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ON WEIGHTED INEQUALITIES WITH GEOMETRIC MEAN OPERATOR

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Abstract

We give a characterization of pairs of weights for the validity of weighted inequalities involving certain generalized geometric mean operators generated by some Volterra integral operators, which include the Hardy averaging operator and the Riemann–Liouville integral operators. The estimations of the constants are also discussed. Our results generalize the work done by J. A. Cochran, C.-S. Lee, H. P. Heinig, B. Opic, P. Gurka, and L. Pick.

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1. Introduction

Let $0 < b \le \infty$, $\Omega = \{(x, t) \in \mathbb{R}^2 \mid 0 < t < x < b\}$, and $\phi : \Omega \mapsto (0, \infty)$. Consider the Volterra integral operator

$$T_\phi f(x) := \int_0^x \phi(x, t) f(t)\, dt, \qquad f \ge 0, \tag{1.1}$$

with ϕ satisfying the following conditions:

(Φ1) $\int_0^x \phi(x, t)\, dt = 1$ for all $0 < x < b$;

(Φ2) for any $r > 0$, there exists $M(r) > 0$ such that

$$\exp\left(\int_0^x \phi(x, t) \log[\phi(x, t)^{-1} t^{r-1}]\, dt\right) \ge M(r) x^r \quad \forall\, 0 < x < b.$$

The geometric mean operator generated by T_ϕ is defined by

$$G_\phi f(x) := \exp[T_\phi \log f(x)]. \tag{1.2}$$

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In the case that ϕ is homogeneous of degree -1, conditions (Φ1) and (Φ2) are satisfied if

$$\int_0^1 \phi(1, t)\, dt = 1 \quad \text{and} \quad 0 < \exp\left(\int_0^1 \phi(1, t) \log[\phi(1, t)^{-1} t^{r-1}]\, dt\right) < \infty.$$

In particular, they hold for $\phi(x, t) = \alpha t^{\alpha-1}/x^\alpha$ and $\phi(x, t) = \alpha(x - t)^{\alpha-1}/x^\alpha$, where $\alpha > 0$.

This paper deals with the exponential inequality

$$\left(\int_0^b (G_\phi f(x))^q u(x)\, dx\right)^{1/q} \le C\left(\int_0^b f(x)^p v(x)\, dx\right)^{1/p}, \tag{1.3}$$

where $0 < p, q < \infty$, and u, v are measurable functions defined on $(0, b)$, almost everywhere finite and positive. A considerable number of works are devoted to the study of (1.3). We refer the reader to Heinig *et al.* (see [7]), Opic and Gurka [22], Pick and Opic [26], and Persson and Stepanov [24] for $\phi(x, t) = 1/x$, Cochran and Lee [4], Čižmešija and Pečarić [3], Jain and Singh [12], and Jarrah and Singh [13] for $\phi(x, t) = \alpha t^{\alpha-1}/x^\alpha$, Jain *et al.* [10] and Jain *et al.* [11] for $\phi(x, t) = h(t)/\int_0^x h(y)\, dy$, Heinig *et al.* [9], Heinig [8], and Love [15, 16] for ϕ to be homogeneous of degree -1, Nassyrova *et al.* [20] and Persson *et al.* [25] for ϕ satisfying the Oinarov condition, and Kaijser *et al.* (see [14]) for general ϕ.

The purpose of this paper is to extend the results in [7, 22, 26] to more general ϕ. Furthermore, we discuss some applications of our main result to the case that ϕ is homogeneous of degree -1, including $\phi(x, t) = \alpha t^{\alpha-1}/x^\alpha$ and $\alpha(x - t)^{\alpha-1}/x^\alpha$ for $\alpha > 0$. Our results are generalizations of works of [4, 7, 8, 9, 22, 26].

Throughout this paper we assume that all functions are measurable on their domains, and u, v given in (1.3) are almost everywhere finite and positive. For $0 < p < \infty$ and $\eta \ge 0$, define

$$L^+_{p,\eta} := \left\{ f : (0, b) \mapsto [0, \infty] \;\middle|\; \int_0^b f(x)^p \eta(x)\, dx < \infty \right\}.$$

If $\eta \equiv 1$, we write L^+_p instead of $L^+_{p,\eta}$. For $0 < z < \infty$, we define z^* by $1/z + 1/z^* = 1$. We also take $\exp(-\infty) = 0$, $\log 0 = -\infty$, and $0 \cdot \infty = 0$.

2. Preliminaries

To prove the main results, we need a key tool which is given by Muckenhoupt, Bradley, and Maz'ja (see [2, 18, 19, 23, 27, 29]) as follows.

THEOREM 2.1. *Let $0 < p, q < \infty$, $p > 1$, and $0 < b \le \infty$. Suppose that ρ and η are nonnegative functions, and η^{1-p^*} is locally integrable. Then*

$$\left(\int_0^b \left(\int_0^x f(t)\, dt\right)^q \rho(x)\, dx\right)^{1/q} \le C\left(\int_0^b f(x)^p \eta(x)\, dx\right)^{1/p} \tag{2.1}$$

holds for all $f \in L^+_{p,\eta}$ *if and only if* $A < \infty$, *where*

$$A = \begin{cases} \sup\limits_{0<\xi<b} \left(\int_\xi^b \rho(x)\, dx \right)^{1/q} \left(\int_0^\xi \eta(x)^{1-p^*}\, dx \right)^{1/p^*} & \text{if } p \le q, \\ \left\{ \int_0^b \left(\int_x^b \rho(t)\, dt \right)^{r/q} \left(\int_0^x \eta(t)^{1-p^*}\, dt \right)^{r/q^*} \eta(x)^{1-p^*}\, dx \right\}^{1/r} & \text{if } q < p, \end{cases} \tag{2.2}$$

and $1/r = 1/q - 1/p$. *Moreover, the best constant* C *in* (2.1) *satisfies*

$$\begin{cases} A \le C \le \left(1 + \dfrac{q}{p^*}\right)^{1/q} \left(1 + \dfrac{p^*}{q}\right)^{1/p^*} A & \text{if } 1 < p \le q < \infty, \\ q^{1/q} \left(\dfrac{p^* q}{r}\right)^{1/q^*} A \le C \le q^{1/q} (p^*)^{1/q^*} A & \text{if } 1 \le q < p < \infty, \\ q^{1/q} (p^*)^{1/q^*} A \le C \le q^{1/q} A & \text{if } 0 < q < 1 < p < \infty. \end{cases} \tag{2.3}$$

The following Lemma 2.2 deals with the existence of $G_\phi f(x)$ for $f \in L^+_{p,v}$.

LEMMA 2.2. *Let* $p > 0$, ϕ *satisfy* (Φ1), *and* $\int_0^x \phi(x,t) \log \phi(x,t)\, dt$ *be finite for all* $0 < x < b$. *Suppose that* v *is almost everywhere finite and positive, and* (2.4) *holds:*

$$T_\phi \log(1/v)(x) \text{ is well defined and } T_\phi \log(1/v)(x) < \infty \text{ for all } 0 < x < b. \tag{2.4}$$

Then, for all $f \in L^+_{p,v}$, $G_\phi f(x)$ *exists and is finite for all* $0 < x < b$.

PROOF OF LEMMA 2.2 We first prove that if $h \in L^+_1$, then $G_\phi h(x)$ exists for all $0 < x < b$. Suppose that $\int_0^b h(t)\, dt < \infty$. Then, for any $0 < x < b$, $\int_0^x h(t)\, dt = \int_0^x \phi(x,t)\phi(x,t)^{-1} h(t)\, dt < \infty$. By [6, Theorem 187], $\int_0^x \phi(x,t) \log[\phi(x,t)^{-1} h(t)]\, dt$ is well defined and

$$\exp\left(\int_0^x \phi(x,t) \log[\phi(x,t)^{-1} h(t)]\, dt \right) = \lim_{r \to 0+} \left\{ \int_0^x \phi(x,t) (\phi(x,t)^{-1} h(t))^r\, dt \right\}^{1/r}$$

exists and is finite. Since

$$T_\phi \log h(x) = \int_0^x \phi(x,t) \log \phi(x,t)\, dt + \int_0^x \phi(x,t) \log[\phi(x,t)^{-1} h(t)]\, dt,$$

we see that

$$G_\phi h(x) = \exp\left(\int_0^x \phi(x,t) \log \phi(x,t)\, dt \right) \exp\left(\int_0^x \phi(x,t) \log[\phi(x,t)^{-1} h(t)]\, dt \right)$$

exists and is finite for all $0 < x < b$. For $f \in L^+_{p,v}$, let $h = f^p v$ and hence $h \in L^+_1$. Since $-\infty \le T_\phi \log h(x) < \infty$,

$$T_\phi \log f(x) = \frac{1}{p} (T_\phi \log h(x) + T_\phi \log(1/v)(x))$$

and $G_\phi f(x) = G_\phi h(x)^{1/p} G_\phi (1/v)(x)^{1/p}$ exists and is finite for all $0 < x < b$. □

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For any $s > 0$, let $h^s = f^p v$. Then by a similar argument given in the proof of Lemma 2.2, we see that $G_\phi f(x) = G_\phi h(x)^{s/p} G_\phi(1/v)(x)^{1/p}$. This implies the following lemma.

LEMMA 2.3. *Let $0 < p, q < \infty$ and ϕ, v be given in Lemma 2.2. Then (1.3) holds for all $f \in L^+_{p,v}$ if and only if, for any $s > 0$,*

$$\left(\int_0^b (G_\phi h(x))^{sq/p} w(x)\,dx\right)^{p/(sq)} \le C^{p/s}\left(\int_0^b h(x)^s\,dx\right)^{1/s} \tag{2.5}$$

holds for all $h \in L^+_s$ with the same best constant C as in (1.3). Here

$$w(x) = G_\phi(1/v)(x)^{q/p} u(x). \tag{2.6}$$

3. Main results

Let $0 < p, q < \infty$, $\delta > 1$, and w be given by (2.6). We define

$$A_\delta := \begin{cases} \sup\limits_{0<\xi<b} \xi^{(\delta-1)/p}\left(\displaystyle\int_\xi^b x^{-\delta q/p} w(x)\,dx\right)^{1/q} & \text{if } p \le q, \\ \left\{\displaystyle\int_0^b \left(\int_x^b t^{-\delta q/p} w(t)\,dt\right)^{p/(p-q)} x^{(\delta q-p)/(p-q)}\,dx\right\}^{(p-q)/(pq)} & \text{if } q < p. \end{cases} \tag{3.1}$$

Our main result can be described as follows.

THEOREM 3.1. *Let $0 < p, q < \infty$, ϕ satisfy (Φ1) to (Φ2), and (2.4) hold. Then (1.3) holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. Moreover, the best constant C in (1.3) satisfies*

$$\sup_{\delta>1} L_\delta A_\delta \le C \le \inf_{\delta>1} U_\delta A_\delta, \tag{3.2}$$

where

$$U_\delta = \begin{cases} \inf\limits_{s>1}\left(\dfrac{p+(s-1)q}{p}\right)^{1/q}\left(\dfrac{p+(s-1)q}{(\delta-1)q}\right)^{(s-1)/p} M\left(\dfrac{\delta}{s}\right)^{-s/p} & \text{if } p \le q, \\ \inf\limits_{s>1}\left(\dfrac{sq}{p}\right)^{1/q}\left(\dfrac{s-1}{\delta-1}\right)^{s/p-1/q} \max\{(s^*)^{s/p-1/q}, 1\} M\left(\dfrac{\delta}{s}\right)^{-s/p} & \text{if } q < p, \end{cases} \tag{3.3}$$

and

$$L_\delta = \begin{cases} \left(\dfrac{\delta-1}{\delta-1+\exp(-\varepsilon\delta)}\right)^{1/p} & \text{if } p \le q, \\ \left(\dfrac{\delta q-q}{p}\right)^{1/q} \min\left(M_l^{(\delta q-p)/(p(p-q))}, M_u^{(\delta q-p)/(p(p-q))}\right) & \text{if } q < p. \end{cases} \tag{3.4}$$

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Here $\varepsilon \geq 0$ is the largest number that satisfies

$$\int_0^t \phi(x, z)\, dz \geq \varepsilon t \phi(x, t) \quad \forall (x, t) \in \Omega$$

and M_l, M_u are positive constants that satisfy

$$M_l x \leq \exp\biggl(\int_0^x \phi(x, t) \log t \, dt\biggr) \leq M_u x \quad \forall\, 0 < x < b.$$

If ϕ is homogeneous of degree -1 and $\int_0^1 \phi(1, t)\, dt = 1$, then

$$\begin{aligned}\exp&\biggl(\int_0^x \phi(x, t) \log[\phi(x, t)^{-1} t^{r-1}]\, dt\biggr)\\ &= x^r \exp\biggl(\int_0^1 \phi(1, t) \log[\phi(1, t)^{-1} t^{r-1}]\, dt\biggr),\end{aligned}$$

and $(\Phi 2)$ is satisfied with $M(r) = \exp(\int_0^1 \phi(1, t) \log[\phi(1, t)^{-1} t^{r-1}]\, dt)$ if this constant exists and is positive. Therefore, we now apply Theorem 3.1 to the case that ϕ is homogeneous of degree -1 and where it satisfies $(\Phi\mathrm{H}1)$–$(\Phi\mathrm{H}3)$:

$(\Phi\mathrm{H}1)$ $\int_0^1 \phi(1, t)\, dt = 1$;
$(\Phi\mathrm{H}2)$ $M_1 = \exp(\int_0^1 \phi(1, t) \log \phi(1, t)\, dt) < \infty$;
$(\Phi\mathrm{H}3)$ $M_2 = \exp(\int_0^1 \phi(1, t) \log t\, dt) > 0$.

For such a case, $(\Phi 1)$ and $(\Phi 2)$ are satisfied. We may choose

$$M(\delta/s) = \exp\biggl(\int_0^x \phi(x, t) \log[\phi(x, t)^{-1} t^{\delta/s - 1}]\, dt\biggr) x^{-\delta/s} = M_1^{-1} M_2^{\delta/s - 1}$$

and $M_l = M_u = M_2$. The following theorem can be obtained from Theorem 3.1.

THEOREM 3.2. *Let $0 < p, q < \infty$, and let ϕ be homogeneous of degree -1 and satisfy $(\Phi\mathrm{H}1)$–$(\Phi\mathrm{H}3)$. Suppose that (2.4) holds. Then (1.3) holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. The estimation of C can be obtained by (3.2)–(3.4) with*

$$M(\delta/s) = M_1^{-1} M_2^{\delta/s - 1}, \qquad M_l = M_u = M_2. \tag{3.5}$$

In the case when $p \leq q$,

$$U_\delta = \begin{cases} M_1^{1/p} M_2^{(1-\delta)/p} & \text{if } 1 < \delta \leq \dfrac{M_1 M_2 p e}{q} + 1, \\ \biggl(\dfrac{(\delta - 1) q}{M_1 M_2 p}\biggr)^{1/q} M_1^{1/p} M_2^{(1-\delta)/p} e^{(1-\delta)/(M_1 M_2 p e)} & \text{if } \delta > \dfrac{M_1 M_2 p e}{q} + 1, \end{cases} \tag{3.6}$$

and hence $U_\delta \leq M_1^{1/p} M_2^{(1-\delta)/p}$.

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In [9, Theorem 2.1] Heinig *et al.* gave a sufficient condition for (1.3) to hold if $b=\infty$, ϕ is homogeneous of degree -1, and ϕ satisfies (ΦH1) and (ΦH3). On the other hand, Heinig [8] proved that in the case $p=q=1$, by adding some other conditions, the sufficient condition given in [9, Theorem 2.1] is also necessary. Our result yields a complete characterization of u and v in (1.3) for ϕ satisfying (ΦH1)–(ΦH3).

Consider the particular case $b=\infty$, $u(x)=x^m$, and $v(x)=x^n$. Then $w(x)$ defined by (2.6) reduces to $w(x)=M_2^{-nq/p}x^{m-(nq/p)}$. For $q<p$, $A_\delta=\infty$ for all $\delta>1$. If $p\le q$ and $(m+1)/q=(n+1)/p$, then $A_\delta=M_2^{-n/p}(p/(\delta q-q))^{1/q}$. Since $U_\delta\le M_1^{1/p}M_2^{(1-\delta)/p}$, by (3.2) and (3.3),

$$C\le M_1^{1/p}M_2^{-n/p}\inf_{\delta>1}M_2^{(1-\delta)/p}\left(\frac{p}{\delta q-q}\right)^{1/q}=M_1^{1/p}M_2^{-n/p}(-e\log M_2)^{1/q}.$$

Therefore,

$$\left(\int_0^\infty (G_\phi f(x))^q x^m\,dx\right)^{1/q}\le M_1^{1/p}M_2^{-n/p}(-e\log M_2)^{1/q}\left(\int_0^\infty f(x)^p x^n\,dx\right)^{1/p}. \tag{3.7}$$

The following corollary considers the case when $\phi(x,t)=\alpha t^{\alpha-1}/x^\alpha$, where $\alpha>0$. For such a case,

$$M_1=\alpha e^{1/\alpha-1},\quad M_2=e^{-1/\alpha},\quad \varepsilon=1/\alpha. \tag{3.8}$$

COROLLARY 3.3. *Let $0<p,q<\infty$ and $\alpha>0$. Suppose that u, v are almost everywhere finite and positive, and (2.4) holds with $\phi(x,t)=\alpha t^{\alpha-1}/x^\alpha$. Then*

$$\left(\int_0^b\left\{\exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q}\le C\left(\int_0^b f(x)^p v(x)\,dx\right)^{1/p} \tag{3.9}$$

holds for all $f\in L^+_{p,v}$ if and only if $A_\delta<\infty$ for all $\delta>1$. The estimation of C can be obtained by (3.2)–(3.4) with (3.5) and (3.8).

Consider the case $b=\infty$. For $\alpha=1$, Corollary 3.3 reduces to [22, Theorem] and [26, Corollary 3.10]. For general α, inequality (3.9) was also investigated in [9, 12, 13]. In [9, Theorem 2.2], it was shown that (3.9) holds for all $f\in L^+_{p,v}$ if and only if $A_{\alpha+1}<\infty$ (in the case when $p\le q$) or $A_{p(\alpha+1)/q}<\infty$ (in the case when $q<p$). Corollary 3.3 contains these results; in addition, it also provides an estimation of C. If $p\le q$ and $(m+1)/q=(n+1)/p$, then by (3.7),

$$\begin{aligned}&\left(\int_0^\infty\left\{\exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right)\right\}^q x^m\,dx\right)^{1/q}\\&\quad\le\alpha^{1/p-1/q}\exp(1/q+(n-\alpha+1)/(\alpha p))\left(\int_0^\infty f(x)^p x^n\,dx\right)^{1/p}.\end{aligned} \tag{3.10}$$

If $p = q = 1$ and $m = n$, then (3.10) is the well-known Cochran–Lee's inequality.

We can also apply Theorem 3.2 to the case when $\phi(x, t) = \alpha(x - t)^{\alpha-1}/x^{\alpha}$, where $\alpha > 0$. In this case,

$$M_1 = \alpha e^{1/\alpha - 1}, \quad M_2 = e^{-\gamma - \Gamma'(\alpha+1)/\Gamma(\alpha+1)}, \quad \varepsilon = \begin{cases} 0 & \text{for } 0 < \alpha < 1, \\ 1 & \text{for } \alpha \geq 1. \end{cases} \tag{3.11}$$

The constant M_2 can be obtained by the following equalities:

$$\log M_2 = \alpha \int_0^1 z^{\alpha-1} \log(1 - z)\, dz = -\alpha \int_0^1 \sum_{n=1}^{\infty} \frac{z^{n+\alpha-1}}{n}\, dz = -\gamma - \frac{\Gamma'(\alpha + 1)}{\Gamma(\alpha + 1)},$$

where γ is the Euler constant and $\Gamma(x)$ is the gamma function. The last equality is based on [1, Theorem 1.2.5]. We have the following corollary.

COROLLARY 3.4. *Let $0 < p, q < \infty$ and $\alpha > 0$. Suppose that u, v are almost everywhere finite and positive, and (2.4) holds with $\phi(x, t) = \alpha(x - t)^{\alpha-1}/x^{\alpha}$. Then*

$$\begin{aligned} &\left(\int_0^b \left\{\exp\left(\frac{\alpha}{x^{\alpha}} \int_0^x (x - t)^{\alpha-1} \log f(t)\, dt\right)\right\}^q u(x)\, dx\right)^{1/q} \\ &\qquad \leq C\left(\int_0^b f(x)^p v(x)\, dx\right)^{1/p} \end{aligned} \tag{3.12}$$

holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. The estimation of C can be obtained by (3.2)–(3.4) with (3.5) and (3.11).

Another type of characterization can also be found in [20, Theorem 5.1] for the case when $0 < p, q < \infty$, $\alpha > 0$ and in [8, Corollary 3.1] for the case when $p = q = 1$, $\alpha \geq 1$.

4. Proof of Theorem 3.1

We first prove the sufficient part. Suppose that $A_\delta < \infty$ for all $\delta > 1$. Condition $(\Phi 2)$ ensures that $\int_0^x \phi(x, t) \log \phi(x, t)\, dt$ is finite for all $0 < x < b$. By Lemmas 2.2 and 2.3, inequality (1.3) holds for all $f \in L^+_{p,v}$ if and only if (2.5) holds for all $h \in L^+_s$, where $s > 1$. Since $\int_0^x \phi(x, t) \log[\phi(x, t)^{-1} t^{\delta/s-1} h(t)]\, dt$ is well defined, by Jensen's inequality and $(\Phi 2)$,

$$\begin{aligned} G_\phi h(x) &\leq \exp\left(-\int_0^x \phi(x, t) \log[\phi(x, t)^{-1} t^{\delta/s-1}]\, dt\right) \int_0^x t^{\delta/s-1} h(t)\, dt \\ &\leq M(\delta/s)^{-1} x^{-\delta/s} \int_0^x t^{\delta/s-1} h(t)\, dt. \end{aligned} \tag{4.1}$$

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This implies that

$$\begin{aligned}\int_0^b &(G_\phi h(x))^{sq/p} w(x)\,dx \\ &\le M(\delta/s)^{-sq/p} \int_0^b \left(\int_0^x t^{\delta/s-1} h(t)\,dt\right)^{sq/p} x^{-\delta q/p} w(x)\,dx.\end{aligned}$$

Replace $p, q, f(t), \rho(x)$, and $\eta(x)$ in Theorem 2.1 by $s, sq/p, t^{\delta/s-1}h(t)$, $x^{-\delta q/p}w(x)$, and $x^{s-\delta}$, respectively. Then (2.5) holds and

$$C \le \begin{cases} \left(\dfrac{p+(s-1)q}{p}\right)^{1/q}\left(\dfrac{p+(s-1)q}{(\delta-1)q}\right)^{(s-1)/p} M(\delta/s)^{-s/p} A_\delta & \text{for } p \le q, \\ \left(\dfrac{sq}{p}\right)^{1/q}\left(\dfrac{s-1}{\delta-1}\right)^{s/p-1/q} \max\{(s^*)^{s/p-1/q}, 1\} M(\delta/s)^{-s/p} A_\delta & \text{for } q < p. \end{cases} \tag{4.2}$$

Since (4.2) is true for arbitrary $s > 1$ and $\delta > 1$, we have the upper estimations of C given in (3.2) and (3.3).

In the following we prove the necessary part of Theorem 3.1. The idea is based on the proof of [24, Lemma 1] and [26, Lemma 3.2]. Consider the case $p \le q$. By Lemma 2.3, the inequality

$$\left(\int_0^b (G_\phi h(x))^{q/p} w(x)\,dx\right)^{1/q} \le C\left(\int_0^b h(x)\,dx\right)^{1/p} \tag{4.3}$$

holds for all $h \in L_1^+$ with the same constant C as in (1.3). Let $\xi > 0$, $\delta > 1$, and let $\varepsilon \ge 0$ be the largest number that satisfies $\int_0^t \phi(x,z)\,dz \ge \varepsilon t \phi(x,t)$ for all $(x,t) \in \Omega$. Let

$$h(t) = \chi_{(0,\xi)}(t)\xi^{-1} + \chi_{(\xi,b)}(t) e^{-\varepsilon\delta}\xi^{\delta-1} t^{-\delta}.$$

Then

$$\left(\int_0^b h(x)\,dx\right)^{1/p} \le \left(\frac{\delta - 1 + e^{-\varepsilon\delta}}{\delta - 1}\right)^{1/p}. \tag{4.4}$$

On the other hand, for $\xi < x < b$,

$$\begin{aligned}\int_0^x \phi(x,t)\log h(t)\,dt &= \int_0^\xi \phi(x,t)\log \xi^{-1}\,dt + \int_\xi^x \phi(x,t)\log[e^{-\varepsilon\delta}\xi^{\delta-1}t^{-\delta}]\,dt \\ &= -\log\xi - \delta\int_\xi^x \phi(x,t)\log\left[\frac{t}{\xi}\right]dt \\ &\quad - \varepsilon\delta\int_\xi^x \phi(x,t)\,dt.\end{aligned}$$

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Moreover, since

$$\begin{aligned}\int_{\xi}^{x} \phi(x,t)\log\left[\frac{t}{\xi}\right]dt &= \log\left[\frac{x}{\xi}\right]\int_{\xi}^{x}\phi(x,t)\,dt + \int_{\xi}^{x}\phi(x,t)\left(-\int_{t}^{x}\frac{1}{y}\,dy\right)dt\\ &= \log\left[\frac{x}{\xi}\right]\int_{\xi}^{x}\phi(x,t)\,dt - \int_{\xi}^{x}\frac{1}{y}\int_{\xi}^{y}\phi(x,t)\,dt\,dy\\ &= \log\left[\frac{x}{\xi}\right] - \int_{\xi}^{x}\frac{1}{y}\int_{0}^{y}\phi(x,t)\,dt\,dy,\end{aligned}$$

we have

$$\begin{aligned}\int_{0}^{x}\phi(x,t)\log h(t)\,dt = &-\log\xi - \delta\log\left[\frac{x}{\xi}\right]\\ &+\delta\int_{\xi}^{x}\left(\frac{1}{t}\int_{0}^{t}\phi(x,z)\,dz\right) - \varepsilon\phi(x,t)\,dt\\ \geq\ &\log[\xi^{\delta-1}x^{-\delta}].\end{aligned}$$

This shows that $G_\phi h(x) \geq \xi^{\delta-1}x^{-\delta}$ for $\xi < x < b$ and

$$\int_{0}^{b}(G_\phi h(x))^{q/p}w(x)\,dx \geq \xi^{(\delta-1)q/p}\int_{\xi}^{b}x^{-\delta q/p}w(x)\,dx. \tag{4.5}$$

By (4.3)–(4.5),

$$C\left(\frac{\delta-1+e^{-\varepsilon\delta}}{\delta-1}\right)^{1/p} \geq \xi^{(\delta-1)/p}\left(\int_{\xi}^{b}x^{-\delta q/p}w(x)\,dx\right)^{1/q}. \tag{4.6}$$

Since (4.6) holds for all $0 < \xi < b$,

$$C \geq \left(\frac{\delta-1}{\delta-1+e^{-\varepsilon\delta}}\right)^{1/p}A_\delta. \tag{4.7}$$

Inequality (4.7) is true for all $\delta > 1$, and we have the lower estimation given in (3.2) and (3.4).

Consider the case $q < p$. Let $\{b_n\}$ be an increasing sequence which converges to b and

$$w_n(x) = [\min(w(x), n)]\chi_{(0,b_n)}(x) + [\min(w(x), x^{-2q/r})]\chi_{[b_n,b)}(x),$$

where $1/r = 1/q - 1/p$. For $\delta > 1$, define

$$h_n(x) = x^{(\delta q-p)/(p-q)}\left(\int_{x}^{b}t^{-\delta q/p}w_n(t)\,dt\right)^{p/(p-q)} \quad \text{for } 0 < x < b.$$

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By the dual Hardy inequality (see [23, Theorem 6.2]), we see that

$$\int_0^b h_n(x)\,dx \le \left(\frac{p}{\delta q - q}\right)^{p/(p-q)} \int_0^b w_n(x)^{p/(p-q)}\,dx < \infty$$

and $G_\phi h_n(x)$ exists and is finite for all $0 < x < b$. Replace h by h_n in (4.3). Since $w_n \le w$,

$$\left(\int_0^b (G_\phi h_n(x))^{q/p} w_n(x)\,dx\right)^{1/q} \le C\left(\int_0^b h_n(x)\,dx\right)^{1/p}. \tag{4.8}$$

By $(\Phi 2)$,

$$\begin{aligned} G_\phi h_n(x) &\ge \left\{\exp\left(\int_0^x \phi(x,t)\log[t^{(\delta q-p)/(p-q)}]\,dt\right)\right\}\left(\int_x^b t^{-\delta q/p} w_n(t)\,dt\right)^{p/(p-q)} \\ &\ge \widetilde{M}^p x^{(\delta q-p)/(p-q)}\left(\int_x^b t^{-\delta q/p} w_n(t)\,dt\right)^{p/(p-q)}, \end{aligned}$$

where $\widetilde{M} = \min(M_l^{(\delta q-p)/(p(p-q))},\ M_u^{(\delta q-p)/(p(p-q))})$. Therefore,

$$\begin{aligned} &\int_0^b (G_\phi h_n(x))^{q/p} w_n(x)\,dx \\ &\quad\ge \widetilde{M}^q \int_0^b \left(\int_x^b t^{-\delta q/p} w_n(t)\,dt\right)^{q/(p-q)} x^{(\delta q^2-pq)/(p^2-pq)} w_n(x)\,dx \\ &\quad= \frac{(\delta q - q)\widetilde{M}^q}{p}\int_0^b \left(\int_x^b t^{-\delta q/p} w_n(t)\,dt\right)^{p/(p-q)} x^{(\delta q-p)/(p-q)}\,dx. \end{aligned}$$

By (4.8),

$$C \ge \left(\frac{\delta q - q}{p}\right)^{1/q} \widetilde{M}\left\{\int_0^b \left(\int_x^b t^{-\delta q/p} w_n(t)\,dt\right)^{p/(p-q)} x^{(\delta q-p)/(p-q)}\,dx\right\}^{(p-q)/(pq)}.$$

Let $n \to \infty$. Since $w_n \uparrow w$, we have $C \ge ((\delta q - q)/p)^{1/q}\widetilde{M} A_\delta$. This holds for all $\delta > 1$, so we have the lower estimation given in (3.2) and (3.4). This completes the proof.

5. Concluding remarks

REMARK 5.1. In [17, Theorem 2], Manakov showed that if $1 < p < q < \infty$, $b = \infty$, and $\int_0^\infty \eta(x)^{1-p^*}\,dx = \infty$, then the upper estimation of C given in (2.3) can be replaced by

$$C \le \left(\frac{\Gamma(q/\tau)}{\Gamma(1+1/\tau)\Gamma((q-1)/\tau)}\right)^{\tau/q} A, \qquad \text{for } \tau = q/p - 1. \tag{5.1}$$

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Hence if $0 < p < q < \infty$ then U_δ given in (3.3) can be replaced by

$$U_\delta = \inf_{s>1}\left(\frac{s-1}{\delta-1}\right)^{(s-1)/p}\left(\frac{\Gamma(sq/(q-p))}{\Gamma(q/(q-p))\Gamma((sq-p)/(q-p))}\right)^{(q-p)/(pq)} M\left(\frac{\delta}{s}\right)^{-s/p}. \tag{5.2}$$

REMARK 5.2. Suppose that $0 < p, q < \infty$, $b = \infty$, u and v are positive and finite almost everywhere, ϕ satisfies (Φ1) and (Φ2), and (2.4) holds. Define $w(x) = G_\phi(1/v)(x)^{q/p}u(x)$. By Theorem 3.1 and the results in [20, 24, 25, 28], the following (1)–(5) are equivalent.

(1)

$$\left(\int_0^\infty (G_\phi f(x))^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p}, \quad f \in L^+_{p,v}. \tag{5.3}$$

(2) A_δ defined by (3.1) is finite for all $\delta > 1$.

(3) For all $\delta > 1$,

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^{\delta q/p} w(x)\,dx\right)^{p/(\delta q)} \le C\left(\int_0^\infty f(x)^\delta\,dx\right)^{1/\delta}, \quad f \in L^+_\delta. \tag{5.4}$$

(4) For all $\delta > 1$,

$$\left(\int_0^\infty f(x)^{\delta q/p} w(x)\,dx\right)^{p/(\delta q)} \le C\left(\int_0^\infty f(x)^\delta dx\right)^{1/\delta}, \quad f \in L^+_\delta \text{ and } f\downarrow, \tag{5.5}$$

where $f\downarrow$ means f is nonincreasing.

(5) The constant B is finite, where B is defined by (5.6):

$$B := \begin{cases} \sup_{\xi>0} \xi^{-1/p}\left(\int_0^\xi w(x)\,dx\right)^{1/q} & \text{if } p \le q, \\ \left\{\int_0^\infty \left(\frac{1}{x}\int_0^x w(t)\,dt\right)^{p/(p-q)} dx\right\}^{(p-q)/(pq)} & \text{if } q < p. \end{cases} \tag{5.6}$$

The constant C that occurs in (5.3)–(5.5) may be different. More equivalent conditions can also be found in [5, 21], but we leave the details to the reader.

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