A CLASS OF COMPACT RIGID 0-DIMENSIONAL SPACES

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A topological space is called "rigid" if its autohomeomorphism group is trivial. In (**1**), de Groot and McDowell showed that there are rigid, 0-dimensional spaces of arbitrarily high cardinality but left open the question of whether or not there are *compact*, rigid, 0-dimensional spaces of arbitrarily high cardinality, pointing out that an affirmative answer implies the existence of arbitrarily large Boolean rings with trivial automorphism groups. In this paper we construct a class of rigid, 0-dimensional spaces X^α of arbitrary infinite cardinality and show that their Stone-Čech compactifications βX^α are also rigid, thus answering the above question affirmatively.

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For every ordinal number β, let $X_\beta = \{\beta\} \times [0, \omega^\beta[$. For the least ordinal number α of any given infinite cardinality, let $\mathscr{A}_\alpha$ be the set of all non-limit ordinal numbers less than α and let $X^\alpha = \bigcup\{X_\beta \colon \beta \in \mathscr{A}_\alpha\}$. We claim that there is an injection $\varphi\colon X^\alpha \to \mathscr{A}_\alpha$ such that $\varphi((\beta, \gamma)) > \beta$ for all $(\beta, \gamma) \in X^\alpha$. To see this, observe that since card $\mathscr{A}_\alpha$ = card α = card $\alpha \cdot$ card α, $\mathscr{A}_\alpha = \bigcup\{\mathscr{B}_\beta \colon \beta < \alpha\}$, where the $\mathscr{B}_\beta$ are disjoint sets of cardinality card α. Furthermore, since α is the first ordinal number of cardinality card α, we have card$(\mathscr{B}_\beta - [0, \beta])$ = card α for all $\beta < \alpha$. However, card$(\omega^\beta) \leqq$ card $\omega \leqq$ card α for $\beta < \omega$ and card (ω^β) = card $\beta <$ card α for $\omega \leqq \beta < \alpha$. Hence, the desired φ can be obtained by letting $\varphi|X_\beta$, $\beta < \alpha$, be any injection into $\mathscr{B}_\beta - [0, \beta]$. Note that since card $X^\alpha \geqq$ card $\mathscr{A}_\alpha$ = card α, the existence of such a φ shows that card X^α = card α. Now, given such a φ, we partially order X^α by requiring $x \leqq y$ if and only if there is a finite sequence $x_1, x_2, \ldots, x_n \in X^\alpha$ such that

(1) $x_1 = x$,

(2) x_n and y belong to the same X_β with $x_n \leqq y$ in the natural order on X_β,

(3) $1 < k \leqq n \Rightarrow x_{k-1} \in X_{\varphi(x_k)}$.

As an immediate consequence of the fact that the sequence $x_1, x_2, \ldots, x_n$ is uniquely determined except for length by x, we note for future reference that any two elements of X^α with a common predecessor must be comparable.

In all that follows we write X for X^α whenever it seems convenient to do so.

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Let $\mathscr{I}_X$ denote the set of all $Y \subseteq X$ such that no distinct $x, y \in Y$ are comparable. For $x \in X$ and $Y \in \mathscr{I}_X$, let

$$\langle x, Y\rangle = \{z \in X: z \leqq x \wedge (y \in Y \Rightarrow z \nleqq y)\}.$$

Then $\mathscr{B}_X = \{\langle x, Y\rangle: x \in X, Y \in \mathscr{I}_X\}$ is a base for a topology on X. For, given any $\langle x_1, Y_1\rangle, \langle x_2, Y_2\rangle \in \mathscr{B}_X$ and $z \in \langle x_1, Y_1\rangle \cap \langle x_2, Y_2\rangle$, let

$$Y_3 = \{x \in Y_1: y \in Y_2 \Rightarrow x \nleqq y\}$$

and $Y_4 = \{x \in Y_2: y \in Y_3 \Rightarrow x \nleqq y\}$. Then one readily verifies that

$$Y_3 \cup Y_4 \in \mathscr{I}_X$$

and $z \in \langle z, Y_3 \cup Y_4\rangle \subseteq \langle x_1, Y_1\rangle \cap \langle x_2, Y_2\rangle$. Moreover, with the topology generated by $\mathscr{B}_X$, X is T_1. For, given distinct $x, y \in X$, either $x \nleqq y$ in which case $x \notin \langle y, \emptyset\rangle \in \mathscr{N}(y)$, or $y \nleqq x$ in which case $x \notin \langle y, \{x\}\rangle \in \mathscr{N}(y)$. Furthermore, any two disjoint closed subsets F_1 and F_2 of X are separated by a partition (i.e., X is the disjoint union of two open-closed sets E_1 and E_2 with $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$). To see this, choose for each $x \in F_1$ a $\langle x, Y_x\rangle \in \mathscr{N}(x)$ such that $\langle x, Y_x\rangle \cap F_2 = \emptyset$. Then to show that $E_1 = \bigcup\{\langle x, Y_x\rangle: x \in F_1\}$, $E_2 = X - E_1$ is the desired partition, it clearly suffices to show that E_1 is closed. However, for any $x' \notin E_1$ there is a $\langle x', Y'\rangle \in \mathscr{N}(x')$ such that $\langle x', Y'\rangle \cap F_1 = \emptyset$. Now if $z \in \langle x', Y'\rangle \cap E_1$, then $z \in \langle x', Y'\rangle \cap \langle x, Y_x\rangle$ for some $x \in F_1$. Therefore, since x and x' have the common predecessor z, they must be comparable. However, if $x < x'$, then, since $x \notin \langle x', Y'\rangle$, there is a $y \in Y'$ such that $x \leqq y$, and hence $z \leqq y$, which contradicts $z \in \langle x', Y'\rangle$. Similarly, $x' < x$ leads to a contradiction. Hence, $\langle x', Y'\rangle \cap E_1 = \emptyset$ so that E_1 is closed, as asserted. Now it follows immediately from the above observations that X is completely regular and (**2**, Theorem 16.17) 0-dimensional in the sense of (**2**). As a consequence (**2**, Theorem 16.11), $\dim \beta X = 0$.

LEMMA 1. *Every well-ordered set A has a cofinal subset B such that*
(1) *no $C \subseteq B$ with* card $C <$ card B *is cofinal,*
(2) *if $b \in B$, then* $\text{card}\{c \in B: c < b\} < \text{card } B$.

Proof. Let B' be a cofinal subset of A of least cardinality. Let β be the first ordinal number with card β = card B' and let $f: [0, \beta[\rightarrow B'$ be any bijection. We define $g: [0, \beta[\rightarrow B'$ by induction. Suppose that $\gamma < \beta$ and $g(\delta)$ has been defined for all $\delta < \gamma$. If γ is a limit ordinal number, let

$$g(\gamma) = \sup_{B'}\{g(\delta): \delta < \gamma\},$$

which exists since $\text{card}\{g(\delta): \delta < \gamma\} \leqq \text{card } \gamma < \text{card } \beta = \text{card } B'$ and B' contains no cofinal subset of lower cardinality. If $\gamma = \delta + 1$, let $g(\gamma) = f(\delta)$ if $f(\delta) > g(\delta)$ and let $g(\gamma) = \inf\{b \in B': g(\delta) < b\}$ if $f(\delta) \leqq g(\delta)$. Then g is an isomorphism and $g([0, \beta[)$ is cofinal in B' so that $B = g([0, \beta[)$ works.

Notation. We write "f: $(A, a) \sim (B, b)$" for "f is a homeomorphism of A onto B with $f(a) = b$".

Definition. We say that $Y \subseteq X$ "borders" $x \in X$ provided
(1) $x \in Y$,
(2) $y \in Y \wedge y < z < x \Rightarrow z \in Y$, and
(3) $z < x \Rightarrow z \leqq y < x$ for some $y \in Y$.

LEMMA 2. *Suppose that f: $([0, \beta], \beta) \sim (T, x)$, where β is a non-zero ordinal number and $T \subseteq X$. Suppose that $Y \subseteq X$ borders x. Then there is a $\beta' < \beta$ such that $f(]\beta', \beta]) \subseteq Y$.*

Proof. Suppose the contrary. Then, since $f(\beta) = x \in Y$, $[0, \beta[$ must contain a cofinal subset B such that $f(B) \cap Y = \emptyset$. Since $\langle x, \emptyset\rangle \in \mathscr{N}(x)$ implies that $f(]\gamma, \beta]) \subseteq \langle x, \emptyset\rangle$ for some $\gamma < \beta$, we can assume that $f(B) \subseteq \langle x, \emptyset\rangle$. Moreover, by Lemma 1, we can assume that B has properties (1) and (2) of Lemma 1. Now suppose that there is a $\beta_0 \in B$ such that for every $\gamma \in B$ there is a $g(\gamma) \in B \cap [0, \beta_0[$ such that $f(\gamma)$ and $f(g(\gamma))$ are comparable. Then for some $\delta_0 < \beta_0$, $g^{-1}(\delta_0)$ is cofinal in B; for otherwise, $\sup_B g^{-1}(\delta)$ would exist for all $\delta \in B \cap [0, \beta_0[$, in which case $C = \{\sup_B g^{-1}(\delta)\colon \delta \in B \cap [0, \beta_0[\}$ would be a cofinal subset of B with card $C <$ card B. Now, since Y borders x and $f(\delta_0) < x$, there is a $y \in Y$ such that $f(\delta_0) \leqq y < x$. Consider any $\gamma \in g^{-1}(\delta_0)$. Then $f(\delta_0)$ and $f(\gamma)$ are comparable. If $f(\gamma) \leqq f(\delta_0)$, then $f(\gamma) \leqq y$. If $f(\delta_0) < f(\gamma)$, then y and $f(\gamma)$ must be comparable since they have the common predecessor $f(\delta_0)$. However, if $y < f(\gamma)$, then $y < f(\gamma) < x$ so that $f(\gamma) \in Y$, which contradicts $f(B) \cap Y = \emptyset$. Hence, $f(\gamma) \leqq y$ for all $\gamma \in g^{-1}(\delta_0)$. Then

$$f(g^{-1}(\delta_0)) \cap \langle x, \{y\}\rangle = \emptyset$$

which is impossible since $\langle x, \{y\}\rangle \in \mathscr{N}(x)$ and $g^{-1}(\delta_0)$ is cofinal in $[0, \beta[$. Therefore, no such β_0 exists. Hence, B contains a cofinal subset C with $f(C) \in \mathscr{I}_X$. Since $x \notin f(B) \subseteq \langle x, \emptyset\rangle$, it follows that $\langle x, f(C)\rangle \in \mathscr{N}(x)$ and $f^{-1}(\langle x, f(C)\rangle)$ does not meet the cofinal subset C of $[0, \beta[$, which is impossible.

Now for any topological space Y, let β be the first ordinal number with card $Y <$ card β, and, for any $\gamma \in [0, \beta]$, let Y^γ be the subspace of Y defined inductively as follows: let $Y^0 = Y$; if γ is a non-zero limit ordinal number, let $Y^\gamma = \bigcap\{Y^\delta\colon \delta < \gamma\}$; if $\gamma = \delta + 1$, let Y^γ be the set of all non-isolated points of the space Y^δ. Define h_Y: $Y \to [0, \beta]$ by setting

$$h_Y(x) = \sup\{\gamma \leqq \beta\colon x \in Y^\gamma\},$$

and define k_Y: $Y \to [0, \beta]$ by setting

$$k_Y(x) = \inf\{\gamma \leqq \beta\colon T \subseteq Y \wedge \delta > \gamma \Rightarrow ([0, \omega^\delta], \omega^\delta) \nsim (T, x)\}.$$

Clearly, both $h_Y(x)$ and $k_Y(x)$ are invariant under homeomorphism. Moreover, if $U \in \mathscr{N}(x)$, then $k_U(x) = k_Y(x)$, since $([0, \omega^\delta], \omega^\delta) \sim (]\delta', \omega^\delta], \omega^\delta)$ for all $\delta' < \omega^\delta$, $\delta > 0$. Hence, $k_Y(x) \neq k_Y(y)$ implies that $(U, x) \nsim (V, y)$ for any $U \in \mathscr{N}(x)$, $V \in \mathscr{N}(y)$.

THEOREM 1. *Suppose that α is the first ordinal number of any given infinite cardinality and $X = X^\alpha$ is the 0-dimensional space of cardinality* card α *described*

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above. Then for any $x \in X$ we have $k_X(x) = \varphi(x)$. Consequently, for any distinct $x, y \in X$, $U \in \mathcal{N}(x) \wedge V \in \mathcal{N}(y) \Rightarrow (U, x) \nsim (V, y)$. In particular, X is rigid.

Proof. Suppose that f: $([0, \omega^\delta], \omega^\delta) \sim (T, x)$ for some $T \subseteq X$. If $x \in X_\gamma$, let $A = [(\gamma, 0), x]$ and $B = [(\varphi(x), 0), x]$. Then $Y = A \cup B$ borders x. Hence, by Lemma 2 there is a $\beta' < \omega^\delta$ such that $f(]\beta', \omega^\delta]) \subseteq Y$. Since we merely wish to show that $\delta \leqq \varphi(x)$, we can assume that $\delta \neq 0$. Then

$$([0, \omega^\delta], \omega^\delta) \sim (]\beta', \omega^\delta], \omega^\delta);$$

thus we can assume that $T \subseteq Y$. Now since $\gamma < \varphi(x)$, we have $h_Y(x) = \varphi(x)$. Moreover, since $T \subseteq Y$, it follows that $h_T(x) \leqq h_Y(x)$. Therefore,

$$\delta = h_{[0,\omega^\delta]}(\omega^\delta) = h_T(x) \leqq \varphi(x).$$

Hence, $k_X(x) \leqq \varphi(x)$. But clearly $([0, \omega^{\varphi(x)}], \omega^{\varphi(x)}) \sim (X_{\varphi(x)} \cup \{x\}, x)$. Hence, $k_X(x) = \varphi(x)$.

LEMMA 3. *Suppose that Y is a completely regular space such that $Y - \{x\}$ is not C^*-embedded in Y for any $x \in Y$. Then Y and βY have isomorphic autohomeomorphism groups.*

Proof. Clearly, it suffices to show that any autohomeomorphism f of βY carries Y onto Y. Thus, suppose that $Y' = f(Y) \neq Y$. Then we can assume that there is an $x \in Y - Y'$ (otherwise, replace Y and f by Y' and f^{-1}, respectively). Now (**2**, problem 9N.1), $Y - \{x\}$ is C^*-embedded in $\beta Y - \{x\}$. However, $Y' \subseteq \beta Y - \{x\}$ and $\beta Y' = \beta Y$ so that $\beta Y - \{x\}$ is C^*-embedded in βY. Therefore, $Y - \{x\}$ is C^*-embedded in βY, and hence in Y, contrary to hypothesis.

Remark. The essence of the above proof is given in (**2**) as a hint for (**2**, problem 9N.3).

THEOREM 2. *Suppose that α is the first ordinal number of any given infinite cardinality and $X = X^\alpha$ is the space described above. Then $\beta X = \beta X^\alpha$ is a compact, rigid, 0-dimensional space of cardinality $2^{2^{\text{card}\,\alpha}}$.*

Proof. The rigidity of βX will follow from Theorem 1 and Lemma 3 provided we show that $X - \{x\}$ is not C^*-embedded in X for any $x \in X$. Thus, consider any $x \in X$. Then $\varphi(x) = \beta + 1$ for some ordinal number β, and hence $\omega^{\varphi(x)} = \omega^\beta \cdot \omega$. Now define f: $X - \{x\} \rightarrow \mathbf{R}$ by setting $f(y) = (-1)^n$, where n is the least integer such that $y \leqq (\varphi(x), \omega^\beta \cdot n)$, if such an integer exists, and $f(y) = 0$ otherwise. Then $f \in C^*(X - \{x\})$ but f cannot be extended to x.

To verify the cardinality of βX^α we require a $D \in \mathcal{I}_X$ such that card $D =$ card α. Clearly, we may take $D = \{(\varphi((1, n)), 0)$: $n < \omega\}$ if $\alpha = \omega$. If $\alpha > \omega$, then we assert that $\mathcal{D} = \mathcal{A}_\alpha - \varphi(X^\alpha)$ has card $\mathcal{D} =$ card α. Suppose the contrary; let β be the first infinite ordinal number with card $\beta \geqq$ card $\mathcal{D}$ and let γ be the first ordinal number with card $\gamma >$ card β. Then $\gamma \leqq \alpha$ so that

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card$([0, \gamma[\, \cap \mathscr{A}_\alpha) = $ card γ. However, $\mathscr{D}_0 = [0, \gamma[\, \cap \mathscr{D}$ has card $\mathscr{D}_0 \leqq$ card β so that, defining $\mathscr{D}_n$ inductively by setting $\mathscr{D}_n = [0, \gamma[\, \cap \varphi(\bigcup\{X_\delta \colon \delta \in \mathscr{D}_{n-1}\})$, we have card $\mathscr{D}_n \leqq$ card $\mathscr{D}_{n-1} \cdot$ card $\beta \leqq$ card β inductively for $n < \omega$. Therefore, since $[0, \gamma[\, \cap \mathscr{A}_\alpha = \bigcup\{\mathscr{D}_n \colon n < \omega\}$, we have

$$\text{card } \gamma = \text{card}([0, \gamma[\, \cap \mathscr{A}_\alpha) = \sum\{\text{card } \mathscr{D}_n \colon n < \omega\} \leqq \text{card } \omega \cdot \text{card } \beta = \text{card } \beta,$$

which is a contradiction. Hence, we may take $D = \{(\beta, 0) \colon \beta \in \mathscr{D}\}$ if $\alpha > \omega$. Now since $D \in \mathscr{I}_X$ and any two elements of X with a common predecessor must be comparable, we have $\langle x, \emptyset \rangle \cap \langle y, \emptyset \rangle = \emptyset$ for distinct $x, y \in D$. Then $\langle x, \emptyset \rangle \cap D = \{x\}$ for any $x \in D$ so that D is discrete. Moreover, for any $x \notin D$, either $\langle x, D \rangle$ or $\langle x, \emptyset \rangle$ must be a neighbourhood of x disjoint from D, so that D is closed. Now if we inspect the argument that $F_1 = D$ and $F_2 = \emptyset$ are separated by a partition, we see that $\bigcup\{\langle x, \emptyset \rangle \colon x \in D\}$ is closed in X. Thus, any $f \in C^*(D)$ can be extended to $g \in C^*(X)$ by setting $g(y) = f(x)$ if $y \in \langle x, \emptyset \rangle$ for some $x \in D$, and $g(y) = 0$ otherwise. Hence, D is C^*-embedded in X so that $\beta D \subseteq \beta X$. Therefore, since D is discrete and card $D =$ card $X^\alpha =$ card α, we have $2^{2^{\text{card } \alpha}} =$ card $\beta D \leqq$ card $\beta X^\alpha \leqq 2^{2^{\text{card } \alpha}}$.

COROLLARY. *The Boolean ring of open-closed subsets of βX^α has trivial automorphism group and cardinality $2^{\text{card } \alpha}$.*

Proof. By the arguments used in the proof of the above theorem we see that the $\bigcup\{\langle x, \emptyset \rangle \colon x \in D'\}$, $D' \subseteq D$, are $2^{\text{card } \alpha}$ distinct open-closed subsets of X^α. Therefore, since X^α cannot have more than $2^{\text{card } \alpha}$ open-closed subsets, and since $Y \to \text{cl}_{\beta X} Y$ is a one-to-one correspondence between the open-closed subsets of X and the open-closed subsets of βX, it follows that βX^α has exactly $2^{\text{card } \alpha}$ open-closed subsets.

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