A REMARK ON FLAT AND PROJECTIVE MODULES

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It is the purpose of this note to give some characterizations of flat and projective modules, partly in ideal theoretical terms, partly in terms of the exterior product of a module ("puissance extérieure"); cf. **(1)**.

We shall consider left modules over a ring R with identity element and without proper zero divisors. The left module M is called flat if $X \otimes_R M$ is an exact functor on the category of right R-modules X. If M is flat over a commutative domain R, M is necessarily torsion-free. Therefore when looking for flatness of a module M over a commutative domain, one may assume from the start that M is torsion-free.

In the following theorem, we shall not restrict ourselves to commutative rings R, but the modules concerned have to be torsion-free, which, of course, should mean that $rm = 0$ implies $r = 0$ or $m = 0$.

Before stating the theorem, we remark that if $\mathfrak{a}$ is a right ideal of R, $\mathfrak{a}M$ means the Z-module consisting of all finite sums $\sum a_i m_i$ where $a_i \in \mathfrak{a}, m_i \in M$.

THEOREM 1. *Let R be a ring with identity element and without proper zero divisors, and let M be a torsion-free left R-module. Then M is flat if and only if*

$$(*) \qquad (\mathfrak{a} \cap \mathfrak{b})M = \mathfrak{a}M \cap \mathfrak{b}M$$

for all right ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R.

Proof. The necessity may be shown as in **(2**, p. 32, Proposition 6**)**. In fact, for any right ideal $\mathfrak{a}$ of R, $\phi(\mathfrak{a})$ (in Bourbaki's notation) may be identified with $\mathfrak{a}M$ by the canonical mapping from $R \otimes_R M$ to M.

Conversely, let M be a torsion-free left R-module for which $(*)$ is satisfied for all right ideals $\mathfrak{a}$ and $\mathfrak{b}$. To prove that M is flat, we shall show that any linear relation in M is a consequence of linear relations in R **(2**, p. 43, Corollary 1**)**, i.e. for any linear relation $\sum r_i m_i = 0$ there exists a finite set of elements $\bar{m}_j \in M$, $\bar{r}_{ij} \in R$ such that

$$m_i = \textstyle\sum_j \bar{r}_{ij} \bar{m}_j \quad \text{and} \quad \sum_i r_i \bar{r}_{ij} = 0$$

for all i and j, respectively.

We shall show this by induction on n. For $n = 1$, $r_1 m_1 = 0$ implies $r_1 = 0$ or $m_1 = 0$, so that there is nothing to prove. Let us now assume that any linear relation in M with $n - 1$ terms is a consequence of linear relations in R. We shall then prove that this also holds for any linear relation

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$$\sum_{i+1}^{n} r_i m_i = 0$$

with n terms.

If all the coefficients r_i are zero, there is nothing to prove; so we may assume that at least one of them, say r_n, is not zero.

Let $\mathfrak{a}$ be the right ideal $\mathfrak{a} = r_1 R + \ldots + r_{n-1} R$ and $\mathfrak{b}$ be the right ideal $\mathfrak{b} = r_n R$. Obviously

(1) $$r_n m_n = -\sum r_1 m_1 - \ldots - \sum r_{n-1} m_{n-1}$$

is an element of $\mathfrak{a}M \cap \mathfrak{b}M$ and hence, because of (*), an element of $(\mathfrak{a} \cap \mathfrak{b})M$. Therefore $r_n m_n$ admits a representation of the form

(2) $$r_n m_n = \sum_j a_j \bar{m}_j, \qquad a_j \in \mathfrak{a} \cap \mathfrak{b}.$$

As an element of $\mathfrak{a}$ each coefficient a_j may be written as a right linear combination of $r_1, \ldots, r_{n-1}$

(3) $$a_j = r_1 x_{ij} + \ldots + r_{n-1} x_{n-1,j}.$$

Inserting this in (1) and (2), we obtain

$$r_1(m_1 + \sum_j x_{1j} \bar{m}_j) + \ldots + r_{n-1}(m_{n-1} + \sum_j x_{n-1,j} \bar{m}_j) = 0.$$

We have thus obtained a linear relation with $n - 1$ terms; by the inductive assumption, we can find elements $\hat{r}_{ik} \in R$, $1 \leqslant i \leqslant n - 1$, and elements $\hat{m}_k \in M$ such that

$$m_i + \sum_j x_{ij} \bar{m}_j = \sum_k \hat{r}_{ik} \hat{m}_k, \qquad 1 \leqslant i \leqslant n - 1,$$

and

(4) $$\sum_{i=1}^{n-1} r_i \hat{r}_{ik} = 0 \qquad \text{for all } k.$$

As an element of $\mathfrak{b} = r_n R$, a_j can be written $a_j = r_n x_{n,j}$. Now, $r_n \neq 0$ and M is torsion-free, so (2) implies

$$m_n = \sum_j x_{n,j} \bar{m}_j.$$

$m_1, \ldots, m_n$ are now linear combinations of the $\bar{m}_j$ and $\hat{m}_k$, namely

$$m_i = \sum_j (-x_{ij}) \bar{m}_j + \sum_k \hat{r}_{ik} \hat{m}_k, \qquad 1 \leqslant i \leqslant n - 1,$$
$$m_n = \sum_j x_{n,j} \bar{m}_j.$$

(3) implies

$$r_1(-x_{ij}) + \ldots + r_{n-1}(-x_{n-1,j}) + r_n x_{n,j} = -a_j + a_j = 0$$

for all j. Because of (4), (5) is now a linear representation of the m_i, $1 \leqslant i \leqslant n$, of the desired form.

THEOREM 2. *For any ring R without proper zero divisors the following conditions are equivalent:*

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(i) *The weak global dimension of R is at most one, i.e.* $\mathrm{Tor}_2^R(A, B) = 0$ *for all right R-modules A and all left R-modules B.*

(ii) $(\mathfrak{a} \cap \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} \cap \mathfrak{b}\mathfrak{c}$ *for all right ideals $\mathfrak{a}$ and $\mathfrak{b}$, and all left ideals $\mathfrak{c}$.*

(iii) $\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) = \mathfrak{a}\mathfrak{b} \cap \mathfrak{a}\mathfrak{c}$ *for all left ideals $\mathfrak{b}$ and $\mathfrak{c}$, and all right ideals $\mathfrak{a}$.*

Proof. (i) $\Rightarrow$ (ii). For any left ideal $\mathfrak{c}$ we have a short exact sequence

$$0 \to \mathfrak{c} \to R \to R/\mathfrak{c} \to 0$$

from which we infer

$$\mathrm{Tor}_1^R(A, \mathfrak{c}) \simeq \mathrm{Tor}_2^R(A, R/\mathfrak{c}) = 0$$

for an arbitrary right R-module A. Thus $\mathfrak{c}$ is flat. Since R has no proper zero divisor, $\mathfrak{c}$ is torsion-free, and (ii) thus follows from Theorem 1.

(ii) $\Rightarrow$ (i). By Theorem 1, (ii) implies that any left ideal $\mathfrak{c}$ is flat. As before, this involves $\mathrm{Tor}_2^R(A, R/\mathfrak{c}) \simeq \mathrm{Tor}_1^R(A, \mathfrak{c}) = 0$ for any right R-module A. Let

$$0 \to K \to F \to A \to 0$$

be a short exact sequence where F is a free right R-module. From

$$\mathrm{Tor}_1^R(K, R/\mathfrak{c}) \simeq \mathrm{Tor}_2^R(A, R/\mathfrak{c}) = 0$$

it follows that K is flat (**2**, Chapter I, §4, Proposition **1**). Hence for any left R-module B, we have

$$\mathrm{Tor}_2^R(A, B) \simeq \mathrm{Tor}_1^R(K, B) = 0.$$

(i) $\Leftrightarrow$ (iii) follows from the equivalence (i) $\Leftrightarrow$ (ii) in view of the left–right symmetry in the definition of the weak global dimension.

Remark. If R is moreover assumed to be left Noetherian, then the weak global dimension of R is equal to the left global dimension (**7**, Theorem 20, p. 154). Hence (ii) and (iii) are characterizations of rings whose left global dimension is $\leqslant 1$, i.e. of left hereditary rings (**4**, VI, Proposition 2.8).

In the following part of this note we shall restrict ourselves to modules over commutative rings R without proper zero divisors, i.e. to modules over integral domains.

Let K be the quotient field of the integral domain R. If M is a torsion-free R-module, it may be embedded in a vector space over K, viz. $K \otimes_R M$. We begin by giving a necessary condition for a torsion-free R-module M with $\dim_K(K \otimes_R M) < \infty$ to be flat over R.

THEOREM 3. *Let M be a torsion-free module over the integral domain R. If* $\dim_K(K \otimes_R M) = d < \infty$ *and M is R-flat, then* $\overset{d+1}{\wedge} M = 0$, *where* $\wedge M$ *denotes the exterior product ("puissance extérieure"); cf.* (**1**, §5.5).

Proof. Let $m_1, \ldots, m_{d+1}$ be any $d + 1$ elements of M. Since

$$\dim_K(K \otimes_R M) < d + 1,$$

there exists a non-trivial linear relation

$$r_1 m_1 + \ldots + r_{d+1} m_{d+1} = 0, \qquad r_1, \ldots, r_{d+1} \text{ not all } 0.$$

Because M is R-flat, there exist elements $\bar{m}_j \in M$,

$$\bar{r}_{ij} \in R \ (1 \leqslant i \leqslant d+1, 1 \leqslant j \leqslant n),$$

for which

$$m_i = \sum_{j+1}^{n} \bar{r}_{ij} \bar{m}_j, \qquad \sum_{i=1}^{d+1} r_i \bar{r}_{ij} = 0$$

for all i and j, respectively. We may assume that $d + 1 \leqslant n$, since otherwise we could formally insert elements $\bar{m}_j$ with coefficients $\bar{r}_{ij} = 0$.

The element $m_1 \wedge \ldots \wedge m_{d+1} \in \overset{d+1}{\wedge} M$ can be expressed as

$$(6) \qquad m_1 \wedge \ldots \wedge m_{d+1} = \sum_{i_1, \ldots, i_{d+1}} \bar{r}_{1,i_1} \ldots \bar{r}_{d+1,i_{d+1}} \bar{m}_{i_1} \wedge \ldots \wedge \bar{m}_{i_{d+1}},$$

where $(i_1, \ldots, i_{d+1})$ runs through all ordered sets of $d + 1$ integers (equal or different) between 1 and n.

If two of the i's are equal, the corresponding term in (6) vanishes according to the definition of the exterior product. It therefore suffices to let $(i_1, \ldots, i_{d+1})$ run through all ordered sets of $d + 1$ mutually distinct integers between 1 and n.

Consider the terms in (6) where $(i_1, \ldots, i_{d+1})$ runs through the permutations of $d + 1$ fixed mutually distinct integers $1 \leqslant n_1 < \ldots < n_{d+1} \leqslant n$. Then

$$\bar{m}_{i_1} \wedge \ldots \wedge \bar{m}_{i_{d+1}} = \epsilon \bar{m}_{n_1} \wedge \ldots \wedge \bar{m}_{n_{d+1}}$$

where $\epsilon = +1$ or -1 according as $(i_1, \ldots, i_{d+1})$ is an even or odd permutation of $(n_1, \ldots, n_{d+1})$. Hence the sum of the corresponding terms in (6) is

$$\det(\bar{r}_{in_i}) m_{n_1} \wedge \ldots \wedge m_{n_{d+1}}.$$

Since $(r_1, \ldots, r_{d+1})$ is a non-trivial solution of the system of the $d + 1$ linear equations

$$\sum_{i=1}^{d+1} x_i \bar{r}_{in_i} = 0,$$

we see that $\det(\bar{r}_{in_i}) = 0$. This holds for any system $1 \leqslant n_1 < \ldots < n_{d+1} \leqslant n$ of $d + 1$ mutually distinct integers; hence the sum (6) vanishes. Since $\overset{d+1}{\wedge} M$ is generated by all elements of the form $m_1 \wedge \ldots \wedge m_{d+1}$, the proof of $\overset{d+1}{\wedge} M = 0$ is complete.

Before stating the next theorem, we remark that the exterior product has the usual localization property. In fact, let S be any multiplicatively closed set of elements of R containing the identity element but not containing 0. For any module M let M_S be the module of formal quotients $[m/s]$ with respect to S, considered as a module over the quotient ring R_S; cf. (**7**, 8.6). It is then readily checked that

$$\left[\frac{m_1}{s_1}\right] \wedge \ldots \wedge \left[\frac{m_n}{s_n}\right] \rightarrow \left[\frac{m_1 \wedge \ldots m_n}{s_1 \ldots s_n}\right]$$

defines an isomorphism of $\overset{n}{\wedge} M_S$ onto $(\overset{n}{\wedge} M)_S$.

We shall now prove

THEOREM 4. *Let M be a torsion-free module over the integral domain R, and suppose $\dim_K(K \otimes_R M) = d < \infty$, where K is the quotient field of R. Then M is R-projective if and only if M is finitely generated and $\overset{d+1}{\wedge} M = 0$.*

Proof. If M is projective, M is also flat, so that Theorem 3 implies $\overset{d+1}{\wedge} M = 0$. Moreover, since $K \otimes_R M$ is a finitely generated K-module, it follows from (**3**, §5.5, Proposition 9) that M is a finitely generated R-module.

Conversely, let M be a finitely generated R-module for which $\overset{d+1}{\wedge} M = 0$. Since M is finitely generated, it suffices to show that the local components $M_\mathfrak{m}$ are free $R_\mathfrak{m}$-modules for any maximal ideal $\mathfrak{m}$ in R; cf. (**2**, p. 138, Theorem 1).

By the remark preceding Theorem 4, we have $\overset{d+1}{\wedge} M_\mathfrak{m} = (\overset{d+1}{\wedge} M)_\mathfrak{m} = 0$ for any maximal ideal $\mathfrak{m}$. Clearly $\dim_K(K \otimes_{R_\mathfrak{m}} M_\mathfrak{m}) = d$ so that the finitely generated module $M_\mathfrak{m}$ cannot be generated by less than d elements. Let $(m_1, \ldots, m_\nu)$ be a minimal set of generators in the sense that no element m_i is superfluous. We have only to show that $\nu \leqslant d$, since it then follows that $m_1, \ldots, m_\nu$ will be an independent set of generators, i.e. a free $R_\mathfrak{m}$-base.

Assume $\nu > d$. If $a_1 m_1 + \ldots + a_\nu m_\nu = 0$, $a_i \in R_\mathfrak{m}$, then any a_i must belong to $\mathfrak{m}R_\mathfrak{m}$; otherwise a_i would be a unit and m_i an $R_\mathfrak{m}$-linear combination of the remaining m_j, i.e. m_i would be superfluous in the set of generators. This means that for any element $a_1 m_1 + \ldots + a_\nu m_\nu \in M_\mathfrak{m}$, the a's are uniquely determined (mod $\mathfrak{m}R_\mathfrak{m}$). Consequently, an alternating multilinear mapping ϕ of $M_\mathfrak{m}{}^{d+1}$ into $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ is obtained by setting

$$\phi(a_{11} m_1 + \ldots + a_{1\nu} m_\nu, \ldots, a_{d+1,1} m_1 + \ldots + a_{d+1,\nu} m_\nu) = \det_{1 \leqslant i,j \leqslant d+1} (a_{ij}) \pmod{\mathfrak{m}R_\mathfrak{m}}.$$

Since $\phi(m_1, \ldots, m_{d+1}) = 1 \pmod{\mathfrak{m}R_\mathfrak{m}}$, ϕ does not vanish identically. Thus ϕ induces a non-vanishing homomorphism of $\overset{d+1}{\wedge} M_\mathfrak{m}$ into $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$, which means that $\overset{d+1}{\wedge} M_\mathfrak{m} \neq 0$, contradicting our assumption on M.

Since a flat module over an integral domain is necessarily torsion-free, the following corollary is a consequence of Theorems 3 and 4.

COROLLARY 1; cf. (**5**). *A finitely generated flat module over an integral domain is projective.*

Combining Theorem 1 and Corollary 1, we obtain

COROLLARY 2. *A finitely generated torsion-free module over an integral domain R is projective if and only if $(\mathfrak{a} \cap \mathfrak{b})M = \mathfrak{a}M \cap \mathfrak{b}M$ for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R.*

It is obvious that the assumption of M being torsion-free is essential for the validity of Corollary 2. However, for an arbitrary module M, we obtain by passing to the factor module of M with respect to its torsion module M_T

COROLLARY 2′. *A finitely generated module M over an integral domain R is the direct sum of a torsion module and a projective module if and only if there exists a non-zero ideal $\mathfrak{c}$ (dependent on M) such that*

$$(*) \qquad (\mathfrak{a} \cap \mathfrak{b})M = \mathfrak{a}M \cap \mathfrak{b}M$$

holds for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ contained in $\mathfrak{c}$.

Proof. If $M \simeq P \oplus M_T$ with projective P, then M_T is finitely generated; thus there is a non-zero element $c \in R$ with $cM_T = 0$ and $(*)$ is satisfied for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ contained in (c). To prove the "if" part, it suffices to show that

$$(\mathfrak{a} \cap \mathfrak{b})M/M_T \supseteq \mathfrak{a}M/M_T \cap \mathfrak{b}M/M_T$$

for all ideals $\mathfrak{a}$ and $\mathfrak{b}$, for by Corollary 2 this implies that

$$0 \to M_T \to M \to M/M_T \to 0$$

is split exact.

Any element $\bar{x} \in \mathfrak{a}M/M_T \cap \mathfrak{b}M/M_T$ has the form

$$\bar{x} = \sum_i a_i \bar{m}_i = \sum_i b_i \bar{m}_i, \qquad a_i \in \mathfrak{a},\ b_i \in \mathfrak{b},\ \bar{m}_i \in M/M_T.$$

Choosing representatives $x, m_i \in M$, we obtain

$$x = \sum_i a_i m_i + m' = \sum_i b_i m_i + m'', \qquad m' \text{ and } m'' \in M_T.$$

Let c be a non-zero element in $\mathfrak{c}$ such that $cm' = cm'' = 0$. Then

$$cx = \sum_i ca_i m_i = \sum_i cb_i m_i \in c\mathfrak{a}M \cap c\mathfrak{b}M.$$

Since $c\mathfrak{a} \subseteq \mathfrak{c}$ and $c\mathfrak{b} \subseteq \mathfrak{c}$, we infer that

$$c\mathfrak{a}M \cap c\mathfrak{b}M = (c\mathfrak{a} \cap c\mathfrak{b})M = c(\mathfrak{a} \cap \mathfrak{b})M$$

and thus

$$cx = \sum_j cd_j m_j, \qquad d_j \in \mathfrak{a} \cap \mathfrak{b}.$$

Hence $x - \sum_j d_j m_j \in M_T$ or

$$\bar{x} = \sum_j d_j \bar{m}_j \in (\mathfrak{a} \cap \mathfrak{b})M/M_T.$$

If R is a Prüfer ring, i.e. a semi-hereditary integral domain, then any ideal $\mathfrak{a}$ in R is a flat R-module; cf. (**4**, VII, Proposition 4.2). Therefore, by Theorem 3, $\overset{2}{\wedge}\mathfrak{a} = 0$ for any ideal $\mathfrak{a}$ considered as an R-module. Conversely, if $\overset{2}{\wedge}\mathfrak{a} = 0$ for any ideal $\mathfrak{a}$ in an integral domain R, then by Theorem 4, any finitely generated ideal in R is a projective R-module and R is thus a Prüfer ring. In other words

COROLLARY 3. *An integral domain R is a Prüfer ring if and only if $\overset{2}{\wedge}\mathfrak{a} = 0$ for any ideal $\mathfrak{a}$ in R.*

This may be generalized to arbitrary commutative rings with an identity element.

THEOREM 5. *The ideals of a commutative ring R with an identity element form a distributive lattice, i.e.*

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c} \qquad \textit{for any three ideals } \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \textit{ in } R$$

if and only if $\overset{2}{\wedge}\mathfrak{a} = 0$ *for every ideal* $\mathfrak{a}$ *in* R.

Proof. The lattice of ideals in R is distributive if and only if, for any maximal ideal $\mathfrak{m}$ in R, the ideals in the generalized quotient ring $R_{\mathfrak{m}}$ are totally ordered with respect to set inclusion **(6)**.

First, let us assume that any $R_{\mathfrak{m}}$ has the above property. On account of the localization principle, it suffices to prove $\overset{2}{\wedge}\mathfrak{a}' = 0$ for any ideal $\mathfrak{a}'$ of $R_{\mathfrak{m}}$. If $a, b \in R_{\mathfrak{m}}$, we have either $(a) \subseteq (b)$ or $(b) \subseteq (a)$. If $a = br$, $r \in R_{\mathfrak{m}}$, say, then $a \wedge b = r(b \wedge b) = 0$. Thus $\overset{2}{\wedge}\mathfrak{a}' = 0$, and the "only if" part is proved.

Conversely, if $\overset{2}{\wedge}\mathfrak{a} = 0$ for every ideal $\mathfrak{a}$ in R, any ideal in $R_{\mathfrak{m}}$ has the same property. To prove that the ideals of $R_{\mathfrak{m}}$ are totally ordered, it suffices to show that for any two elements a and b of $R_{\mathfrak{m}}$, we have either $(a) \subseteq (b)$ or $(b) \subseteq (a)$.

If this were not the case, then, for suitable a and b, (a, b) would be a minimal system of generators for the ideal $\mathfrak{a}' = aR_{\mathfrak{m}} + bR_{\mathfrak{m}}$. Just as in the proof of Theorem 4, we could define an alternating bilinear mapping ϕ of $\mathfrak{a}' \times \mathfrak{a}'$ into $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ by setting

$$\phi(ax_1 + by_1, ax_2 + by_2) = (x_1 y_2 - x_2 y_1) \pmod{\mathfrak{m}R_{\mathfrak{m}}}.$$

Since $\phi(a, b) = 1 \pmod{\mathfrak{m}R_{\mathfrak{m}}}$, $a \wedge b$ were not zero in $\mathfrak{a}' \wedge \mathfrak{a}'$, contradicting our assumption.

Remark. By combining Theorem 5 and Corollary 3, we see that an integral domain is a Prüfer ring if and only if its ideals form a distributive lattice. This may be regarded as a generalization of the well-known theorem that a Noetherian domain is a Dedekind domain if and only if the ideals form a distributive lattice.

Added in proof. F-injectivity has been studied by D. F. Sanderson under the name "F-Divisibility." See his article (*A generalization of divisibility and injectivity in modules*, Can. Math. Bull., *8* (1965), 506–513) for the construction of the F-injective hull.

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