ON THE EVALUATION OF SOME EXPRESSIONS CONCERNING THE PBIB DESIGNS

Robert Cléroux

1. Introduction and summary. Consider a two associate class partially balanced incomplete block (PBIB) design [2] with parameters of the first kind $t, b, r, k, \lambda_1, \lambda_2, n_1, n_2$ and parameters of the second kind p^i_{jk}, $i, j, k = 1, 2$. Let the letters m, p, ℓ, s represent treatments and define A^i_m = class of i^{th} associates of treatment m, λ_{mps} = number of times the treatments m, p and s all occur together in the same block, $\lambda_{mp\ell s}$ = number of times the treatments m, p, ℓ and s all occur together in the same block,

$$A_{01} = \{m, p, s \mid p \in A^1_m, s \in A^1_m, p \neq s\},$$

$$A_{04} = \{m, p, \ell, s \mid p \in A^1_m, s \in A^1_\ell, m \neq s, m \neq \ell, p \neq \ell, p \neq s\},$$

$$A_{05} = \{m, p, s \mid p \in A^1_m, s \in A^2_m\}.$$

The purpose of this paper is to give explicit expressions for $\sum_{A_{01}} \lambda_{mps}$, $\sum_{A_{04}} \lambda_{mp\ell s}$ and $\sum_{A_{05}} \lambda_{mps}$ for some classes of PBIB designs. These expressions are useful when one studies the robustness of the F-test in these designs, as was seen in [5], by the line of approach used by Rao [7] and Giri [6].

2. Singular group divisible designs. A two associate class PBIB design is group divisible [3] if the number of treatments is $t = mn$, so that treatments can be separated into m groups of n, two treatments in the same group being

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first associates and two treatments belonging to different groups being second associates. Clearly we have $n_1 = n-1$, $n_2 = n(m-1)$, $(p^1_{jk}) = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix}$ and $(p^2_{jk}) = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix}$.

The design is singular if $\lambda_1 = r$. We can prove the following:

LEMMA 1. In any two associate class PBIB design of the singular group divisible type, $k = k^*n$ where k^* is an integer

The proof of this lemma is found in Clatworthy [4].

Note that the case $k^* = 1$ or $\lambda_2 = 0$ is without any interest since then the design is disconnected.

THEOREM 1. For any two associate class PBIB design of the singular group divisible type, we have

(i) $\sum_{A_{01}} \lambda_{mps} = bk(n-1)(n-2)$.

(ii) $\sum_{A_{04}} \lambda_{mp\ell s} = bk(n-1)(n-2)(n-3) + m(m-1)n^2(n-1)^2 \lambda_2$

(iii) $\sum_{A_{05}} \lambda_{mps} = mn^2(m-1)(n-1) \lambda_2$.

Proof. (i) Consider a point $(m, p, s) \in A_{01}$. Since $p \in A^1_m$, treatments p and m belong to the same group. Similarly s and m belong to the same group; then m, p and s belong to the same group and are first associates by pairs. Since $\lambda_1 = r$ treatments m and p occur together in r blocks and similarly for m and s. Therefore treatments m, p and s occur together in r blocks because treatment m cannot occur in more than r blocks. It follows that $\lambda_{mps} = r$. Moreover we have $m\binom{n}{3} 3! = mn(n-1)(n-2)$ points in A_{01}. Therefore

$$\sum_{A_{01}} \lambda_{mps} = mn(n-1)(n-2)r = tr(n-1)(n-2) = bk(n-1)(n-2) .$$

(ii) Define A'_{04} and A''_{04} to be the following sets of the association scheme:

$$A'_{04} = \{(m, p, \ell, s) \in A_{04} \mid m \text{ and } \ell \text{ belong to the same group}\}$$

$$A''_{04} = \{(m, p, \ell, s) \in A_{04} \mid m \text{ and } \ell \text{ belong to different groups}\} .$$

Then A'_{04} and A''_{04} are disjoint sets, $A_{04} = A'_{04} + A''_{04}$ and

$$(2.1) \qquad \sum_{A_{04}} \lambda_{mp\ell s} = \sum_{A'_{04}} \lambda_{mp\ell s} + \sum_{A''_{04}} \lambda_{mp\ell s} .$$

Consider $(m, p, \ell, s) \in A'_{04}$. Then the four treatments are first associates by pairs since they belong to the same group. Since $\lambda_1 = r$ and since no treatment can occur more than r times in the design, the four treatments must occur together in r blocks, that is, $\lambda_{mp\ell s} = r$. Moreover there are $m\binom{n}{4} 4! = mn(n-1)(n-2)(n-3)$ points in A'_{04}. Then

$$\sum_{A'_{04}} \lambda_{mp\ell s} = mn(n-1)(n-2)(n-3)r = tr(n-1)(n-2)(n-3)$$

$$(2.2) \qquad = bk(n-1)(n-2)(n-3).$$

Consider now a point $(m, p, \ell, s) \in A''_{04}$. Treatments m and p belong to the same group and treatments ℓ and s belong to another group. Then treatments m and ℓ, m and s, ℓ and p and s and p are second associates. Since $\lambda_1 = r$ all the treatments in the same group occur together in r blocks. By lemma 1, if $k^* > 1$ then $k > n$. Therefore in all the blocks containing treatments m and p, one or more other groups of treatments will occur. And among these groups, the group containing treatments ℓ and s will occur as many times with m and p as m and ℓ can occur together, that is λ_2 times. Then $\lambda_{mp\ell s} = \lambda_2$. Moreover there are $m(m-1) [\binom{n}{2} 2!]^2 = m(m-1)n^2(n-1)^2$ points in A''_{04}.

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Then

(2.3) $$\sum_{A''_{04}} \lambda_{mp\ell s} = m(m-1)n^2(n-1)^2 \lambda_2 \quad .$$

Combining (2.1), (2.2) and (2.3) yields the desired result.

(iii) Consider $(m, p, s) \in A_{05}$. Since $\lambda_1 = r$, treatments m and p occur together in r blocks. Any block containing m will therefore also contain p, so that treatment s will occur with m and p as many times as it can occur with m, that is λ_2 times, since $s \in A^2_m$. Hence $\lambda_{mps} = \lambda_2$. Moreover there are $mn^2(m-1)(n-1)$ points in A_{05}; thus

$$\sum_{A_{05}} \lambda_{mps} = mn^2(m-1)(n-1)\lambda_2. \qquad \text{QED}$$

3. <u>Semi-regular group divisible designs</u>. A two associate class PBIB design is of the semi-regular group divisible type [3] if it is group divisible and if $\lambda_1 < r$ and $\lambda_2 = \frac{rk}{t}$. The parameters $n_1, n_2, p^1_{jk}, p^2_{jk}$; $j, k = 1, 2$, are the same as in the preceeding section. Bose and Connor [1] prove the following:

LEMMA 2. <u>In any two associate class PBIB design of the semi-regular group divisible type</u>, k <u>is divisible by</u> m. <u>Moreover</u>, <u>if</u> $k = cm$, <u>then each block contains</u> c <u>treatments of each group</u>.

We can now prove the following:

THEOREM 2. <u>In any two associate class PBIB design of the semi-regular group divisible type</u>, <u>we have</u>

(i) $\sum_{A_{01}} \lambda_{mps} = bk(c-1)(c-2)$.

(ii) $\sum_{A_{04}} \lambda_{mp\ell s} = bk(c-1)[(c-2)(c-3) + c(m-1)(c-1)]$.

(iii) $\sum_{A_{05}} \lambda_{mps} = bkc(m-1)(c-1)$.

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Proof. (i) Let $(m, p, s) \in A_{01}$ and write

$$\delta_i(m, p, s) = \begin{cases} 1 & \text{if the } i^{th} \text{ block contains treatments } m, p \text{ and } s \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda_{mps} = \sum_{i=1}^{b} \delta_i(m, p, s)$ and $\sum_{A_{01}} \lambda_{mps} = \sum_{i=1}^{b} \sum_{A_{01}} \delta_i(m, p, s)$.

We are going to compute $\sum_{A_{01}} \delta_i(m, p, s)$, the number of points of A_{01} which occur in block i. By lemma 2, block i contains c treatments of each group; and $(m, p, s) \in A_{01}$ if and only if m, p and s belong to the same group. Thus $\delta_i(m, p, s)$ will take the value 1 in A_{01} as many times as we will find, in block i, triplets belonging to the same group, that is $m\binom{c}{3} 3! = mc(c-1)(c-2) = k(c-1)(c-2)$. And that is independent of i. Hence $\sum_{A_{01}} \lambda_{mps} = bk(c-1)(c-2)$.

(ii) Define A'_{04} and A''_{04} as in theorem 1. Then $\sum_{A_{04}} \lambda_{mp\ell s} = \sum_{A'_{04}} \lambda_{mp\ell s} + \sum_{A''_{04}} \lambda_{mp\ell s}$. Write

$$\delta_i(m, p, \ell, s) = \begin{cases} 1 & \text{if block } i \text{ contains treatments } m, p, \ell \text{ and } s \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda_{mp\ell s} = \sum_{i=1}^{b} \delta_i(m, p, \ell, s)$ and

$$(3.1) \quad \sum_{A_{04}} \lambda_{mp\ell s} = \sum_{i=1}^{b} \sum_{A'_{04}} \delta_i(m, p, \ell, s) + \sum_{i=1}^{b} \sum_{A''_{04}} \delta_i(m, p, \ell, s).$$

We have $(m, p, \ell, s) \in A'_{04}$ if and only if treatments m, p, ℓ and s belong to the same group. Then $\delta_i(m, p, \ell, s)$ will take

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the value 1 in A'_{04} as many times as we will find in block i, points belonging to the same group. By lemma 3, this number is $m\binom{c}{4} 4! = mc(c-1)(c-2)(c-3) = k(c-1)(c-2)(c-3)$.

Therefore $\sum_{A'_{04}} \delta_i(m, p, \ell, s) = k(c-1)(c-2)(c-3)$ and

$$(3.2) \qquad \sum_{i=1}^{b} \sum_{A'_{04}} \delta_i(m, p, \ell, s) = bk(c-1)(c-2)(c-3).$$

For any point $(m, p, \ell, s) \in A''_{04}$, treatments m and p belong to one group and treatments ℓ and s to another. Therefore $\delta_i(m, p, \ell, s)$ will take the value 1 in A''_{04} as many times as we will find in block i, points with two coordinates in one group and with the other two in another group. By lemma 2, this number is $\binom{m}{2} 2! \left[\binom{c}{2} 2!\right]^2 = m(m-1)c^2(c-1)^2$ $= kc(m-1)(c-1)^2$.

Therefore $\sum_{A''_{04}} \delta_i(m, p, \ell, s) = kc(m-1)(c-1)^2$ and

$$(3.3) \qquad \sum_{i=1}^{b} \sum_{A''_{04}} \delta_i(m, p, \ell, s) = bkc(m-1)(c-1)^2.$$

Finally (3.1), (3.2), and (3.3) yield the desired result.

(iii) Let $(m, p, s) \in A_{05}$ and $\delta_i(m, p, s)$ be as defined in (i) Then $\sum_{A_{05}} \lambda_{mps} = \sum_{i=1}^{b} \sum_{A_{05}} \delta_i(m, p, s)$. But for $(m, p, s) \in A_{05}$, m and p belong to the same group and s to another group. Therefore $\delta_i(m, p, j)$ will take the value 1 in A_{05} as many times as we will find in block i, triplets with two components belonging to the same group and the other to another group. By lemma 2, this number is

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$$\binom{m}{2} 2! \binom{c}{2} 2! \; c = m(m-1)c^2(c-1) = kc(m-1)(c-1)$$

Hence

$$\sum_{A_{05}} \delta_i(m, p, s) = kc(m-1)(c-1)$$

and $$\sum_{i=1}^{b} \sum_{A_{05}} \delta_i(m, p, s) = bkc(m-1)(c-1) \qquad \text{QED.}$$

4. Simple designs. A two associate class PBIB design is simple if $\lambda_1 = 0$ or $\lambda_2 = 0$. The case $\lambda_1 = 0$ is a trivial case. Although designs with $\lambda_2 = 0$ can become designs with $\lambda_1 = 0$ by interchanging the names of the associate classes, the following theorem is of practical importance.

THEOREM 3. *In any two associate class PBIB designs with* $\lambda_2 = 0$ *we have*

(i) $\sum_{A_{01}} \lambda_{mps} = bk(k-1)(k-2)$.

(ii) $\sum_{A_{04}} \lambda_{mp\ell s} = bk(k-1)(k-2)(k-3)$.

(iii) $\sum_{A_{05}} \lambda_{mps} = 0$.

The proof is similar to that of theorem 3.

5. Designs of other types. Formulas for $\sum_{A_{01}} \lambda_{mps}$, $\sum_{A_{04}} \lambda_{mp\ell s}$ and $\sum_{A_{05}} \lambda_{mps}$ have not been obtained for the following types of designs: regular group divisible, triangular, latin square and cyclic. But designs of these types being at the same time simple or having $k = 2$ are covered by the four previous theorems. It is also worth noting that the values of $\sum_{A_{01}} \lambda_{mps}$, $\sum_{A_{04}} \lambda_{mp\ell s}$ and $\sum_{A_{05}} \lambda_{mps}$ can be obtained by enumeration although that is laborious and sometimes practically impossible.

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REFERENCES

1. R. C. Bose and W. S. Connor, Combinatorial properties of group divisible incomplete block designs, Ann. Math. Stat., 23, 1952, pp. 367-383.

2. R. C. Bose and K. R. Nair, Partially balanced incomplete block designs. Sankhya, 4, 1939, pp. 337-372.

3. R. C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes. J. Amer. Stat. Ass., 47, 1952, pp. 151-184.

4. W. H. Clatworthy, Contributions on partially balanced incomplete block designs with two associate classes. U. S. Nat. Bur. of Standards, Applied Math. Series, 47, 1956.

5. R. Cleroux, Sur la robustesse du test F dans l'analyse intrabloc du schēma ā blocs incomplet partiellement ēquilibrē et ā deux classes associēes". (Doctoral dissertation, Universitē de Montrēal, 1965).

6. N. C. Giri, On the F-test in the Intrablock analysis of a class of PBIB designs. J. Amer. Stat. Ass., 60, 1965, pp. 285-293.

7. P. V. Rao, The F-test in the Intrablock Analysis of a class of PBIB designs. Ann. Instit. Statist. Math., 15, 1963, pp. 25-36.

University of Montreal

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