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Summation Formulae for Coefficients of L-functions

Dedicated to the memory of F. V. Atkinson

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Abstract. With applications in mind we establish a summation formula for the coefficients of a general Dirichlet series satisfying a suitable functional equation. Among a number of consequences we derive a generalization of an elegant divisor sum bound due to F. V. Atkinson.

1 Introduction

Let $a(n)$, $n \geqslant 1$, denote a sequence of complex numbers, that is an arithmetic function. One of the most basic goals of number theory is the establishment of asymptotic formulae, as accurate as possible, for the summatory function

$$A(x) = \sum_{n \leqslant x} a(n).$$

It is a happy fact, exemplified over and over again, that when the sequence $a(n)$ is of great arithmetic interest, the generating series

$$\mathcal{A}(s) = \sum_{1}^{\infty} a(n) n^{-s} \tag{1.1}$$

is endowed with valuable analytic properties which offer tools to help achieve this goal. An important early example of such a tool is the well-known summation formula of Voronoi in the case of $a(n) = \tau(n)$, the divisor function. See the recent survey by Miller and Schmid [MS] for an excellent modern account of this and more recent developments. In this paper we shall establish a summation formula à la Voronoi for the coefficients of a general Dirichlet series which satisfies a suitable functional equation.

Suppose the coefficients of the series (1.1) have a bound $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$ so that $\mathcal{A}(s)$ is holomorphic in $\operatorname{Re} s > 1$. Suppose $\mathcal{A}(s)$ has analytic continuation to the whole complex plane, where it is holomorphic except possibly for a pole at $s = 1$ (not necessarily a simple pole). Assume we have a functional equation

$$\mathcal{A}(1-s) = w\gamma(s)\mathcal{B}(s) \quad \text{if } \operatorname{Re} s > 1, \tag{1.2}$$

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where w is a complex number, the root number, with $|w| = 1$ where $\mathcal{B}(s)$ is given by a series

$$\mathcal{B}(s) = \sum_{1}^{\infty} b(n) n^{-s} \tag{1.3}$$

with $b(n) \ll n^{\varepsilon}$, and $\gamma(s)$ is holomorphic in $\operatorname{Re} s > \frac{1}{2}$.

Here, although our results are more general, we have in mind products of the Riemann zeta function and the Dirichlet L-functions, as well as automorphic L-functions of any degree. In such cases the resulting series $\mathcal{A}(s)$ is known to satisfy the above conditions with $\gamma(s)$ having the shape

$$\gamma(s) = \left(\pi^{-m} D\right)^{s-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\Big(\frac{s+\kappa_j}{2}\Big) \Gamma\Big(\frac{1-s+\kappa_j}{2}\Big)^{-1}, \tag{1.4}$$

where D is a positive integer called the conductor, and the "spectral parameters" κ_j are complex numbers having $\operatorname{Re} \kappa_j \geqslant 0$. Here

$$k = \operatorname{Re} \sum_{j=1}^{m} \kappa_j \tag{1.5}$$

is an integer, the weight. We call m the degree of $\mathcal{A}(s)$. By Stirling's formula

$$\Gamma(\sigma + it) = \sqrt{2\pi} (it)^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2} t} \Big(\frac{t}{e}\Big)^{it} \Big\{ 1 + O\Big(\frac{1}{t}\Big) \Big\}$$

with $t > 0$ and any real σ (the implied constant depending on σ), one derives

$$\begin{aligned} \gamma(\sigma + it) &= \omega (Qt)^{m(\sigma - \frac{1}{2})} (Qt/e)^{imt} \left\{1 + O\left(t^{-1}\right)\right\}, \\ \gamma(\sigma - it) &= \overline{\omega} (Qt)^{m(\sigma - \frac{1}{2})} (e/Qt)^{imt} \left\{1 + O\left(t^{-1}\right)\right\}, \end{aligned} \tag{1.6}$$

if $\sigma \geqslant \frac{1}{2}$ and $t \geqslant 1$, where ω is a complex number of modulus $|\omega| = 1$ and $2\pi Q \geqslant 1$. Specifically,

$$\omega = e\Big(\frac{2k - m}{8}\Big) \quad \text{where } e(u) = e^{2\pi i u}, \tag{1.7}$$

$$Q = \sqrt[m]{D}/2\pi\ . \tag{1.8}$$

Hence one gets

$$\gamma(s) \ll (Q|s|)^{m(\sigma - \frac{1}{2})}, \quad \text{if } \sigma \geqslant \tfrac{1}{2}\ . \tag{1.9}$$

We do not really require $\gamma(s)$ to be exactly as in (1.4)—essentially the estimates (1.6) are sufficient.

As for our goal, the evaluation of the sum

$$A(x)=\sum_{n\leqslant x}a(n),$$

one expects by contour integration that the main term should be

$$R(x)=\operatorname*{res}_{s=1}\mathcal{A}(s)x^s/s. \tag{1.10}$$

Indeed we show the following result.

Proposition 1.1 *For $x\geqslant D^{\frac12}$ we have*

$$A(x)=R(x)+O\big(D^{\frac{1}{m+1}}x^{\frac{m-1}{m+1}+\varepsilon}\big), \tag{1.11}$$

with any $\varepsilon>0$, where the implied constant depends only on ε and the spectral parameters $\kappa_1,\kappa_2,\dots,\kappa_m$.

However, we wish to make the error term smaller. This goal is possible in full generality provided the main term is appended by a dual sum

$$B(x,N)=\sum_{n\leqslant N}b(n)n^{-\frac{m+1}{2m}}\cos\Big(2\pi m\Big(\frac{nx}{D}\Big)^{\frac1m}+\frac{\pi\ell}{4}\Big). \tag{1.12}$$

Here $\ell=m-3-2k$ and N is at our disposal. The cosine factor in the dual sum (1.12) depends slightly on the weight k; more precisely it depends only on $\ell=m-3-2k$ modulo 8. The longer the dual sum which is used, the stronger the error term that is possible. Our main result is

Theorem 1.2 *Let $\mathcal{A}(s)$ be a Dirichlet series of degree $m\geqslant 2$ satisfying the functional equation* (1.2) *with gamma factors given by* (1.4). *Then for $x\geqslant D^{\frac12}$ we have*

$$A(x)=R(x)+w(\pi m)^{-\frac12}D^{\frac{1}{2m}}x^{\frac{m-1}{2m}}B(x,N)+O\big(D^{\frac1m}N^{-\frac1m}x^{\frac{m-1}{m}+\varepsilon}\big), \tag{1.13}$$

where N is any number with $1\leqslant N\leqslant x$. The implied constant depends only on ε and the spectral parameters $\kappa_1,\kappa_2,\dots,\kappa_m$.

Proof of Proposition 1.1 This follows from (1.13) by the trivial estimation

$$B(x,N)\ll N^{\frac{m-1}{2m}+\varepsilon} \tag{1.14}$$

on choosing $N=D^{\frac{1}{m+1}}x^{\frac{m-1}{m+1}}$. ■

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2 Proof of Main Theorem

By [T, Lemma 3.12] we have

$$A(x) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \mathcal{A}(s)\frac{x^s}{s}\,ds + O\Big(\frac{x^{1+2\varepsilon}}{T}\Big) \tag{2.1}$$

for any $1 \leqslant T \leqslant x$, where $\alpha = 1+\varepsilon$. Here we move the integration to the parallel segment with $\operatorname{Re} s = 1-\alpha = -\varepsilon$. We pass poles at $s = 1$ with residue $R(x)$ and at $s = 0$ with residue $\mathcal{A}(0)$ satisfying

$$\mathcal{A}(0) \ll Q^{\frac{m}{2}}. \tag{2.2}$$

We estimate the integrals over the horizontal segments $-\varepsilon \leqslant \operatorname{Re} s \leqslant 1+\varepsilon$, $\operatorname{Im} s = \pm T$ using (1.2), (1.6) and the convexity principle. We obtain

$$A(x) = R(x) + I(x) + O\Big(Q^{\frac{m}{2}} + T^{-1}\Big(x + \big(QT\big)^{\frac{m}{2}}\Big)x^{2\varepsilon}\Big) \tag{2.3}$$

where $I(x)$ is the integral from (2.1) but with α replaced by $1-\alpha$. Changing the variable s into $1-s$ and applying the functional equation (1.2) we return $I(x)$ to the original segment on the line $\operatorname{Re} s = \alpha = 1+\varepsilon$, namely

$$I(x) = \frac{w}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \gamma(s)\mathcal{B}(s)\frac{x^{1-s}}{1-s}\,ds.$$

Next we introduce the series (1.3) and integrate termwise obtaining

$$I(x) = wx\sum_{1}^{\infty} b(n)c(nx) \tag{2.4}$$

where

$$c(y) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{\gamma(s)}{1-s}y^{-s}\,ds. \tag{2.5}$$

First we estimate $c(y)$ for $y > 2(QT)^m$. Introducing (1.6) to (2.5) we derive by partial integration (there is no stationary point in this region) the following bound:

$$c(y) \ll y^{-\alpha}T^{-1}(QT)^{m(\alpha-\frac{1}{2})}, \quad \alpha = 1+\varepsilon. \tag{2.6}$$

This bound for $y = nx$ shows that the terms of (2.4) with $nx > 2(QT)^m$ contribute $O(T^{-1}(QT)^{\frac{m}{2}}x^{\varepsilon})$, a quantity which is already present in the error term of (2.3).

Now there remains the finite sum

$$I_0(x) = wx\sum_{nx\leqslant 2(QT)^m} b(n)c(nx). \tag{2.7}$$

Here we are going to evaluate $c(nx)$ more precisely by the stationary phase method. For technical simplifications we move the integration in (2.5) from $\operatorname{Re} s = \alpha = 1+\varepsilon$ to $\operatorname{Re} s = \beta = \frac{1}{2} + \frac{1}{m}$. Actually, to avoid a pole at $s = 1$ we only move the segments with $1 \leqslant |t| \leqslant T$. We obtain

$$c(y) = \frac{1}{2\pi i} \fint_{\beta - iT}^{\beta+iT} \frac{\gamma(s)}{1-s} y^{-s}\, ds + O\big(y^{-\alpha} T^{-1} (QT)^{m(\alpha - \frac{1}{2})} + y^{-\beta} T^{-1} (QT)^{m(\beta-\frac{1}{2})} \big).$$

Here the error term comes from a trivial estimation of the relevant integrals over the horizontal segments $\operatorname{Im} s = \pm T$ and $\operatorname{Im} s = \pm 1$. The symbol $\fint$ indicates that the segment $\beta + it$ with $|t| < 1$ is removed from the integration. Note that the second part of the above error term is absorbed by the first one, because $y \leqslant 2(QT)^m$. Now we use the approximation (1.6) together with $(1-s)^{-1} = it^{-1} + O(t^{-2})$ to derive

$$c(y) = \pi^{-1} y^{-\beta} Q \operatorname{Re} i\omega \int_1^T (Qt/e)^{imt} y^{-it}\, dt + O\left((yT)^{-1} (QT)^{\frac{m}{2}} x^{\varepsilon} \right). \tag{2.8}$$

Lemma 2.1 *If $2 \leqslant z \leqslant 2T$ then*

$$\int_1^T \left(\frac{t}{ez} \right)^{imt} dt = \sqrt{\frac{2\pi z}{m}}\, e^{\frac{\pi i}{4} - imz} + O\Big(\Big(\frac{1}{\sqrt{T}} + \Big| \log \frac{z}{T} \Big| \Big)^{-1} \Big) \tag{2.9}$$

where the main term exists only if $z \leqslant T$.

Proof This follows from [T, Lemma 4.6]. ■

Applying Lemma 2.1 for $z = Q^{-1} y^{\frac{1}{m}}$ we find by (2.8) that

$$c(y) = \left(\frac{2Q}{\pi m} \right)^{\frac{1}{2}} y^{-\frac{m+1}{2m}} \operatorname{Re}\big(\omega\, e^{\frac{3\pi i}{4} - \frac{im}{Q} y^{\frac{1}{m}}} \big) + O\big(\Delta(y) + (yT)^{-1} (QT)^{\frac{m}{2}} x^{\varepsilon} \big) \tag{2.10}$$

where

$$\Delta(y) = Q y^{-\beta} \left(\frac{1}{\sqrt{T}} + \left| \log \frac{(QT)^m}{y} \right| \right)^{-1} \tag{2.11}$$

and the main term exists only if $y \leqslant (QT)^m$. Recall that we assumed $2 \leqslant z \leqslant 2T$, so that we have (2.10) only for $(2Q)^m \leqslant y \leqslant (2QT)^m$. But the integral (2.9) is bounded if $0 < z < 2$, and so is the integral in (2.8) if $0 < y < (2Q)^m$. Therefore (2.8) yields

$$c(y) \ll Q y^{-\beta} + (yT)^{-1} (QT)^{\frac{m}{2}} x^{\varepsilon}. \tag{2.12}$$

This bound shows that (2.10) also holds for $y < (2Q)^m$ because, in this range, $(yT)^{-1}(QT)^{\frac{m}{2}}$ is the dominating term.

Applying (2.10) with $y = nx$ to all terms of the sum (2.7) we get

$$I_0(x) = w\Big(\frac{2Q}{\pi m}\Big)^{\frac{1}{2}} x^{\frac{m-1}{2m}} B_0(x,T) + O\Big(\Big(E + T^{-1}(QT)^{\frac{m}{2}}\Big)x^{\varepsilon}\Big)$$

where

$$B_0(x,T) = \sum_{nx \leqslant (QT)^m} b(n) n^{-\frac{m+1}{2m}} \operatorname{Re}\big(\omega e^{\frac{3\pi i}{4} - \frac{im}{Q}(nx)^{\frac{1}{m}}}\big),$$

$$E = xQ \sum_{nx \leqslant 2(QT)^m} (nx)^{-\beta}\Big(\frac{1}{\sqrt{T}} + \Big|\log\frac{(QT)^m}{nx}\Big|\Big)^{-1}.$$

It remains to estimate the error term E. We proceed as follows:

$$\begin{aligned}\sum_{n \leqslant 2N} n^{-\beta}\Big(\frac{1}{M} + \Big|\log\frac{n}{N}\Big|\Big)^{-1} &\ll \int_0^{2N} u^{-\beta}\Big(\frac{1}{M} + \Big|\log\frac{u}{N}\Big|\Big)^{-1} du + N^{-\beta}M + 1\\ &= N^{1-\beta}\int_0^2 u^{-\beta}\Big(\frac{1}{M} + |\log u|\Big)^{-1} du + N^{-\beta}M + 1\\ &\ll N^{-\beta}(M + N\log 2M).\end{aligned}$$

Hence

$$\begin{aligned}E &\ll xQ\sqrt{T}(QT)^{-\beta m} + Q(QT)^{(1-\beta)m}\log x\\ &= xT^{-\frac{1}{2}}(QT)^{-\frac{m}{2}} + T^{-1}(QT)^{\frac{m}{2}}\log x.\end{aligned}$$

Gathering the above results we find that

$$A(x) = R(x) + w\Big(\frac{2Q}{\pi m}\Big)^{\frac{1}{2}} x^{\frac{m-1}{2m}} B_0(x,T) + O\Big(\Big(x + (QT)^{\frac{m}{2}}\Big)T^{-1}x^{\varepsilon}\Big)$$

for all $1 \leqslant T \leqslant x$. For ω, Q given in (1.7), (1.8) we have $B_0(x,T) = B(x,N)$ where $N = x^{-1}(QT)^m$. Under the assumption $N \leqslant x$ the above error term is $O(T^{-1}x^{1+\varepsilon})$ and the proof of Theorem 1.2 is complete.

3 Special Cases, General Conjectures, Another Method

In the case of degree two the main theorem becomes the following.

Proposition 3.1 *If $\mathcal{A}(s)$ has degree $m = 2$ then*

$$(3.1)\qquad \sum_{n \leqslant x} a(n) = R(x) + O\big((Dx/N)^{\frac{1}{2}}x^{\varepsilon}\big) + \frac{w}{\sqrt{2\pi}}(Dx)^{\frac{1}{4}} \sum_{n \leqslant N} b(n) n^{-\frac{3}{4}} \cos\Big(4\pi\sqrt{\frac{nx}{D}} - \frac{\pi}{4}(2k+1)\Big)$$

where $R(x)$ is given by (1.10). *Hence*

$$\sum_{n\leqslant x} a(n) = R(x) + O\big(D^{\frac{1}{3}} x^{\frac{1}{3}+\varepsilon}\big). \tag{3.2}$$

The implied constant depends only on ε and κ_1, κ_2.

In the case of degree three the main theorem becomes the following.

Proposition 3.2 *If $\mathcal{A}(s)$ has degree $m = 3$ then*

$$\sum_{n\leqslant x} a(n) = R(x) + O\big((Dx^2/N)^{\frac{1}{3}} x^{\varepsilon}\big) + \frac{w}{\sqrt{3\pi}} D^{\frac{1}{6}} x^{\frac{1}{3}} \sum_{n\leqslant N} b(n) n^{-\frac{2}{3}} \cos\Big(6\pi\sqrt[3]{\frac{nx}{D}} - \frac{\pi k}{2}\Big). \tag{3.3}$$

Hence

$$\sum_{n\leqslant x} a(n) = R(x) + O\big(D^{\frac{1}{4}} x^{\frac{1}{2}+\varepsilon}\big). \tag{3.4}$$

The implied constant depends only on ε and $\kappa_1, \kappa_2, \kappa_3$.

In real life $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are L-functions associated with automorphic forms. In this world one should expect a considerable cancellation of the terms in $B(x, N)$ to take effect, improving the trivial bound (1.14) to the best possible extent.

Conjecture 1 *For any automorphic series $\mathcal{B}(s)$ we have*

$$B(x, N) \ll (DNx)^{\varepsilon}. \tag{3.5}$$

This conjecture would imply the following. (Ignore the condition $N \leqslant x$ in (1.13) since this can be relaxed by more work.)

Conjecture 2 *For any automorphic series $\mathcal{A}(s)$ we have*

$$\sum_{n\leqslant x} a(n) = R(x) + O\big(D^{\frac{1}{2m}} x^{\frac{m-1}{2m}+\varepsilon}\big). \tag{3.6}$$

Note that the Riemann hypothesis for $\mathcal{A}(s)$ yields (3.6) with inferior error term $O(x^{\frac{1}{2}}(Dx)^{\varepsilon})$.

Our technology, based solely on the functional equation, does not allow one to improve the trivial bound for the dual sum $B(x, N)$ (the functional equation is involutary). Indeed, the exponent $\frac{m-1}{m+1}$ in the error term of (1.11) is so far the best known in general. In particular for $m = 2$ it remains a great open problem to improve the bound

$$\sum_{n \leqslant x} a(n) \ll x^{\frac{1}{3}},$$

when $\mathcal{A}(s)$ is a Hecke L-function associated to a cusp form.

Fortunately, one can achieve improvements in a few of the most basic cases, which although quantitatively slight, are important for applications, including those in [FI] which originally motivated this work. These may be achieved by employing estimates for exponential sums and/or bounds for bilinear forms.

In principle, such ideas can be exercised for any $\mathcal{A}(s)$ which factors into L-functions of lower degree. However, in this case another method which uses the mean-value theorem for squares of the factors of $\mathcal{A}(s)$ also yields interesting results. Suppose that

$$\mathcal{A}(s) = \mathcal{A}_1(s)\mathcal{A}_2(s)$$

where $\mathcal{A}_1(s), \mathcal{A}_2(s)$ are series of degree m_1, m_2 respectively, satisfying $m_1 + m_2 = m \geqslant 3$. The mean value theorem yields

$$\sum_{n \leqslant x} a(n) = R(x) + O\big(D^{\frac{1}{2}} x^{\theta(m_1,m_2)+\varepsilon}\big).$$

Here we have

$$\theta(0, m) = \frac{m}{m+1},$$

which is worse than our exponent $\frac{m-1}{m+1}$ in general,

$$\theta(1, m-1) = \frac{m-1}{m+1},$$

which agrees with our exponent, and

$$\theta(m_1, m_2) = \frac{m-2}{m} \quad \text{if } m_1, m_2 \geqslant 2,$$

which is better than our exponent.

4 Improvements for Products of Three L-Functions

We desire to have

$$A(x) = R(x) + O\big(D^{\vartheta} x^{\theta+\varepsilon}\big) \tag{4.1}$$

with some $\vartheta > 0$ and, to surpass the Riemann hypothesis slightly, with some $\theta < \frac{1}{2}$. This is already achieved with $\theta = \frac{1}{3}$ for general series of degree two. Therefore we consider only $\mathcal{A}(s)$ of degree three (all cases of higher degree seem hopelessly difficult). As we mentioned in the previous section we must assume that $\mathcal{A}(s)$ factors. We shall consider in detail $\mathcal{A}(s)$ which factors completely into degree one L-functions.

The other case of factors having degree one and two can also be successfully treated, but only with additional arguments.

Specifically we assume that

$$\mathcal{A}(s) = L(s,\chi_1)L(s,\chi_2)L(s,\chi_3) \tag{4.2}$$

where $\chi_j (\operatorname{mod} D_j)$ are primitive characters. In this case

$$a(n) = \sum_{n_1 n_2 n_3 = n} \chi_1(n_1)\chi_2(n_2)\chi_3(n_3). \tag{4.3}$$

Clearly our basic conditions are satisfied. By the functional equation for each $L(s,\chi_j)$ separately, we get the functional equation (1.2) with $b(n) = \overline{a(n)}$ and the conductor $D = D_1 D_2 D_3$. The root number w is the product of normalized Gauss sums

$$w = i^{-k}\tau(\chi_1)\tau(\chi_2)\tau(\chi_3)D^{-\frac{1}{2}},$$

while the weight k is the number of odd characters

$$k = \frac{1}{2}\sum_{j=1}^{3}\bigl(1-\chi_j(-1)\bigr).$$

We come back to (1.13) or, more specifically, (3.3). Our problem reduces to a non-trivial estimation of the dual sum

$$B(x,N) = \sum_{n\leqslant N} \overline{a}(n) n^{-\frac{2}{3}} \cos(6\pi(nx/D)^{\frac{1}{3}} - \pi k/2), \tag{4.4}$$

where $a(n)$ is given by (4.3). First we estimate the triple sums

$$\sum_{n_1}{}' \sum_{n_2}{}' \sum_{n_3 \leqslant N}{}' \chi_1(n_1)\chi_2(n_2)\chi_3(n_3) e\Bigl(3\Bigl(\frac{n_1 n_2 n_3 x}{D}\Bigr)^{\frac{1}{3}}\Bigr) \tag{4.5}$$

where $\sum'$ restricts the variable n_j to an interval $N_j < n_j \leqslant N_j'$ with $N_j < N_j' \leqslant 2N_j$ and $N_1 N_2 N_3 \leqslant N$. We lock two of the variables and apply to a single sum in the free variable the following quite general result.

Lemma 4.1 *Let F be a smooth function on the interval $[L,2L]$ with derivatives satisfying $|F^{(r)}| \asymp FL^{-r}$ for $r = 2,3,4$. Then, for any subinterval I of $[L,2L]$, we have*

$$\sum_{\ell\in I} e\bigl(F(\ell)\bigr) \ll F^{\frac{1}{14}} L^{\frac{5}{7}} + F^{-\frac{1}{2}} L. \tag{4.6}$$

Proof This is essentially Theorem 2.9 for $q = 2$ in [GK]; see also [T, Theorem 5.13] for $k = 4$. ■

Note that we do not require $|F'| \asymp FL^{-1}$ as assumed in [GK] so we get the term $F^{-\frac{1}{2}}L$ in (4.6) rather than $F^{-1}L$. This suffices for our applications and, because the first derivative estimate is not required, we can add to $F(\ell)$ any linear term $\alpha\ell$ with α real without changing the result. This feature allows us to derive the bound

$$\sum_{\ell \in I} \chi(\ell) e(F(\ell)) \ll \sqrt{D} L\left(F^{\frac{1}{14}} L^{-\frac{2}{7}} + F^{-\frac{1}{2}}\right) \tag{4.7}$$

for any primitive character $\chi (\operatorname{mod} D)$ (just expand χ into additive characters by means of the Gauss sum $\tau(\chi)$ and use $|\tau(\chi)| = \sqrt{D}$). By (4.7) we obtain

$$\sum_{n_j}{}' \chi_j(n_j) e\left(3\left(\frac{n_1 n_2 n_3 x}{D}\right)^{\frac{1}{3}}\right) \ll \sqrt{D_j} N_j \left\{ N_j^{-\frac{2}{7}} \left(\frac{xN_1 N_2 N_3}{D}\right)^{\frac{1}{42}} + \left(\frac{D}{x N_1 N_2 N_3}\right)^{\frac{1}{6}} \right\}.$$

Hence, summing trivially over the remaining two variables we bound the triple sum, say $C(N_1, N_2, N_3)$, given by (4.5) as

$$C(N_1, N_2, N_3) \ll \sqrt{D} N_1 N_2 N_3 \left\{ N_j^{-\frac{2}{7}} \left(\frac{x N_1 N_2 N_3}{D}\right)^{\frac{1}{42}} + \left(\frac{D}{x N_1 N_2 N_3}\right)^{\frac{1}{6}} \right\}.$$

Clearly the best result is obtained if N_j is chosen to be the largest of N_1, N_2, N_3, so that $N_j \geqslant (N_1 N_2 N_3)^{\frac{1}{3}}$. In the normal case that $x \geqslant D$, this gives

$$C(N_1, N_2, N_3) \ll D^{\frac{10}{21}} (N_1 N_2 N_3)^{\frac{13}{14}} x^{\frac{1}{42}}. \tag{4.8}$$

Using (4.8) we derive by (4.4) that

$$B(x, N) \ll (D^{20} N^{11} x)^{\frac{1}{42}} (\log x)^3. \tag{4.9}$$

This together with (1.13) or (3.3) yields

$$A(x) = R(x) + O(D^{\frac{1}{6}} x^{\frac{1}{3}} (D^{20} N^{11} x)^{\frac{1}{42}} (\log x)^3 + (Dx^2/N)^{\frac{1}{3}} x^{\varepsilon}).$$

Choosing $N = (x/D)^{\frac{13}{25}}$ we arrive at (4.1) with $\vartheta = \frac{38}{75}$ and $\theta = \frac{37}{75}$. Note that the condition $x \geqslant D$ is no longer needed because the result is trivial (just) if $x < D$.

For easy reference we state the established asymptotic as a self-contained theorem.

Theorem 4.2 *Let $\chi_j (\operatorname{mod} D_j)$ be primitive characters and denote $D = D_1 D_2 D_3$. For any $x \geqslant 1$ we have*

$$\sum_{n_1 n_2 n_3 \leqslant x} \chi_1(n_1) \chi_2(n_2) \chi_3(n_3) = R(x) + O\left(D^{\frac{38}{75}} x^{\frac{37}{75} + \varepsilon}\right), \tag{4.10}$$

where the main term is given by

$$R(x) = \operatorname{res}_{s=1} L(s, \chi_1) L(s, \chi_2) L(s, \chi_3) x^s / s \tag{4.11}$$

and the implied constant depends only on ε.

Our Theorem 4.2 is a direct generalization of the elegant result of F. V. Atkinson [A] for $\tau_3(n)$ which is the case $D = 1$. In this particular case the result has since been improved, the record small error term being $O(x^{\frac{43}{96}+\varepsilon})$ due to G. Kolesnik[K].

If $\chi_2 = \chi_3 = 1$ are the trivial characters so that $D_2 = D_3 = 1$, $L(s,\chi_2) = L(s,\chi_3) = \zeta(s)$ and $\chi_1 = \chi$ is primitive of conductor $D_1 = D > 1$, then

$$R(x) = L(1,\chi)x\log x + \big(L'(1,\chi) + (2\gamma-1)L(1,\chi)\big)\,x, \tag{4.12}$$

where γ is the Euler constant.

5 Some Generalizations

The most essential ingredient in our arguments in the general case is the functional equation (1.2) which connects our primary Dirichlet series $\mathcal{A}(s)$ with its unique dual series $\mathcal{B}(s)$. There are, however, situations where the dual side of $\mathcal{A}(s)$ (the right side of (1.2) contains more than one Dirichlet series. Think of the matrix functional equation derived from the Eisenstein series for congruence groups with several parabolic subgroups. An important case of multiple duals appears for derivatives of L-functions. Here one gets a relevant functional equation, say for $L'(s,\chi)$, by differentiating that for $L(s,\chi)$:

$$L'(1-s,\chi) = -w\gamma(s)L'(s,\overline{\chi}) - w\gamma'(s)L(s,\overline{\chi}).$$

Note that $\gamma'(s)$ is not a product of gamma functions, nevertheless it does satisfy a Stirling type asymptotic formula very similar to that for $\gamma(s)$ (the main term for $\gamma'(s)$ is obtained by differentiating the main term for $\gamma(s)$ with respect to t). Our arguments with the stationary phase method still work, but of course producing slightly different dual sums. Estimating these various dual sums trivially (or alternatively, estimating earlier the relevant integrals before computing them by the stationary phase method), we find that the crude formula (1.11) remains valid without change.

We conclude this paper by stating one result of this kind.

Theorem 5.1 *Let $\chi_1(\operatorname{mod} D_1)$ and $\chi_2(\operatorname{mod} D_2)$ be primitive characters. Let $a(n)$ be the coefficients of $\mathcal{A}(s) = -L(s,\chi_1)L'(s,\chi_2)$, that is*

$$a(n) = \sum_{n_1 n_2 = n} \chi_1(n_1)\chi_2(n_2)\log n_2. \tag{5.1}$$

For $x \geqslant 1$ we have

$$\sum_{n\leqslant x} a(n) = R(x) + O\big((D_1 D_2 x)^{\frac{1}{3}} x^{\varepsilon}\big)\,, \tag{5.2}$$

where

$$R(x) = \operatorname{res}_{s=1} -L(s,\chi_1)L'(s,\chi_2)x^s/s. \tag{5.3}$$

The implied constant depends only on ε.

If χ_2 is the trivial character so $D_2 = 1$ and $L(s,\chi_2) = \zeta(s)$, and if $\chi_1 = \chi$ is primitive of conductor $D_1 = D > 1$ then

$$R(x) = L(1,\chi)x\log x + \big(L'(1,\chi) - L(1,\chi)\big)\,x. \tag{5.4}$$

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