since A and B are simple, it follows from part (1) of Proposition 4.2 that $\check{\alpha}$ and $\check{\beta}$ are aperiodic. Thus, by the equivalence between (1') and (3') in Theorem 9 in [30], $\phi \circ \check{\alpha} \circ \phi^{-1}$ and $\check{\beta}$, and thus $\check{\alpha}$ and $\check{\beta}$, are cocycle conjugate[2]. It follows that their dual actions are conjugate, and hence $(A, \alpha) \cong (B, \beta)$ as desired.

We turn to the last part of the statement. Fix a unital $KK^{\mathbb{T}}$-equivalence $\rho \in KK^{\mathbb{T}}((A, \alpha), (B, \beta))$. Arguing as in the first part using Baaj-Skandalis duality, we obtain a unital $KK^{\mathbb{Z}}$-equivalence

$$\eta \in KK^{\mathbb{Z}}\big((A^\alpha, \check{\alpha}), (B^\beta, \check{\beta})\big).$$

Denote by $\mathcal{F}(\eta) \in KK(A^\alpha, B^\beta)$ the KK-equivalence that η induces under the forgetful functor $KK^{\mathbb{Z}} \to KK$. Similarly, we let $\mathcal{F}(\eta \rtimes \mathbb{Z}) \in KK(A, A)$ denote the KK-equivalence that $\eta \rtimes \mathbb{Z}$ induces, which can be canonically identified with $\mathcal{F}(\rho)$. Using naturality of the functors involved, we deduce that the diagram

$$\begin{array}{ccccc} K_j(A^\alpha) & \longrightarrow & K_j(A) & \longrightarrow & K_{1-j}(A^\alpha) \\ \downarrow{\scriptstyle \mathcal{F}(\eta)_j} & & \downarrow{\scriptstyle \mathcal{F}(\rho)_j} & & \downarrow{\scriptstyle \mathcal{F}(\eta)_{1-j}} \\ K_j(B^\beta) & \longrightarrow & K_j(B) & \longrightarrow & K_{1-j}(B^\beta), \end{array}$$

commutes. Since all vertical maps are isomorphisms and $\mathcal{F}(\eta)_0([1_{A^\alpha}]) = [1_{B^\beta}]$, we deduce that $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) \cong (\mathrm{Ext}_*(\beta), [1_{B^\beta}])$.

We now prove the converse, so assume that A and B satisfy the UCT. By part (4) of Proposition 4.2, A^α and B^β also satisfy the UCT. Since $\check{\alpha}$ and $\check{\beta}$ are approximately inner, an isomorphism $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) \cong (\mathrm{Ext}_*(\beta), [1_{B^\beta}])$ is equivalent to $\check{\alpha}$ and $\check{\beta}$ having isomorphic Pimsner-Voiculescu 6-term exact sequences (in a unit-preserving way); see the comments before Definition 4.5. It thus follows from Theorem 2.12 in [28] that $(A^\alpha, \check{\alpha})$ is unitally $KK^{\mathbb{Z}}$-equivalent to $(B^\beta, \check{\beta})$. Again by Baaj-Skandalis duality, it follows that (A, α) is unitally $KK^{\mathbb{T}}$-equivalent to (B, β). This finishes the proof. ☐

In the context of Theorem 4.6, the assumption that the diagram (4.1) from Definition 4.5 commutes cannot be dropped, and it is not enough to have isomorphisms $K_*(A^\alpha) \cong K_*(B^\beta)$ and $K_*(A) \cong K_*(B)$. In the next example, we construct a Kirchberg

[2] Nakamura uses the terminology "outer conjugate" to mean what we call "cocycle conjugate".

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algebra A satisfying the UCT, and two actions $\alpha, \gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ with the Rokhlin property, such that $A^\alpha \cong A^\gamma$ but α and γ are not conjugate. In particular, the example shows that an isomorphism of the $K^{\mathbb{T}}$-theory cannot in general be lifted to a $KK^{\mathbb{T}}$-equivalence.

In preparation for our construction, we introduce some notation. If B is a unital C^*-algebra and $\varphi \in \mathrm{Aut}(B)$ is an approximately inner automorphism, then the Pimsner-Voiculescu exact sequence for φ reduces to the short exact sequences

$$0 \longrightarrow K_j(B) \xrightarrow{K_j(\iota)} K_j(B \rtimes_\varphi \mathbb{Z}) \longrightarrow K_{1-j}(B) \longrightarrow 0\,, \tag{4.2}$$

for $j = 0, 1$. We denote by $\eta_j(\varphi) \in \mathrm{Ext}(K_{1-j}(B), K_j(B))$ the class of the above extension. Note that if $\varphi, \psi \in \mathrm{Aut}(B)$ are cocycle conjugate automorphisms, then $\eta_j(\varphi) = \eta_j(\psi)$.

Example 4.7. Let p be a prime number. Let $\xi_0 \in \mathrm{Ext}^1_{\mathbb{Z}}\big(\mathbb{Z}\big[\frac{1}{p}\big], \mathbb{Z}\big)$ be a nontrivial class, and fix a representing extension

$$0 \to \mathbb{Z} \to E_0 \to \mathbb{Z}\big[\tfrac{1}{p}\big] \to 0.$$

Set

$$K = \mathbb{Q} \oplus \bigoplus_{n=1}^{\infty} \big(\mathbb{Z} \oplus E_0 \oplus \mathbb{Z}\big[\tfrac{1}{p}\big]\big) \quad \text{and} \quad E = K \oplus K.$$

Then K and E are torsion free and abelian. Using that E_0 is a non-trivial extension of $\mathbb{Z}\big[\frac{1}{p}\big]$ by $\mathbb{Z}$, fix a non-trivial extension

$$0 \to K \to E \to K \to 0, \tag{4.3}$$

and write $\xi \in \mathrm{Ext}(K, K)$ for the induced class. Since K is torsion free, it follows that the extension in (4.3) is pure. Note that there is also an isomorphism $E \cong K \oplus K$. (In other words, and this is a crucial ingredient in the construction, the group E can be written in two non-equivalent ways as an extension of K by itself.)

Set $k_0 = (1_{\mathbb{Q}}, 0, 0, \ldots) \in K$ and $e_0 = (k_0, k_0) \in E$. Since K is torsion-free, we may use Elliott's classification of simple $\mathrm{A}\mathbb{T}$-algebras with real rank zero (see the comments before Proposition 3.2.7 in [38]), to find a simple, unital $\mathrm{A}\mathbb{T}$-algebra C satisfying $K_0(C) \cong K_1(C) \cong K$ with $[1_C]$ corresponding to $k_0 \in K$. Use the case $i = 1$ of Theorem 3.1 in [24] to find an approximately inner automorphism $\varphi \in \mathrm{Aut}(C)$ with $\eta_0(\varphi) = 0$ and $\eta_1(\varphi) = \xi \in \mathrm{Ext}(K, K)$. Then

$$K_0(C \rtimes_\varphi \mathbb{Z}) \cong E \cong K_1(C \rtimes_\varphi \mathbb{Z}).$$

The proof of Theorem 3.1 in [24] is in fact constructive, and the argument used to prove the case $i = 1$ shows that φ can be chosen to be approximately representable. Indeed, it is shown in Subsection 3.11 of [24] that there is an increasing sequence $(C_n)_{n\in\mathbb{N}}$ of unital subalgebras of C with $C = \varinjlim C_n$, and unitaries $u_n \in C_n$ satisfying $\varphi = \varinjlim \mathrm{Ad}(u_n)$.

Claim: *φ is aperiodic.* Arguing by contradiction, suppose that there exist $n \geq 1$ and $u \in \mathcal{U}(C)$ such that $\varphi^n = \mathrm{Ad}(u)$. Set $v = u\varphi(u)\cdots\varphi^{n-1}(u) \in C$. Then $\varphi^{n^2} = \mathrm{Ad}(v)$ and v is φ^{n^2}-invariant. Moreover, we have

$$\begin{aligned}
\varphi(v) &= \varphi(u)\varphi^2(u)\cdots\varphi^{n-1}(u)\varphi^n(u) = \varphi(u)\varphi^2(u)\cdots\varphi^{n-1}(u)u \\
&= u\varphi^{-n}\big(\varphi(u)\varphi^2(u)\cdots\varphi^{n-1}(u)\big) = u\varphi\big(\varphi^{-n}(u)\big)\varphi^2\big(\varphi^{-n}(u)\big)\cdots\varphi^{n-1}\big(\varphi^{-n}(u)\big) \\
&= u\varphi(u)\cdots\varphi^{n-1}(u) = v,
\end{aligned}$$

and thus v is φ-invariant. Denote by D the twisted crossed product of C by φ with respect to the twist induced by v. By Theorem 2.4 in [32], the crossed product $C \rtimes_\varphi \mathbb{Z}$ is isomorphic to the induced algebra $\mathrm{Ind}^{\mathbb{T}}_{(n^2\mathbb{Z})^\perp}(D)$. By compactness of $\mathbb{T}$, this induced algebra is isomorphic to $C(\mathbb{T}, D)$. In particular,

$$E \cong K_0(C \rtimes_\varphi \mathbb{Z}) \cong K_1(C \rtimes_\varphi \mathbb{Z}) \cong K_0(D) \oplus K_1(D). \tag{4.4}$$

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We proceed to compute the K-theory of D. Set $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{Z}))$. By Theorem 3.4 in [34], there is an automorphism φ_0 of $C \otimes \mathcal{K}$ whose crossed product is isomorphic to D (this is the so-called Packer-Raeburn stabilisation trick). An explicit formula for φ_0 is given at the beginning of the proof on page 301 (see Equation (3.1)), which shows that we can choose φ_0 to be unitarily equivalent to $\varphi \otimes \mathrm{id}_{\mathcal{K}}$ (see also the top line on page 302). It follows that D is isomorphic to $(C \otimes \mathcal{K}) \rtimes_{\varphi \otimes \mathrm{id}_{\mathcal{K}}} \mathbb{Z} \cong (C \rtimes_\varphi \mathbb{Z}) \otimes \mathcal{K}$, and thus has the same K-groups as $C \rtimes_\varphi \mathbb{Z}$. That is, $K_0(D) \cong K_1(D) \cong E$.

Combining the above with (4.4), we deduce that $E \cong E \oplus E$. This is, however, not the case: the largest divisible subgroup of E is $\mathbb{Q}^2$, while the largest divisible subgroup of $E \oplus E$ is $\mathbb{Q}^4$. This contradiction shows that φ is aperiodic, proving the claim.

Set $B = C \rtimes_\varphi \mathbb{Z}$ and $\beta = \widehat{\varphi}\colon \mathbb{T} \to \mathrm{Aut}(B)$. Then β has the Rokhlin property by Proposition 2.8, since φ is approximately representable. Moreover, B is unital, separable, nuclear, satisfies the UCT, and is simple since φ is aperiodic. Set $A = B \otimes \mathcal{O}_\infty$ and $\alpha = \beta \otimes \mathrm{id}_{\mathcal{O}_\infty}$. Then A is a unital Kirchberg algebra satisfying the UCT, and α has the Rokhlin property. Note that

$$A^\alpha = (C \rtimes_\varphi \mathbb{Z})^\beta \otimes \mathcal{O}_\infty = C \otimes \mathcal{O}_\infty,$$

and thus A^α is the unique unital Kirchberg algebra satisfying the UCT with K-theory given by $K_0(A^\alpha) \cong K_1(A^\alpha) \cong K$, with unit class k_0.

We now wish to realize A in a different way as a crossed product by an approximately representable automorphism, in such a way such that the associated Pimsner-Voiculescu exact sequence splits into two copies of the trivial extension of K by itself.

Let $\psi_0 \in \mathrm{Aut}(\mathcal{O}_\infty)$ be an aperiodic, approximately representable automorphism of $\mathcal{O}_\infty$ (see, for example, Proposition 3.3 in [12]). Set $\psi = \mathrm{id}_{A^\alpha} \otimes \psi_0$, which we identify with an aperiodic, approximately representable automorphism of A^α. Since $K_1(\mathcal{O}_\infty) = 0$, both classes $\eta_0(\psi_0)$ and $\eta_1(\psi_0)$ are trivial. It follows that the same is true for $\eta_0(\psi) = \eta_1(\psi) = 0$ in $\mathrm{Ext}(K, K)$. The crossed product $A^\alpha \rtimes_\psi \mathbb{Z}$ is therefore a unital Kirchberg algebra satisfying the UCT with both K-groups isomorphic to E, and unit class e_0. It follows from the classification of Kirchberg algebras (specifically Theorem 4.2.4 in [36]) that there is an isomorphism $A \cong A^\alpha \rtimes_\psi \mathbb{Z}$. Denote by $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ the action that $\widehat{\psi}$ induces on A via this identification. Then γ has the Rokhlin property because ψ is approximately representable. Moreover,

$$A^\gamma \cong (A^\alpha \rtimes_\psi \mathbb{Z})^{\widehat{\psi}} = A^\alpha.$$

It follows that $K_*^{\mathbb{T}}(A, \alpha) \cong K_*^{\mathbb{T}}(A, \gamma)$ as groups (in fact, α and γ have isomorphic crossed products and fixed point algebras). This group isomorphism is automatically an isomorphism of $R(\mathbb{T})$-modules, since the action of $R(\mathbb{T}) \cong \mathbb{Z}[x, x^{-1}]$ on $K_*^{\mathbb{T}}$ is determined by the dual automorphism, which for Rokhlin actions is approximately inner (by Proposition 2.8) and hence trivial on K-theory.

We denote by $\iota^\alpha\colon A^\alpha \to A$ and $\iota^\gamma\colon A^\alpha \to A$ the induced inclusions of A^α into A as the α- and γ-fixed point algebras, respectively.

Claim: *there is no $KK^{\mathbb{T}}$-equivalence (unital or otherwise) between (A, α) and (A, γ).* This argument is in part similar to the one used in the proof of Theorem 4.6. Arguing by contradiction, let us assume that there is a $KK^{\mathbb{T}}$-equivalence $\rho \in KK^{\mathbb{T}}((A, \alpha), (A, \gamma))$. Using the Baaj-Skandalis duality, we get a $KK^{\mathbb{Z}}$-equivalence

$$\rho \rtimes \mathbb{T} \in KK^{\mathbb{Z}}\big((A \rtimes_\alpha \mathbb{T}, \widehat{\alpha}), (A \rtimes_\gamma \mathbb{T}, \widehat{\gamma})\big).$$

Since $(A \rtimes_\alpha \mathbb{T}, \widehat{\alpha}) \sim_{KK^{\mathbb{Z}}} (A^\alpha, \check{\alpha})$ and $(A \rtimes_\alpha \mathbb{T}, \widehat{\gamma}) \sim_{KK^{\mathbb{Z}}} (A^\alpha, \check{\gamma})$ by Takai duality, we identify $\rho \rtimes \mathbb{T}$ with a $KK^{\mathbb{Z}}$-equivalence

$$\kappa \in KK^{\mathbb{Z}}\big((A^\alpha, \check{\alpha}), (A^\alpha, \check{\gamma})\big).$$

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Denote by $\mathcal{F}(\kappa) \in KK(A^\alpha, A^\alpha)$ the KK-equivalence that κ induces under the forgetful functor $KK^{\mathbb{Z}} \to KK$. Similarly, we let $\mathcal{F}(\kappa \rtimes \mathbb{Z}) \in KK(A, A)$ denote the KK-equivalence that $\kappa \rtimes \mathbb{Z}$ induces, which can be canonically identified with $\mathcal{F}(\rho)$. By naturality of all the functors involved, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A^\alpha) & \xrightarrow{\iota_*^\alpha} & K_1(A) & \longrightarrow & K_0(A^\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle (\kappa \rtimes \mathbb{Z})_*} & & \downarrow{\scriptstyle \kappa_*} & & \\ 0 & \longrightarrow & K_1(A^\alpha) & \xrightarrow[\iota_*^\gamma]{} & K_1(A) & \longrightarrow & K_0(A^\alpha) & \longrightarrow & 0, \end{array} \tag{4.5}$$

where the vertical maps are all group isomorphisms. Note that the horizontal short exact sequences are the Pimsner-Voiculescu sequences associated to $\check{\alpha}$ and $\check{\gamma}$, respectively, as in (4.2), and thus their Ext-classes are $\eta_1(\check{\alpha})$ and $\eta_1(\check{\gamma})$, respectively. Moreover, $\check{\alpha} = \varphi \otimes \mathrm{id}_{\mathcal{O}_\infty}$ and under the KK-equivalence $C \sim_{KK} C \otimes \mathcal{O}_\infty$ induced by $\mathbb{C} \sim_{KK} \mathcal{O}_\infty$, the class $\eta_1(\check{\alpha})$ corresponds to ξ. Similarly, $\check{\gamma}$ is conjugate to $\mathrm{id}_{A^\alpha} \otimes \psi_0 \in \mathrm{Aut}(A^\alpha \otimes \mathcal{O}_\infty)$, which under the KK-equivalence $A^\alpha \sim_{KK} A^\alpha \otimes \mathcal{O}_\infty$ induced by $\mathbb{C} \sim_{KK} \mathcal{O}_\infty$, the class $\eta_1(\check{\gamma})$ corresponds to $\eta_1(\mathrm{id}_{A^\alpha}) = 0$.

Commutativity of (4.5) implies that $\xi = 0$, which is a contradiction. We conclude that α and γ are not $KK^{\mathbb{T}}$-equivalent. In particular, α and γ are not conjugate.

The above example shows an interesting phenomenon, worth putting into perspective. In Example 10.6 in [39], the authors construct two circle actions on commutative C^*-algebras with isomorphic $K^{\mathbb{T}}$-theory, which are not $KK^{\mathbb{T}}$-equivalent. In their example, the underlying algebras are not even KK-equivalent, so the actions cannot be $KK^{\mathbb{T}}$-equivalent. As informed to us by Schochet, there was until now no known example of two circle actions on the same C^*-algebra, with isomorphic fixed point algebras and crossed products, all satisfying the UCT, and with isomorphic $K^{\mathbb{T}}$-theory, that are not $KK^{\mathbb{T}}$-equivalent. Our construction provides such an example.

Theorem 4.6 states that, for Rokhlin actions on UCT Kirchberg algebras, the class $\mathrm{Ext}(\alpha)$ defined in part (1) of Theorem 3.3 is a complete invariant up to conjugacy. It is then natural to ask for a range result, namely, which Ext-classes arise from Rokhlin actions on UCT Kirchberg algebras. In the next result, we show that the only possible obstructions are the ones we obtained in Theorem 3.3, specifically that the extension is pure.

Theorem 4.8. Let K_0 and K_1 be abelian groups, let $k_0 \in K_0$, and let $\mathcal{E}_0 \in \mathrm{Ext}(K_0, K_1)$ and $\mathcal{E}_1 \in \mathrm{Ext}(K_1, K_0)$ be extensions. The following are equivalent:

(a) There is a Rokhlin action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ on a unital UCT Kirchberg algebra A with $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) \cong (\mathcal{E}_0, \mathcal{E}_1, k_0)$ in the sense of Definition 4.5;

(b) $\mathcal{E}_0$ and $\mathcal{E}_1$ are pure.

Proof. That (a) implies (b) follows from part (1) of Theorem 3.3. Assume that (b) holds, and write $\mathcal{E}_0$ and $\mathcal{E}_1$ explicitly as

$$(\mathcal{E}_0) \quad 0 \longrightarrow K_0 \xrightarrow{\iota_0} G_0 \xrightarrow{\pi_0} K_1 \longrightarrow 0 \qquad \text{and} \qquad (\mathcal{E}_1) \quad 0 \longrightarrow K_1 \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} K_0 \longrightarrow 0.$$

For $j = 0, 1$, let $(K_j^{(n)})_{n\in\mathbb{N}}$ be an increasing sequence of finitely generated subgroups of K_j whose union equals K_j. Without loss of generality, we assume that k_0 belongs to $K_0^{(n)}$ for all $n \in \mathbb{N}$. Fix $j = 0, 1$. Since $\mathcal{E}_j$ is pure, for every $n \in \mathbb{N}$ there exists a (necessarily finitely generated) subgroup $\widetilde{K}_{1-j}^{(n)}$ of G_j such that π_j restricts to an isomorphism $\widetilde{K}_{1-j}^{(n)} \cong K_{1-j}^{(n)}$.

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Let $G_j^{(n)}$ be the subgroup of G_j generated by $\iota_j(K_j^{(n)})$ and $\widetilde{K}_{1-j}^{(n)}$. We denote by $\mathcal{E}_j^{(n)}$ the restricted short exact sequence

$$(\mathcal{E}_j^{(n)}) \qquad 0 \longrightarrow K_j^{(n)} \longrightarrow G_j^{(n)} \longrightarrow K_{1-j}^{(n)} \longrightarrow 0\,,$$

where the maps are the restrictions of the ones for $\mathcal{E}_j$. Fix $n \in \mathbb{N}$. Since $\mathcal{E}_j$ is pure and $G_j^{(n)}$ is finitely generated, it follows that $\mathcal{E}_j^{(n)}$ is isomorphic to the trivial extension. Thus there are isomorphisms

$$G_0^{(n)} \cong K_0^{(n)} \oplus K_1^{(n)} \cong G_1^{(n)}.$$

Under the above identifications, let

$$h_j^{(n)} \colon G_j^{(n)} \cong K_0^{(n)} \oplus K_1^{(n)} \hookrightarrow K_0^{(n+1)} \oplus K_1^{(n+1)} \cong G_j^{(n+1)}$$

be the induced injective map.

Given $n \in \mathbb{N}$, let $\widetilde{B}_n$ be a UCT Kirchberg algebra satisfying

$$\big(K_0(\widetilde{B}_n), K_1(\widetilde{B}_n), [1_B]\big) = \big(K_0^{(n)}, K_1^{(n)}, k_0\big).$$

Fix an aperiodic approximately representable automorphism Φ of $\mathcal{O}_\infty$.[3] Set $B_n = \widetilde{B}_n \otimes \mathcal{O}_\infty$ and $\varphi_n = \mathrm{id}_{\widetilde{B}_n} \otimes \Phi \in \mathrm{Aut}(B_n)$. Then φ_n is approximately representable and aperiodic. Set $A_n = B_n \rtimes_{\varphi_n} \mathbb{Z}$ and let $\alpha^{(n)} \colon \mathbb{T} \to \mathrm{Aut}(A_n)$ denote the dual action of φ_n. Then A_n is a Kirchberg algebra satisfying the UCT by part (4) of Proposition 4.2, and $\alpha^{(n)}$ has the Rokhlin property by Proposition 2.8. Since the K-theory of A_n is finitely generated, it follows from part (3) of Theorem 3.3 that $\mathrm{Ext}_*(\alpha^{(n)})$ is isomorphic to the trivial extension, and hence $(\mathrm{Ext}_*(\alpha^{(n)}), k_0) \cong (\mathcal{E}_0^{(n)}, \mathcal{E}_1^{(n)}, k_0)$.

Use Theorem 4.1.1 in [36] to find a unital homomorphism $\widetilde{\theta}_n \colon \widetilde{B}_n \to \widetilde{B}_{n+1}$ inducing the canonical inclusion

$$(K_0^{(n)}, K_1^{(n)}, k_0) \hookrightarrow \big(K_0^{(n+1)}, K_1^{(n+1)}, k_0\big)$$

at the level of K-theory. Set $\theta_n = \widetilde{\theta}_n \otimes \mathrm{id}_{\mathcal{O}_\infty} \colon B_n \to B_{n+1}$, and note that θ_n is equivariant with respect to the automorphisms φ_n and φ_{n+1}. Let $\rho_n \colon (A_n, \alpha^{(n)}) \to (A_{n+1}, \alpha^{(n+1)})$ be the unital equivariant homomorphism induced by θ_n using duality.

Finally, we set $(A, \alpha) = \varinjlim(A_n, \rho_n, \alpha^{(n)})$. Then A is a Kirchberg algebra satisfying the UCT, and α has the Rokhlin property by part (4) of Theorem 2.5 in [13]. Since $K_j(\rho_n)$ is identified with $h_j^{(n)} \colon G_j^{(n)} \to G_j^{(n+1)}$, it follows that ρ_n induces an embedding $\mathcal{E}_j^{(n)} \hookrightarrow \mathcal{E}_j^{(n+1)}$. Since the direct limit of $\big(\mathcal{E}_j^{(n)}, \rho_n\big)_{n\in\mathbb{N}}$ is isomorphic to $\mathcal{E}_j$, we conclude that $(\mathrm{Ext}_*(\alpha), [1_{A^\alpha}]) = (\mathcal{E}_0, \mathcal{E}_1, k_0)$, as desired. □

The following consequence of Theorem 4.6 and Theorem 4.8 will be needed later.

Corollary 4.9. Up to conjugacy there exists a unique circle action on $\mathcal{O}_2$ with the Rokhlin property.

We turn to Theorems F of the introduction, relating Rokhlin actions on arbitrary C^*-algebras with Rokhlin actions on Kirchberg algebras. We will need to consider a specific automorphism of $\mathcal{O}_2$; its explicit description will be crucial in the proof of Theorem 4.15.

Notation 4.10. For every $n \geq 1$, let $u_n \in M_n \cong \mathcal{B}(\ell^2(\{0, \ldots, n-1\}))$ be the unitary given by $u_n(\delta_j) = \delta_{j+1}$ for $j = 0, \ldots, n-1$, where the subscripts are taken modulo n. (In particular, $u_1 = 1 \in \mathbb{C}$.) Let $\rho_n \colon M_n \to \mathcal{O}_2$ be a unital homomorphism. Fix an isomorphism $\kappa \colon \bigotimes_{n=1}^\infty \mathcal{O}_2 \to \mathcal{O}_2$, and let $\Psi \in \mathrm{Aut}(\mathcal{O}_2)$ be the automorphism satisfying

[3]One can take Φ to be α_t for any $t \neq 0$ in Theorem 1.1 of [3]. Indeed, the comments immediately after the statement show that all α_t are approximately representable, and the fact that α_t, for $t \neq 0$, is aperiodic, is a well-known consequence of the fact that the flow α has the Rokhlin property.

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$\Psi \circ \kappa = \kappa \circ \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\rho_n(u_n))$. One can check with elementary methods that Ψ is aperiodic.

There exist by now a few equivariant $\mathcal{O}_2$-embedding results; see, for example, Proposition 5.3 in [43] or Theorem 4.8 in [8]. We need a more precise version of this embedding result in order to use it in the proof of Theorem 4.15, so we prove it next.

Proposition 4.11. Let B be a unital, separable, exact C^*-algebra, and let $\varphi \in \mathrm{Aut}(B)$ be an approximately inner automorphism. Then there exist a unitary $v \in \mathcal{O}_2$ and a unital, equivariant embedding $(B, \varphi) \to (\mathcal{O}_2, \mathrm{Ad}(v) \circ \Psi)$.

Moreover, the unitary v can be chosen so that there is a unital, equivariant embedding $(\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2}) \to (\mathcal{O}_2, \mathrm{Ad}(v) \circ \Psi)$.

Proof. Fix a unital embedding $\theta\colon B \to \mathcal{O}_2$. Denote by $\pi\colon \ell^\infty(\mathcal{O}_2) \to (\mathcal{O}_2)_\infty$ the canonical quotient map. Let $(u_n)_{n\in\mathbb{N}}$ be any sequence of unitaries in B satisfying $\varphi(b) = \lim\limits_{n\to\infty} \mathrm{Ad}(u_n)(b)$ for all $b \in B$, and set $z = \pi\big((\theta(u_n))_{n\in\mathbb{N}}\big) \in (\mathcal{O}_2)_\infty$. Then the unitary z satisfies $\varphi(b) = z\theta(b)z^*$ for all $b \in B$. By the universal property of the crossed product, there exists a unital embedding $\Theta\colon B \rtimes_\varphi \mathbb{Z} \to (\mathcal{O}_2)_\infty$.

We will show that Θ lifts to a unital completely positive map $B \rtimes_\varphi \mathbb{Z} \to \ell^\infty(\mathcal{O}_2)$ which agrees with θ on the canonical copy of B. Denote by $u \in B \rtimes_\varphi \mathbb{Z}$ the canonical unitary implementing φ For $n \in \mathbb{N}$, let $\phi_n\colon B \rtimes_\varphi \mathbb{Z} \to M_n(B)$ and $\psi_n\colon M_n(B) \to B \rtimes_\varphi \mathbb{Z}$ be the unital completely positive maps given as follows

$$\phi_n(bu^j) = \begin{cases} \sum\limits_{k=j}^{n-1} \varphi^{-k}(b) e_{k,k-j} & \text{if } 0 \leq j < n \\ \sum\limits_{k=0}^{j+n-1} \varphi^{-k}(b) e_{k,k-j} & \text{if } -n < j \leq 0 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi_n(b \otimes e_{j,k}) = \frac{1}{n}\varphi^k(a)u^{k-j}.$$

One readily checks that $\psi_n(\phi_n(bv^j)) = \frac{n-j}{n} bv^j$ if $|j| < n$, and 0 otherwise. In particular, $\psi_n \circ \phi_n$ converges pointwise in norm to $\mathrm{id}_{B\rtimes_\varphi\mathbb{Z}}$, and thus $\Theta \circ \psi_n \circ \phi_n$ converges pontwise in norm to Θ. Since $B \rtimes_\varphi \mathbb{Z}$ is separable, it follows from Theorem 6 in [2] that the set of maps $B \rtimes_\varphi \mathbb{Z} \to (\mathcal{O}_2)_\infty$ which have unital completely positive lifts into $\ell^\infty(\mathcal{O}_2)$ is closed in the point-norm topology. In particular, it suffices to show that for every $n \in \mathbb{N}$, the map $\Theta \circ \psi_n \circ \phi_n$ has a unital completely positive lift which extends θ.

Using nuclearity of M_n, let $\widetilde{\rho}_n\colon M_n \to \ell^\infty(\mathcal{O}_2)$ be any unital compeltely positive lift of the composition

$$M_n \xrightarrow{x \mapsto x \otimes 1_B} M_n(B) \xrightarrow{\psi_n} B \rtimes_\varphi \mathbb{Z} \xrightarrow{\Theta} (\mathcal{O}_2)_\infty.$$

Let $\rho_n\colon M_n(B) \to \ell^\infty(\mathcal{O}_2)$ be the linear map determined by

$$\rho_n(b \otimes e_{j,k}) = \widetilde{\rho}_n(e_{j,j})^{1/2}\theta(b)\widetilde{\rho}_n(e_{k,k})^{1/2}$$

for all $b \in B$ and $j, k = 0, \ldots, n-1$. Then ρ_n is unital. We claim that ρ_n is positive. Since a positive element in $M_n(B)$ is the sum of n elements of the form $\sum_{j,k=0}^{n-1} b_j^* b_k \otimes e_{j,k}$ for some $b_0, \ldots, b_{n-1} \in B$, it suffices to show that ρ_n preserves positivity of such elements. Given $b_0, \ldots, b_{n-1} \in B$, set

$$b = \sum_{j,k=0}^{n-1} b_j^* b_k \otimes e_{j,k} \in M_n(B) \quad \text{and} \quad x = \sum_{j=0}^{n-1} \theta(b_j)\widetilde{\rho}_n(e_{j,j})^{1/2} \in \ell^\infty(\mathcal{O}_2).$$

Then $\rho_n(b) = x^*x \geq 0$, as desired. It follows that ρ_n is positive, and one shows in a similar way that it is completely positive.

Note that $\pi \circ \rho_n = \Theta \circ \psi_n$, and this $\rho_n \circ \phi_n$ is a lift for $\Theta \circ \psi_n \circ \phi_n$ which agrees with θ on the canonical copy of B. Thus there exists a unital completely positive lift for Θ which extends θ.

Using Lemma 2.2 in [20], find a unital embedding $\sigma\colon B \rtimes_\varphi \mathbb{Z} \to \mathcal{O}_2$ satisfying $\sigma(b) = \theta(b)$ for all $b \in B$, and set $w = \sigma(u)$. Consider the following embeddings, where the first one is the restriction of σ to B:

$$(B, \varphi) \hookrightarrow (\mathcal{O}_2, \mathrm{Ad}(w)) \hookrightarrow (\mathcal{O}_2 \otimes \mathcal{O}_2, \mathrm{Ad}(w) \otimes \Psi).$$

Note that $\mathrm{Ad}(w)$ is unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$, and that $\mathrm{id}_{\mathcal{O}_2} \otimes \Psi$ is conjugate to Ψ. It follows that $\mathrm{Ad}(w) \otimes \Psi$ is unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2} \otimes \Psi$ of $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$. Since $\mathrm{id}_{\mathcal{O}_2} \otimes \Psi$ is conjugate to Ψ, and thus there is a unital, equivariant embedding $(B, \varphi) \to (\mathcal{O}_2, \mathrm{Ad}(v) \circ \Psi)$ for some unitary $v \in \mathcal{O}_2$.

Since $(\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2})$ embeds unitally into $(\mathcal{O}_2, \Psi)$, it follows that is also embeds unitally into $(\mathcal{O}_2 \otimes \mathcal{O}_2, \mathrm{Ad}(w) \otimes \Psi)$. Thus the last part of the statement also follows from the construction above. □

As a byproduct of the proposition above, we can easily deduce a $\mathbb{T}$-equivariant version of Kirchberg's $\mathcal{O}_2$-embedding theorem. It should be pointed out that there are other ways of proving this which even apply to all compact groups; see Remark 4.13. We thus do not claim the corollary below to be really new, or as general as it could possibly be.

Corollary 4.12. Let A be a unital, separable, exact C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Denote by $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ the unique action with the Rokhlin property (see Corollary 4.9). Then there exists a unital, equivariant embedding $(A, \alpha) \hookrightarrow (\mathcal{O}_2, \gamma)$.

Proof. Note that A^α is unital, separable and exact, and that the predual automorphism $\check{\alpha} \in \mathrm{Aut}(A^\alpha)$ is approximately inner by Proposition 2.8. Use Proposition 4.11 to find a unitary $v \in \mathcal{O}_2$ and a unital embedding

$$\iota\colon (A^\alpha, \check{\alpha}) \to (\mathcal{O}_2, \mathrm{Ad}(v) \circ \Psi). \tag{4.7}$$

Set $\Phi = \mathrm{Ad}(u) \circ \Psi$. Since Ψ is aperiodic, the same is true for Φ. It follows from parts (3) and (4) of Proposition 4.2 that $\mathcal{O}_2 \rtimes_\Phi \mathbb{Z}$ is a Kirchberg algebra satisfying the UCT, and it has trivial K-theory by the Pimsner-Voiculescu exact sequence. Thus $\mathcal{O}_2 \rtimes_\Phi \mathbb{Z}$ is isomorphic to $\mathcal{O}_2$. Moreover, since Ψ is approximately representable by construction, the same is true for Φ. By Proposition 2.8, it follows that the dual action $\widehat{\Phi}$ has the Rokhlin property, and thus $\widehat{\Phi}$ is conjugate to γ by Corollary 4.9.

Applying crossed products to (4.7), and identifying $(A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z}, \widehat{\check{\alpha}})$ with (A, α) (see Theorem 2.3), and identifying $(\mathcal{O}_2 \rtimes_\Phi \mathbb{Z}, \widehat{\Phi})$ with $(\mathcal{O}_2, \gamma)$ as in the paragraph above, we obtain a unital homomorphism $\widetilde{\iota}\colon (A, \alpha) \to (\mathcal{O}_2, \gamma)$. Finally, $\widetilde{\iota}$ is injective because so is ι and $\mathbb{Z}$ is amenable. □

Remark 4.13. We point out that it is possible to give a simpler proof of Corollary 4.12 which avoids Proposition 4.11, by applying Lemma 4.7 in [1] to any embedding $A \hookrightarrow \mathcal{O}_2$. This argument even has the advantage of working for arbitrary compact groups. We have taken a more indirect approach because Corollary 4.12 by itself is not enough to prove Theorem 4.15, and in its proof we will need to use Proposition 4.11 instead.

Recall that a homomorphism $\pi\colon A \to B$ is said to be *full* if for every $a \in A$ with $a \neq 0$, the ideal in B generated by $\pi(a)$ is all of B. If $\varphi \in \mathrm{Aut}(A)$ and $\psi \in \mathrm{Aut}(B)$ satisfy $\psi \circ \pi = \pi \circ \varphi$, then the $\mathbb{T}$-equivariant homomorphism $\widehat{\pi}\colon A \rtimes_\varphi \mathbb{Z} \to B \rtimes_\psi \mathbb{Z}$ induced by π is full if and only if so is π.

The following easy observation will be used in the proof of Theorem 4.15.

Remark 4.14. If $(A_n, \pi_n)_{n\in\mathbb{N}}$ is an inductive system and π_n is full for every $n \in \mathbb{N}$, then $\varinjlim(A_n, \pi_n)$ is simple.

We are ready to show that every Rokhlin action on a separable, nuclear C^*-algebra is $KK^{\mathbb{T}}$-equivalent to a Rokhlin action on a Kirchberg algebra. Our proof roughly follows

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arguments of Kirchberg, specifically as in Chapter 11 of [19]. However, we need to make the choices carefully and perform some computations in equivariant $KK^{\mathbb{T}}$-theory using the *continuous* Rokhlin property from [12], in order to guarantee that the resulting action on the Kirchberg algebra has the Rokhlin property. (We do not seem to be able to guarantee this using the Cuntz-Pimsner construction from Theorem 2.1 in [27].)

Theorem 4.15. Let A be a separable, nuclear unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there exist a unital Kirchberg algebra D and a circle action $\delta\colon \mathbb{T} \to \mathrm{Aut}(D)$ with the Rokhlin property, such that $(A, \alpha) \sim_{KK^{\mathbb{T}}} (D, \delta)$ unitally.

Proof. Let $v \in \mathcal{O}_2$ be a unitary as in the conclusion of Corollary 4.12, and set $\Phi = \mathrm{Ad}(v) \circ \Psi$. Fix unital embeddings

$$\tau\colon (\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2}) \hookrightarrow (\mathcal{O}_2, \Phi) \quad \text{and} \quad \rho\colon (A^\alpha, \check{\alpha}) \to (\mathcal{O}_2, \Phi).$$

We prove the theorem in two steps. First, assume that there is a unital equivariant embedding $\sigma\colon (\mathcal{O}_2, \Phi) \to (A^\alpha, \check{\alpha})$. Let $s_1, s_2 \in \mathcal{O}_2$ be the canonical generating isometries, and note that $\tau(s_1)$ and $\tau(s_2)$ are Φ-invariant. Set $t_j = \sigma(\tau(s_j))$ for $j = 1, 2$. Then t_j is an isometry, $\check{\alpha}(t_j) = t_j$, and $t_1 t_1^* + t_2 t_2^* = 1$. Define a unital equivariant map $\pi\colon (A^\alpha, \check{\alpha}) \to (A^\alpha, \check{\alpha})$ by

$$\pi(a) = t_1 \sigma(\rho(a)) t_1^* + t_2 a t_2^*$$

for all $a \in A^\alpha$. Then π is full. Indeed, if $a \in A^\alpha \setminus \{0\}$, then $\rho(a) \neq 0$ and thus pure infiniteness of $\mathcal{O}_2$ implies that there exist $x, y \in \mathcal{O}_2$ with $x\rho(a)y = 1_{\mathcal{O}_2}$. Then

$$\sigma(x) t_1^* \pi(a) t_1 \sigma(y) = \sigma(x\rho(a)y) = 1,$$

and thus $\pi(a)$ generates A^α as an ideal, as desired.

Claim 1: *We have* $KK^{\mathbb{Z}}(\pi) = KK^{\mathbb{Z}}(\mathrm{id}_{A^\alpha})$. By Theorem F in the introduction of [12] (see also Corollary 3.10 there), the action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ has the continuous Rokhlin property, and thus it is unitally $KK^{\mathbb{T}}$-equivalent to $\big(C(\mathbb{T}) \otimes \mathcal{O}_2, \mathtt{Lt} \otimes \mathrm{id}_{\mathcal{O}_2}\big)$ by Theorem C of [12]. By Baaj-Skandalis duality, and since the dual of γ is $KK^{\mathbb{Z}}$-equivalent to Φ, we deduce that $(\mathcal{O}_2, \Phi)$ is $KK^{\mathbb{Z}}$-equivalent to $(\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2})$. Thus

$$KK^{\mathbb{Z}}\big((A^\alpha, \check{\alpha}), (\mathcal{O}_2, \Phi)\big) = 0.$$

Since $\sigma \circ \rho$ factors through $(\mathcal{O}_2, \Phi)$, it must be $KK^{\mathbb{Z}}$-trivial. We conclude that $KK^{\mathbb{Z}}(\pi) = KK^{\mathbb{Z}}(\mathrm{id}_{A^\alpha})$.

We deduce from the claim above that π is a $KK^{\mathbb{Z}}$-equivalence, and that $\widehat{\pi}$ is a $KK^{\mathbb{T}}$-equivalence. Set $(C_0, \theta_0) = \varinjlim \big((A^\alpha, \check{\alpha}), \pi\big)$. Then C_0 is simple because π is full (see Remark 4.14), and it is nuclear, unital and separable because so is A^α. Denote by $\Pi\colon (A^\alpha, \check{\alpha}) \to (C_0, \theta_0)$ the canonical equivariant map into the limit.

Claim 2: Π *is a* $KK^{\mathbb{Z}}$*-equivalence.* (This is not immediate from Claim 1, since $KK^{\mathbb{Z}}$ is not a continuous functor.) By the second paragraph on page 287 of [29], it suffices to show that the class $\mathcal{F}(\Pi) \in KK(A^\alpha, C_0)$ that Π induces under the forgetful functor $\mathcal{F}$, is a KK-equivalence. This is contained in Kirchberg's proof (specifically Sections 11.2 and 11.3 in [19]) there), so the claim follows.

Set $(C, \theta) = (C \otimes \mathcal{O}_\infty, \theta_0 \otimes \mathrm{id}_{\mathcal{O}_\infty})$. Then C is a unital Kirchberg algebra, and $(C, \theta) \sim_{KK^{\mathbb{Z}}} (A^\alpha, \check{\alpha})$ unitally, since $(\mathcal{O}_\infty, \mathrm{id}_{\mathcal{O}_\infty}) \sim_{KK^{\mathbb{Z}}} (\mathbb{C}, \mathrm{id}_{\mathbb{C}})$ unitally.

Claim 3: θ *is aperiodic and approximately representable.* It suffices to prove the claim for θ_0. Since (C_0, θ_0) is the direct limit of $(A^\alpha, \check{\alpha})$ and $\check{\alpha}$ is approximately representable, it follows that so is θ_0. Note that there is an isomorphism $C_0 = \varinjlim(A, \widehat{\pi})$. Let $\widehat{\pi}\colon (A, \alpha) \to (A, \alpha)$ be the map induced by π. Then $\widehat{\pi}$ is full, because so is π. It therefore follows from Remark 4.14 that C_0 is simple. By part (1) of Proposition 4.2, we deduce that θ_0 is aperiodic.

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Set $D = C \rtimes_\theta \mathbb{Z}$ and $\delta = \widehat{\theta}$. Since C is a Kirchberg algebra and θ is aperiodic, it follows from part (3) of Proposition 4.2 that D is a unital Kirchberg algebra. Moreover, δ has the Rokhlin property by Proposition 2.8, since θ is approximately representable by Claim 3. Finally, by Claim 2, the equivariant map $\Pi \otimes \mathrm{id}_{\mathcal{O}_\infty}$ is a unital $KK^{\mathbb{Z}}$-equivalente $(A^\alpha, \breve{\alpha}) \sim_{KK^{\mathbb{Z}}} (C, \theta)$. By taking crossed products, we obtain a unital $KK^{\mathbb{T}}$-equivalence $(A, \alpha) \sim_{KK^{\mathbb{T}}} (D, \delta)$. This proves the theorem in the case that there is a unital equivariant embedding $\sigma\colon (\mathcal{O}_2, \Phi) \to (A^\alpha, \breve{\alpha})$.

We turn to the general case. Let $\mathcal{O}_\infty^{(0)}$ be the unique unital UCT Kirchberg algebra which is KK-equivalent to $\mathcal{O}_\infty$ and whose class of the unit is zero.

Claim 4: *there exists an automorphism $\psi \in \mathrm{Aut}(\mathcal{O}_\infty)$ which leaves a copy of $\mathcal{O}_\infty^{(0)}$ invariant, and such that with $\varphi \in \mathrm{Aut}(\mathcal{O}_\infty^{(0)})$ denoting its restriction, φ is aperiodic and admits and a unital equivariant embedding $\iota\colon (\mathcal{O}_2, \Phi) \hookrightarrow (\mathcal{O}_\infty^{(0)}, \varphi)$.* Fix an injective homomorphism $\nu\colon \mathcal{O}_2 \to \mathcal{O}_\infty$, and set $p = 1_{\mathcal{O}_\infty} - \nu(1)$, which is a projection in $\mathcal{O}_\infty$. Adopt the notation of Notation 4.10, in particular we will use the unital homomorphisms $\rho_n\colon M_n \to \mathcal{O}_2$ and the permutation unitaries $u_n \in M_n$. For $n \in \mathbb{N}$, set $w_n = \nu(\rho_n(u_n)) + p$, which is a unitary in $\mathcal{O}_\infty$. Consider the automorphism $\widetilde{\psi} = \bigotimes_{n=1}^\infty \mathrm{Ad}(w_n)$ of $\bigotimes_{n=1}^\infty \mathcal{O}_\infty$, and note that the restriction of $\widetilde{\psi}$ to the invariant subalgebra $\bigotimes_{n=0}^\infty \nu(\mathcal{O}_2)$ can be identified with $\bigotimes_{n=1}^\infty \mathrm{Ad}(\rho_n(u_n))$. Fix an isomorphism $\widetilde{\kappa}\colon \bigotimes_{n=1}^\infty \mathcal{O}_\infty \to \mathcal{O}_\infty$, and let $\mu\colon \mathcal{O}_2 \to \mathcal{O}_\infty$ be the following composition

$$\mathcal{O}_2 \xrightarrow[\cong]{\kappa^{-1}} \bigotimes_{n=1}^\infty \mathcal{O}_2 \xrightarrow{\bigotimes_{n=1}^\infty \nu} \bigotimes_{n=1}^\infty \mathcal{O}_\infty \xrightarrow[\cong]{\widetilde{\kappa}} \mathcal{O}_\infty.$$

Set $w = \mu(v)$ and $\psi = \widetilde{\kappa} \circ \mathrm{Ad}(w) \circ \widetilde{\psi} \circ \widetilde{\kappa}^{-1} \in \mathrm{Aut}(\mathcal{O}_\infty)$. Set $e = \mu(1)$, which is a projection in $\mathcal{O}_\infty$ satisfying $\psi(e) = e$ and $[e] = 0$ in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. In particular, there is an injective homomorphism $j\colon \mathcal{O}_\infty^{(0)} \to \mathcal{O}_\infty$ whose image is $e\mathcal{O}_\infty e$. Note that ψ leaves the image of j invariant. Then $\mu(v)$ is a unitary in $e\mathcal{O}_\infty e$, and we let $\varphi \in \mathrm{Aut}(\mathcal{O}_\infty^{(0)})$ be given by $\varphi = j^{-1} \circ \psi \circ j$. Let $\iota\colon \mathcal{O}_2 \to \mathcal{O}_\infty^{(0)}$ be the unital homomorphism given by $\iota = j^{-1} \circ \mu$. One readily checks that φ and ι satisfy the conditions in the claim.

Claim 5: *any automorphism $\varphi \in \mathrm{Aut}(\mathcal{O}_\infty^{(0)})$ satisfying the conclusion of Claim 4 satisfies $(\mathcal{O}_\infty^{(0)}, \varphi) \sim_{KK^{\mathbb{Z}}} (\mathbb{C}, \mathrm{id}_{\mathbb{C}})$.* Note that there are unital, equivariant homomorphisms $(\mathcal{O}_\infty^{(0)}, \varphi) \to (\mathcal{O}_\infty, \psi)$ and $(\mathbb{C}, \mathrm{id}_{\mathbb{C}}) \to (\mathcal{O}_\infty, \psi)$. Both of these are KK-equivalences, and thus they are $KK^{\mathbb{Z}}$-equivalences by the second paragraph on page 287 of [29].

Let $\varphi \in \mathrm{Aut}(\mathcal{O}_\infty^{(0)})$ be as in the conclusion of Claim 4. Then clearly there is a unital equivariant embedding of $(\mathcal{O}_2, \Phi)$ into $(A^\alpha \otimes \mathcal{O}_\infty^{(0)}, \breve{\alpha} \otimes \varphi)$. Using the first part of this proof, let $\delta_0\colon \mathbb{T} \to \mathrm{Aut}(D_0)$ be an action with the Rokhlin property on a unital Kirchberg algebra D_0 such that (D_0, δ_0) is $KK^{\mathbb{T}}$-equivalent to the dual system of $(A^\alpha \otimes \mathcal{O}_\infty^{(0)}, \breve{\alpha} \otimes \varphi)$. Said dual system is $KK^{\mathbb{T}}$-equivalent to (A, α) by Claim 5, and thus there is a $KK^{\mathbb{T}}$-equivalence $\eta \in KK^{\mathbb{T}}\big((A, \alpha), (D_0, \delta_0)\big)$. Since δ_0 has the Rokhlin property, by Corollary 2.5 there is a canonical isomorphism $K_0(D_0 \rtimes_{\delta_0} \mathbb{T}) \cong K_0(D_0^{\delta_0})$. Combining this with the natural group isomorphism $K_0^{\mathbb{T}}(D_0, \delta_0) \cong K_0(D_0 \rtimes_{\delta_0} \mathbb{T})$ given by Julg's theorem, it follows that there is a natural identification $K_0^{\mathbb{T}}(D_0, \delta_0) \cong K_0(D_0^{\delta_0})$. Since $D_0^{\delta_0}$ is a Kirchberg algebra by part (3) of Proposition 4.2, any element of its K_0-group can be realized by a projection in $D_0^{\delta_0}$. It follows that there exists a projection $q \in D_0^{\delta_0}$ such that $[1_A] \times \eta = [q]$ in $K_0^{\mathbb{T}}(D_0, \delta_0)$. Set $D = qD_0q$ and let $\delta\colon \mathbb{T} \to \mathrm{Aut}(D)$ denote the restriction of δ_0 to D. Then D is a unital Kirchberg algebra, and δ has the Rokhlin property by part (3) of Theorem 2.5 in [13]. Finally, it is clear that η implements a unital $KK^{\mathbb{T}}$-equivalence $(A, \alpha) \sim_{KK^{\mathbb{T}}} (D, \delta)$, as desired. □

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