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Application of the Strong Artin Conjecture to the Class Number Problem

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Abstract. We construct unconditionally several families of number fields with the largest possible class numbers. They are number fields of degree 4 and 5 whose Galois closures have the Galois group A_4, S_4, and S_5. We first construct families of number fields with smallest regulators, and by using the strong Artin conjecture and applying the zero density result of Kowalski–Michel, we choose subfamilies of L-functions that are zero-free close to 1. For these subfamilies, the L-functions have the extremal value at $s = 1$, and by the class number formula, we obtain the extreme class numbers.

1 Introduction

Let $\mathfrak{K}(n, G, r_1, r_2)$ be the set of number fields of degree n with signature (r_1, r_2) whose normal closures have G as their Galois group. Then by the class number formula, the class number h_K for $K \in \mathfrak{K}(n, G, r_1, r_2)$ is given by

$$h_K = \frac{w_K |d_K|^{\frac{1}{2}}}{2^{r_1}(2\pi)^{r_2} R_K} L(1, \rho), \tag{1.1}$$

where w_K is the number of roots of unity in K, d_K is the discriminant of K, R_K is its regulator, and $L(s, \rho) = \zeta_K(s)/\zeta(s)$ is the Artin L-function.

If K has at least one real embedding, $w_K = 2$. If K has no real embedding, then $\phi(w_K) \le n$. Since $\phi(m) \ge \sqrt{m}$, if $m \ne 2, 6$ (*cf.* [20, p. 9]), $w_K \le 4n^2$.

Silverman [23] obtained a lower bound of the regulator R_K of number fields K:

$$R_K > c_n (\log \gamma_n |d_K|)^{r - r_0}, \tag{1.2}$$

where c_n, γ_n are positive constants depending on the degree n of K, $r = r_1 + r_2 - 1$, and r_0 is the maximum of unit ranks of subfields of K.

It is easy to prove that under the Artin conjecture and Generalized Riemann Hypothesis (GRH) for $L(s, \rho)$, $L(1, \rho) \ll (\log\log |d_K|)^{n-1}$. (See the Remark 2.2.) Hence we obtain the conjectural upper bound for the class numbers

$$h_K \ll |d_K|^{\frac{1}{2}} \frac{(\log\log |d_K|)^{n-1}}{(\log |d_K|)^{r - r_0}}.$$

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Here the implied constants depend on n. Now the question is whether the upper bound is sharp. Namely, are there number fields with the largest possible class number of the size

$$|d_K|^{\frac{1}{2}} \frac{(\log\log|d_K|)^{n-1}}{(\log|d_K|)^{r-r_0}} \ ?$$

For real quadratic fields, this is a classical result of Montgomery and Weinberger [14]. Ankeny, Brauer, and Chowla [1] constructed unconditionally, for any n, r_1, r_2, number fields with arbitrarily large discriminants and $h_K \gg |d_K|^{1/2-\epsilon}$. Under the GRH and Artin conjecture for $L(s,\rho)$, Duke [6] constructed totally real fields of degree n whose Galois closures have the Galois group S_n with the largest possible class numbers. Daileda [7] showed Duke's result unconditionally when $n = 3$, and Cho [2] showed it when $n = 4$. He also proved that the Strong Artin Conjecture can replace the hypotheses of the Artin conjecture and the GRH in [6] when $n \geq 5$.

In this paper, we construct unconditionally a family of number fields with the largest possible class numbers. Namely,

(1) $n = 5$, $G = S_5$, $(r_1, r_2) = (1, 2)$,
(2) $n = 4$, $G = S_4$, $(r_1, r_2) = (2, 1)$,
(3) $n = 4$, $G = S_4$, $(r_1, r_2) = (0, 2)$,
(4) $n = 4$, $G = A_4$, $(r_1, r_2) = (0, 2)$.

Note that in all these cases, there are no nontrivial subfields. So $r_0 = 0$; it is clear when $n = 5$. If $G = A_4$, it is clear, since A_4 does not have a subgroup of order 6. If $G = S_4$, the subgroup of order 12 is A_4, but it does not contain a subgroup isomorphic to S_3.

The family of number fields will be obtained as $K_t = \mathbb{Q}[\theta_t]$, where θ_t is a root of $f(x,t) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t) \in \mathbb{Z}[t][x]$. As we see in the class number formula, we first need to construct a family of number fields with the smallest regulators, namely, $R_K \ll (\log|d_K|)^r$, and then find a subfamily for which $L(1,\rho) \gg (\log\log|d_K|)^{n-1}$. Under the assumption of the GRH of $L(s,\rho)$, this can be done by carefully analyzing the set of primes that split completely. Here we need the concept of regular extensions over $\mathbb{Q}(t)$ and use the result of Serre [22] that says that given a regular extension, with a specialization $t \in \mathbb{Z}$, we can choose a subfamily in which almost all small primes split completely. We also need to study square-free and cube-free values of certain polynomials in arithmetic progressions. In particular, in the A_4 case, we need the fact that the number of $1 \leq t < X$, $t \equiv t_M \pmod{M}$ such that $1 + 27t^4$ is cube-free, is $c\frac{X}{M} + O\left(X/M(\log\frac{X}{M})^{3/5}\right)$ for some constant c. This follows easily from the result of Hooley [8, p. 69].

In the absence of the GRH, we have to use the zero density result of [13] to show that in a family of automorphic L-functions, every L-function outside a negligible set is zero-free in a desired region. For this, we need to show that $L(s,\rho)$ is an automorphic L-function of $GL_{n-1}/\mathbb{Q}$. This is called the Strong Artin Conjecture. For the case where $G = S_5$, Calegari [4] showed the modularity of ρ. In the case of $G = A_4$ (resp. S_4), ρ is equivalent to a twist of $\mathrm{Sym}^2(\sigma)$ by a character, where σ is the 2-dimensional representation of $\widetilde{A}_4 \simeq SL_2(\mathbb{F}_3)$, (*resp.* $\widetilde{S}_4 \simeq GL_2(\mathbb{F}_3)$). So ρ is modular. (See [2] for details.)

Here we note that $L(1,\rho) \gg (\log\log|d_K|)^{n-1}$ does not hold when the number field does not come from regular extensions. For example, Daileda [7] showed that for a pure cubic extension given by $f(x,t) = x^3 - t$, $L(1,\rho) \ll \log\log|d_K|$.

In a separate paper [3], we study dihedral and cyclic extensions with large class numbers. In this case, the representation ρ is no longer attached to a cuspidal automorphic representation. We modify the result of [13] to apply in this case. Also, the existence of subfields makes it difficult to obtain number fields with the sharp bound (1.2).

We have not been able to find a family of quintic extensions in $\mathfrak{K}(5, A_5, 1, 2)$ with the smallest possible regulators. (See Remark 8.6.) We also note that if the discriminant is positive, the number of complex roots of a polynomial in $\mathbb{R}[X]$ is a multiple of 4. Hence there are no number fields of type $\mathfrak{K}(5, A_5, 3, 1)$ and $\mathfrak{K}(4, A_4, 2, 1)$. It would be of interest to construct totally real number fields whose Galois closures have A_4 and A_5 as their Galois groups with the largest possible class numbers.

2 Approximation of $L(1,\rho)$ and Zero density Result

We use the following result of Daileda [7], giving the approximation of $\log L(1,\rho)$ as a sum over small primes. Let ρ be an l-dimensional complex representation of a Galois group. We assume that $L(s,\rho)$ is an entire Artin L-function and let N be its conductor. Also, $L(s,\rho)$ has a Dirichlet series

$$L(s,\rho) = \sum_{n=1}^{\infty} \lambda(n) n^{-s}.$$

Proposition 2.1 ([7]) *Let $L(s,\rho)$ and N be as above. Let $6/7 < \alpha < 1$. Suppose that $L(s,\rho)$ is zero-free in the rectangle $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$. If N is sufficiently large, then for any $0 < k < 16/(1-\alpha)$,*

$$\log L(1,\rho) = \sum_{p \leqslant (\log N)^k} \lambda(p) p^{-1} + O_{l,k,\alpha}(1).$$

Remark 2.2 This implies immediately under GRH that $L(1,\rho) \ll (\log\log N)^l$, since $|\lambda(p)| \leq l$ and $\sum_{p\leq x} 1/p = \log\log x + O(1)$.

Due to lack of the GRH, we cannot use the above result directly. We use the following zero density result of Kowalski–Michel to show that in a family of automorphic L-functions, every L-function outside a negligible set is zero-free in a desired region. Let $n \geqslant 1$ be a fixed integer. For all $q \geqslant 1$, let $S(q)$ be a finite set of cuspidal automorphic representations of $GL_n/\mathbb{Q}$ that satisfy the following conditions:

1. The forms $f \in S(q)$ satisfy the Ramanujan–Petersson conjecture at the finite places.
2. There exists $e > 0$ such that for all $f \in S(q)$, the conductor $\mathrm{Cond}(f)$ of f satisfies

$$\mathrm{Cond}(f) \leqslant q^e$$

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3. There exists $d > 0$ such that $|S(q)| \ll q^d$ for all $q \geqslant 1$, the implied constant depending on the family.
4. All the $f \in S(q)$ have the same component at ∞, hence the same gamma factor in the functional equation.

For any cuspidal representation f on $GL_n/\mathbb{Q}$, $\alpha \in \mathbb{R}$, and $T \geq 0$, we let

$$N(f, \alpha, T) = \Big| \big\{ \rho \mid L(f, \rho) = 0, Re(\rho) \geq \alpha, |Im(\rho)| \leq T \big\} \Big|$$

(zeros counted with multiplicity).

Proposition 2.3 ([13]) *Let c_0 be a constant with $c_0 > 5ne/2 + d$. Let $\alpha \geqslant 3/4$ and $T \geqslant 2$. Then*

$$\sum_{f \in S(q)} N(f, \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{2\alpha-1}}$$

for all $q \geqslant 1$ and some $B \geqslant 0$ (depending on the family). The implied constant depends only on the family.

3 Regular Extensions and their Galois Representations

A finite Galois extension E of the rational function field $\mathbb{Q}(t)$ is called regular if $\overline{\mathbb{Q}} \cap E = \mathbb{Q}$. This is equivalent to the fact that $\mathrm{Gal}(E\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(t)) \simeq \mathrm{Gal}(E/\mathbb{Q}(t))$. Suppose that

$$f(x, t) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t) \in \mathbb{Z}[t][x]$$

is an irreducible polynomial of degree n and gives rise to a regular Galois extension over $\mathbb{Q}(t)$ with the Galois group G. Let K_t be a field obtained by adjoining to $\mathbb{Q}$ a root of $f(x, t)$ with a specialization $t \in \mathbb{Z}$ and let $\widehat{K}_t$ be the Galois closure of K_t. Let C be any conjugacy class of G. Serre observed the following important fact, regarding distribution of Frobenius elements in a regular Galois extension [22, p. 45].

Theorem 3.1 *There is a constant $c_f > 0$ depending on f such that for any prime $p \geq c_f$, there is $t_C \in \mathbb{Z}$ so that for any $t \equiv t_C \pmod{p}$, p is unramified in $\widehat{K}_t/\mathbb{Q}$, and* $\mathrm{Frob}_p \in C$.

We write

$$L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)},$$

where $\rho\colon G \to GL_{n-1}(\mathbb{C})$, $H = \mathrm{Gal}(\widehat{K}_t/K_t)$, and $\mathrm{Ind}_H^G 1_H = 1_G \oplus \rho$.

Conjecture 3.2 (Strong Artin Conjecture) *Let ρ be as above. Then ρ is modular, namely, $L(s, \rho, t)$ is an automorphic L-function of $GL_{n-1}/\mathbb{Q}$.*

4 Extreme Class Numbers

In this section, we explain how to obtain the extreme class numbers in a general setting. Let G be a finite group and let $f(x, t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree n whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group G. Let K_t, $\widehat{K}_t$, and $L(s,\rho,t)$ be as in Section 3. The conductor of $L(s,\rho,t)$ is $|d_{K_t}|$. Let

$$L(s,\rho,t)=\sum_{n=1}^{\infty}\lambda_t(n)n^{-s},$$

where $\lambda_t(p) = N_t(p) - 1$ and $N_t(p)$ is the number of solutions of $f(x,t) \equiv 0 \pmod{p}$. Hence, $-1 \le \lambda_t(p) \le n-1$.

Here we restrict ρ to be irreducible in order to apply Proposition 2.3. If G is dihedral or cyclic, ρ is no longer irreducible. In this case, we need to modify Proposition 2.3. We treat this case in a forthcoming paper [3].

We assume Conjecture 3.2 (the Strong Artin Conjecture) for $L(s,\rho,t)$. We expect that the regular Galois extension property implies that the absolute value of a field discriminant will increase with respect to t with a specialization $t \in \mathbb{Z}$.

Assumption 4.1 *If $|t|$ is sufficiently large,* $\log|d_{K_t}| \gg_f \log|t|$.

By the class number formula (1.1) and regulator bound (1.2), we first need to construct a family of number fields with the smallest regulators $R_{K_t} \ll (\log|d_{K_t}|)^{r-r_0}$. Next, we need to make $L(1,\rho,t)$ largest possible. In light of Proposition 2.1, we need to choose t such that $\lambda_t(p) = n-1$ for almost all $p \leqslant (\log|d_{K_t}|)^k$ for some k. Namely, we need to choose t such that almost all $p \leqslant (\log|d_{K_t}|)^k$ split completely in K_t.

Since $f(x,t)$ gives rise to a regular extension over $\mathbb{Q}(t)$, by Theorem 3.1, there is an integer c_f such that for all prime numbers $q \ge c_f$, there is an integer t_q so that for any $t \equiv t_q \pmod{q}$, q splits completely in $\widehat{K}_t$. Now, for given $X \gg 0$, define

$$y=\frac{\log X}{\log\log X} \quad \text{and} \quad M=\prod_{c_f\le q\le y} q.$$

Let t_M be an integer such that $t_M \equiv t_q \pmod{q}$ for all $c_f \le q \le y$. Here $\log M \sim y$, and hence $M \ll X^{\epsilon}$ for any $\epsilon > 0$.

Assume that the discriminant of $f(x,t)$ is a polynomial in t of degree D, and $|d_{K_t}| \le Ct^D$ for some constant C. We define a set $L(X)$ of positive numbers given by

$$L(X)=\left\{\frac{X}{2}<t<X \mid\ t\equiv t_M \pmod{M}, \operatorname{Gal}(\widehat{K}_t/\mathbb{Q})\simeq G\right\}.$$

Under the Strong Artin Conjecture, every t in $L(A)$ gives rise to an automorphic L-function of GL_{n-1} over $\mathbb{Q}$. However, it is possible that different t in $L(A)$ may give rise to the same L-function, namely, $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$. In that case, we say that K_{t_1} and K_{t_2} are arithmetically equivalent. We make the following assumption.

Assumption 4.2 *There exists an integer $m > 0$, depending only on f, such that there are at most m values of t giving rise to the same L-function.*

In order to verify the assumption, we use the following theorem.

Theorem 4.3 (Klingen [12]) *Let $K/\mathbb{Q}$ be a number field of degree $n \leq 11$. Let $\widehat{K}$ be the Galois closure and assume that there exists a non-conjugate field K' which is arithmetically equivalent to K. Then up to conjugacy, only the following 4 cases are possible for $G = \mathrm{Gal}(\widehat{K}/\mathbb{Q})$:*

(i) $n = 7$, $G = GL_3(2)$;
(ii) $n = 8$, $G = \mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/8\mathbb{Z})^{\times}$;
(iii) $n = 8$, $G = GL_2(3)$;
(iv) $n = 11$, $G = PSL_2(11)$.

We will prove that there exists $m > 0$ such that there are at most m isomorphic K_t's for $t \in L(X)$. Hence by the above theorem, they are not arithmetically equivalent and Assumption 4.2 is verified.

Let $\widetilde{L}(X)$ be the set of automorphic L-functions coming from $L(X)$ after removing the possible repetition of the same L-functions among them. In Sections 5 through 8, we consider explicit examples of families of number fields. In those cases, we may have to put more conditions in $L(X)$ in order to satisfy Assumption 4.2, or replace it by some other set. In any case, we show that $X^{1-\epsilon} \ll |\widetilde{L}(X)| \ll X$ for any fixed $\epsilon > 0$.

Let $c_0 = 5(n-1)D/2 + 1$. Choose α with $c_0(1-\alpha)/(2\alpha - 1) < 98/100$. By applying Proposition 2.3 to $\widetilde{L}(X)$ with $e = D$, $d = 1$, and $T = (\log CX^D)^2$, every automorphic L-function excluding exceptional $O(X^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log|d_{K_t}|)^2, \log|d_{K_t}|)^2]$. Let us denote by $\widehat{L}(X)$ the set of the automorphic L-functions with the zero-free region.

Applying Proposition 2.1 to L-functions in $\widehat{L}(X)$, we have

$$\begin{aligned}\log L(1,\rho,t) &= \sum_{q \leqslant (\log|d_{K_t}|)^{1/2}} \lambda_t(q) q^{-1} + O_{n,\alpha}(1) \\ &= (n-1) \sum_{c_f \leqslant q \leqslant (\log|d_{K_t}|)^{1/2}} q^{-1} + O_{n,\alpha}(1) \\ &= (n-1)\log\log\log|d_{K_t}| + O_{n,\alpha}(1),\end{aligned}$$

where we use the fact that $(\log|d_{K_t}|)^{1/2} \leqslant y = \log X/\log\log X$ for large X. So we have $L(1,\rho,t) \gg (\log\log|d_{K_t}|)^{n-1}$. Hence we have the required result

$$h_{K_t} \gg |d_{K_t}|^{\frac{1}{2}} \frac{(\log\log|d_{K_t}|)^{n-1}}{(\log|d_{K_t}|)^{r-r_0}}.$$

5 S_5 Extensions with Signature (1,2)

Let

$$f(x,t) = (x+t)(x^2+5t)(x^2+10t) + t$$

with the discriminant

$$\begin{aligned}\mathrm{Disc}(f(x,t)) = t^4\big(&500000t^{10} + 15000000t^9 + 162350000t^8 + 746700000t^7 \\ &+ 1234759600t^6 + 7714500t^5 - 394744t^4 + 5143500t^3 + 162500t^2 + 3125\big).\end{aligned}$$

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For a non-zero integer t, $f(x,t)$ has one real root and four complex roots.

We claim that the Galois groups of $f(x,t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both S_5. Since $f(x,t)$ is an Eisenstein polynomial for an irreducible element t, it is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ and it is clear that $\mathrm{Disc}(f(x,t))$ is not an square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. If the sextic resolvent has no root in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, then the Galois group is S_5 over both fields. The sextic resolvent of $f(x,t)$ is given by

$$\theta_t(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10}\,\mathrm{Disc}(f(x,t))y$$

where

$$b_2 = 5t^2(24t - 335), \quad b_4 = t^3(400t^3 - 192000t^2 + 661811t - 2400),$$
$$b_6 = 5^2t^3(12400t^5 + 3069000t^4 + 17775t^3 + 168480t^2 - 64t + 2400).$$

If α, a divisor of b_6^2, is a root of $\theta_t(y)$, then

$$(\alpha^3 + b_2\alpha^2 + b_4\alpha + b_6)^2 = 2^{10}\,\mathrm{Disc}(f(x,t))\alpha. \tag{5.1}$$

Since the RHS of (5.1) cannot be a square, it is a contradiction. Hence $f(x,t)$ gives rise to an S_5 regular extension over $\mathbb{Q}(t)$.

Recently, Calegari obtained the modularity of S_5 Galois representations for a special case.

Theorem 5.1 (Calegari [4]) *Let $K/\mathbb{Q}$ be a quintic extension with $\mathrm{Gal}(\widehat{K}/\mathbb{Q}) = S_5$. Furthermore, we assume that*

(i) *the complex conjugation in $\mathrm{Gal}(\widehat{K}/\mathbb{Q}) = S_5$ has the conjugacy class $(12)(34)$;*
(ii) *the extension $\widehat{K}/\mathbb{Q}$ is unramified at 5 and the Frobenius element Frob_5 has the conjugacy class $(12)(34)$.*

If $\rho\colon \mathrm{Gal}(\widehat{K}/\mathbb{Q}) \to GL_4(\mathbb{C})$ is an irreducible representation of dimension 4, then ρ is modular.

Remark 5.2 Calegari observed that the 4-dimensional representation ρ is equivalent to a twist of $As(\sigma)$ by a character, where σ is the 2-dimensional icosahedral representation of $\widetilde{A}_5$ over the quadratic subextension F and As is the Asai lift. He then used the modularity of σ proved by Sasaki [19]. In his thesis [24], Y. Zhang also observed the fact that ρ is twist equivalent to $As(\sigma)$.

Let $K_t = \mathbb{Q}(\theta_1)$ be a quintic field obtained by adjoining the real root θ_1 of $f(x,t)$ to the rational number field $\mathbb{Q}$. We assume that $t \equiv 1 \pmod 5$. Then

$$f(x,t) \equiv x^4(x+1) + 1 \equiv (x+2)(x^2+x+1)(x^2+3x+3) \pmod 5,$$

and the signature of K_t is $(1,2)$. Hence the Galois extensions $\widehat{K}_t/\mathbb{Q}$ satisfy the hypotheses of Theorem 5.1, and Artin L-functions $L(s,\rho,t) = \zeta_{K_t}(s)/\zeta(s)$ are cuspidal automorphic L-functions of $GL_4/\mathbb{Q}$.

We claim that $(\theta_1+t)^5/t$ and $(\theta_1^2+5t)^5/t^2$ are two independent units in K_t. Since $f(x,t) = x^5 + tx^4 + 15tx^3 + 15t^2x^2 + 50t^2x + 50t^3 + t$,

$$\frac{\theta_1^5}{t} = -(\theta_1^4 + 15\theta_1^3 + 15t\theta_1^2 + 50t\theta_1 + 50t^2 + 1).$$

Hence θ_1^5/t is an algebraic integer. From this, it is easy to show that $(\theta_1+t)^5/t$ and $(\theta_1^2+5t)^5/t^2$ are algebraic integers. Now we have

$$\frac{(\theta_1+t)^5}{t}\cdot\frac{(\theta_1^2+5t)^5}{t^2}\cdot\frac{(\theta_1^2+10t)^5}{t^2} = -1.$$

Hence $(\theta_1+t)^5/t$ and $(\theta_1^2+5t)^5/t^2$ are units. To show that they are independent, we need to know the locations of 5 roots of $f(x,t)$. For the case of the real root θ_1, we have $-t-1/t < \theta_1 < -t$. For complex roots, we use the following lemma from [17, p. 9]. See also [21].

Lemma 5.3 *Let f be a polynomial of degree m and $f(\alpha)\neq 0$, $f'(\alpha)\neq 0$. Then for every circle C passing through α, $\alpha - mf(\alpha)/f'(\alpha)$, at least one root of f is inside C, and one root outside of C.*

We apply Lemma 5.3 to $f(x,t)$ with $\alpha = i\sqrt{5t}$, then we can see that another root θ_2 of $f(x,t)$ converges to $i\sqrt{5t}$ as t increases. More precisely,

$$\left|\theta_2 - i\sqrt{5t}\right| = O\Big(\frac{1}{t^{1.5}}\Big).$$

Put $\theta_3 = \overline{\theta_2}$. Apply Lemma 5.3 again to $f(x,t)$ with $\alpha = i\sqrt{10t}$. Then we can find the fourth root θ_4 with

$$\left|\theta_4 - i\sqrt{10t}\right| = O\Big(\frac{1}{t^{1.5}}\Big)$$

and put $\theta_5 = \overline{\theta_4}$.

Assume that $(\theta_1+t)^5/t$ and $(\theta_1^2+5t)^5/t^2$ are dependent. Then $((\theta_1+t)^5/t)^k = ((\theta_1^2+5t)^5/t^2)^l$ for some integers k,l. Then it holds when we replace θ_1 by θ_4, and when we consider the size of θ_1,θ_4, we obtain a contradiction. By definition,

$$R_{K_t} \ll \left|\det\begin{pmatrix}\log\left|\frac{(\theta_1+t)^5}{t}\right| & \log\left|\frac{(\theta_4+t)^5}{t}\right|\\ \log\left|\frac{(\theta_1^2+5t)^5}{t^2}\right| & \log\left|\frac{(\theta_4^2+5t)^5}{t^2}\right|\end{pmatrix}\right|.$$

By the above estimates on θ_1,θ_4, it is clear that $R_{K_t} \ll (\log t)^2$. Since $f(x,t)$ is an Eisenstein polynomial, d_{K_t} is divisible by t^4 if t is square-free. (See [16, p. 60].) Hence $\log d_{K_t} \gg \log t$, and Assumption 4.1 is verified. We have proved the following lemma.

Lemma 5.4 *For a square-free positive integer t, $R_{K_t} \ll (\log d_{K_t})^2$.*

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As described in Section 4, we define a set $L(X)$ of square-free integers:

$$L(X) = \left\{ \frac{X}{2} < t < X \mid t \equiv t_M \pmod{M} \text{ and } t \text{ square-free} \right\}.$$

Each t in $L(X)$ gives rise to an automorphic L-function $L(s, \rho, t) = \zeta_{K_t}(s)/\zeta(s)$ of $GL_4/\mathbb{Q}$. We claim that for a square-free integer t,

$$p \text{ is totally ramified in } K_t \iff p \text{ divides } t.$$

It is known that those primes dividing t totally ramify in K_t. (See [5, Corollary 6.2.4].) Assume that p totally ramifies but does not divide t. This means that $f(x,t) \equiv x^5$ or $(x+a)^5 \pmod{p}$ for $p \nmid t$. We can induce a contradiction by comparing coefficients modulo p. Hence Assumption 4.2 is also verified.

On the other hand, by [7, p. 248], $|L(X)| = c\frac{X}{2M} + O(X^{1/2})$ where

$$c = \frac{6}{\pi^2} \prod_{p|M} (1 - p^{-2})^{-1} \quad \text{and} \quad X^{1-\epsilon} \ll |L(X)| \ll X.$$

Hence by the argument in Section 4, we have the result. We summarize it as follows:

Theorem 5.5 *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(5, S_5, 1, 2)$ with arbitrarily large discriminant d_K for which*

$$h_K > c\, d_K^{\frac{1}{2}} \frac{(\log\log d_K)^4}{(\log d_K)^2}.$$

6 S_4 Extensions with Signature $(2,1)$

Let $t > 1$ be a positive square-free integer and $f(x,t) = x^2(x - 10t)(x - 18t) + t$. Then the discriminant of $f(x,t)$ is

$$\text{Disc}(f(x,t)) = -256t^3(12t+1)(15t-1)(144t^2 - 12t + 1)(225t^2 + 15t + 1) < 0.$$

Since $f(x,t)$ is an Eisenstein polynomial for each prime divisor of t, d_{K_t} is divisible by t^3, and assumption 4.1 is verified. The cubic resolvent $y^3 - 180t^2y^2 - 4ty - 64t^3$ of $f(x,t)$ has the only real root between $180t^2$ and $180t^2 + 1$. Hence $f(x,t)$ gives rise to an S_4 Galois extension for each positive square-free integer t.

Consider $f(x,t) = x^2(x-10t)(x-18t)+t$ over $\overline{\mathbb{Q}}(t)$. It is easy to see that the cubic resolvent $y^3 - 180t^2y^2 - 4ty - 64t^3$, is irreducible over $\overline{\mathbb{Q}}[t]$. By Gauss' Lemma, it is irreducible over $\overline{\mathbb{Q}}(t)$. Hence the Galois group of $f(x,t)$ over $\overline{\mathbb{Q}}(t)$ is S_4. Therefore, $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$.

Note that $f'(x,t) = 4x(x-6t)(x-15t)$, and we can see easily that $f(x,t)$ has two real roots θ_1, θ_2 and two complex roots $\theta_3, \theta_4 = \overline{\theta_3}$. For sufficiently large t, we can see that $10t + 1/t^3 < \theta_1 < 10t + 1/t^2$, $18t - 1/t^2 < \theta_2 < 18t - 1/t^3$. Also by taking $\alpha = 1/t$ and applying Lemma 5.3, we can see that θ_3 and its conjugate θ_4 are inside the circle of radius $1/9$ centered at the origin.

Let $K_t = \mathbb{Q}[\theta_1]$. Then we prove the following lemma.

Lemma 6.1 *θ_1^4/t and $(\theta_1-10t)^4/t$ are independent units in $\mathbb{Z}_{K_t}$.*

Proof Since $\theta_1^4-28t\theta_1^3+180t^2\theta_1^3+t=0$, $\theta_1^4/t=28\theta_1^3-180t\theta_1^2-1$. Hence θ_1^4/t is an algebraic integer. Also

$$\frac{(\theta_1-10t)^2(\theta_1-18t)^2}{t}=\frac{(\theta_1^2-t(28\theta_1-180t))^2}{t}$$
$$=\frac{\theta_1^4}{t}-2\theta_1^2(28\theta_1-180t)^2+t(28\theta_1-180t)^2.$$

So $((\theta_1-10t)^2(\theta_1-18t)^2)/t$ is an algebraic integer. Now we have

$$\frac{\theta_1^4}{t}\cdot\frac{(\theta_1-10t)^2(\theta_1-18t)^2}{t}=1.$$

Hence θ_1^4/t is a unit. By considering $y=x-10t$ or $y=x-18t$, we can see that $(\theta_1-10t)^4/t$ and $(\theta_1-18t)^4/t$ are algebraic integers. We have

$$\frac{\theta_1^8}{t^2}\cdot\frac{(\theta_1-10t)^4}{t}\cdot\frac{(\theta_1-18t)^4}{t}=1.$$

Hence $(\theta_1-10t)^4/t$ is a unit.

Assume that θ_1^4/t and $(\theta_1-10t)^4/t$ are dependent. Then

$$\left(\frac{\theta_1^4}{t}\right)^k=\left(\frac{(\theta_1-10t)^4}{t}\right)^m$$

for some integers k and m. Without loss of generality, we can assume that k is positive. When we consider the size of θ_1, m should be a negative integer. But when we replace θ_1 by θ_2, m should be a positive integer, which induces a contradiction. ■

Lemma 6.2 *For positive square-free t, $R_{K_t}\ll(\log|d_{K_t}|)^2$.*

Proof By definition,

$$R_{K_t}\le\left|\det\begin{pmatrix}\log\left|\frac{\theta_1^4}{t}\right| & \log\left|\frac{\theta_2^4}{t}\right|\\ \log\left|\frac{(\theta_1-10t)^4}{t}\right| & \log\left|\frac{(\theta_2-10t)^4}{t}\right|\end{pmatrix}\right|.$$

By the above estimates on θ_1,θ_2, it is clear that $R_{K_t}\ll(\log t)^2$. Since $t^3\mid d_{K_t}$, we have proved the claim. ■

As described in Section 4, we construct a set $L(X)$ of square-free integers

$$L(X)=\left\{\frac{X}{2}<t<X \mid t\text{ square-free and odd, } t\equiv t_M \pmod M\right\}.$$

As in the Section 5, we can see that $X^{1-\epsilon}\ll|L(X)|\ll X$.

Let $L(s,\rho,t)=\zeta_{K_t}(s)/\zeta(s)$. Then ρ is equivalent to a twist of $\mathrm{Sym}^2(\sigma)$ by a character, where σ is the 2-dimensional representation of $\widetilde{S}_4\simeq GL_2(\mathbb{F}_3)$. Hence ρ is modular and $L(s,\rho,t)$ is a cuspidal automorphic L-function of $GL_3/\mathbb{Q}$. Hence Conjecture 3.2 is true for this case. See [2] for details.

Now we claim that Assumption 4.2 holds, as a consequence of the following lemma, which implies that K_{t_1} and K_{t_2} are not isomorphic if $t_1\neq t_2$.

Lemma 6.3 *For a square-free odd integer t, $p \neq 2$ is totally ramified in K_t if and only if p divides t.*

Proof Since $f(x,t)$ is an Eisenstein polynomial, if $p \mid t$, p is totally ramified. See [5, Corollary 6.2.4]. Conversely, suppose that p is totally ramified and does not divide t. Then $f(x,t) \equiv (x+a)^4 \pmod{p}$. By comparing the coefficients of $f(x,t)$ and $(x+a)^4 \pmod{p}$, we obtain a contradiction. ∎

We have shown that Assumptions 4.1, 4.2, and Conjecture 3.2 hold. Hence we can summarize our result as follows.

Theorem 6.4 *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, S_4, 2, 1)$ with arbitrarily large discriminant d_K for which*

$$h_K > c\,|d_K|^{\frac{1}{2}} \frac{(\log\log|d_K|)^3}{(\log|d_K|)^2}.$$

7 S_4 Extensions with Signature $(0,2)$

Let $t > 1$ be a positive square-free integer and $f(x,t) = x^4 + tx^2 + tx + t$. Then the discriminant $\mathrm{Disc}(f(x,t))$ of $f(x,t)$ is $t^3(12t^2 - 11t + 256)$. Since $f(x,t)$ is an Eisenstein polynomial for each prime divisor of t, d_{K_t} is divisible by t^3, hence Assumption 4.1 is verified.

The cubic resolvent $y^3 - ty^2 - 4ty + 3t^2$ of $f(x,t)$ has three real roots. One of them is located between $t+1$ and $t+2$, hence it is not an integer. So if the cubic resolvent has an integer root, we can show that the integer root should be divisible by t. Since $\pm t, \pm 3t, \pm t^2$, and $\pm 3t^2$ are not a root of the cubic resolvent, the cubic resolvent is irreducible. Hence $f(x,t)$ gives rise to an S_4 Galois extension for each positive square-free integer t.

Consider $f(x,t) = x^4 + tx^2 + tx + t$ over $\overline{\mathbb{Q}}(t)$. It is easy to see that the cubic resolvent $y^3 - ty^2 - 4ty + 3t^2$ is irreducible over $\overline{\mathbb{Q}}[t]$. By Gauss' Lemma, it is irreducible over $\overline{\mathbb{Q}}(t)$. Hence the Galois group of $f(x,t)$ over $\overline{\mathbb{Q}}(t)$ is S_4. Therefore, $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$.

Note that $f'(x,t) = 4x^3 + 2tx + t$ has only one real root x_0, and we can easily check that $f(x_0) > 0$. Hence $f(x,t)$ has four complex roots θ_1, $\theta_2 = \overline{\theta_1}$, θ_3, and $\theta_4 = \overline{\theta_3}$. For sufficiently large t, when we apply Lemma 5.3 with $\alpha = i\sqrt{t}$, we can see that one root lies inside the circle of radius 1 centered at $1 + i\sqrt{t}$. Let θ be the root.

Let $K_t = \mathbb{Q}[\theta]$. Then since $\theta^4/t = -(\theta^2 + \theta + 1)$, θ^4/t is an algebraic integer. Here $N_{K_t/\mathbb{Q}}(\theta) = t$. Hence θ^4/t has norm 1, and it is a unit in $\mathbb{Z}_{K_t}$.

Since $|\theta| \ll \sqrt{t}$, $\log|\theta^4/t| \ll \log t$. Since $t^3 \mid d_{K_t}$, we have the following lemma.

Lemma 7.1 *For positive square-free t, $R_{K_t} \ll \log d_{K_t}$.*

As described in Section 4, we construct a set $L(X)$ of square-free integers,

$$L(X) = \left\{ \frac{X}{2} < t < X \mid\ t \text{ square-free and } t \equiv t_M \pmod{M} \right\},$$

and we can see that $X^{1-\epsilon} \ll |L(X)| \ll X$.

Also, as in the previous sections, we can show that p is totally ramified in K_t if and only if $p \mid t$. Hence K_{t_1} and K_{t_2} are not isomorphic if $t_1 \neq t_2$, and $L(s, \rho, t)$'s are distinct. Therefore, Assumption 4.2 is verified, and we obtain the following theorem.

Theorem 7.2 *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, S_4, 0, 2)$ with arbitrarily large discriminant d_K for which*

$$h_K > c\, d_K^{\frac{1}{2}} \frac{(\log\log d_K)^3}{\log d_K}.$$

8 A_4 Extensions with Signature (0,2)

Consider $f(x,t) = x^4 - 8tx^3 + 18t^2x^2 + 1$, which is considered in [22, p. 44]. Note that $f'(x,t) = 4x(x-3t)^2$. Then $\mathrm{Disc}(f(x,t)) = 16^2(27t^4+1)^2$. We claim that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is a regular extension with Galois group A_4. It is enough to show that the Galois group of $f(x,t)$ over $\overline{\mathbb{Q}}(t)$ is A_4. First, $f(x,t)$ is irreducible over $\overline{\mathbb{Q}}(t)$. By Gauss' Lemma, it is enough to check it over $\overline{\mathbb{Q}}[t]$. It is easy to check that $f(x,t)$ has no root in $\overline{\mathbb{Q}}[t]$. If $f(x,t)$ is a product of two quadratic polynomials, then

$$x^4 - 8tx^3 + 18t^2x^2 + 1 = (x^2 + bx + c)\Big(x^2 + dx + \frac{1}{c}\Big)$$

for some $b, d \in \overline{\mathbb{Q}}[t]$ and $c \in \overline{\mathbb{Q}}$. We can induce a contradiction easily.

Its Ferrari resolvent $\theta(y)$ is $y^3 - 18t^2y^2 - 4y + 8t^2$, and it is irreducible over $\overline{\mathbb{Q}}[t]$. Since the discriminant of $f(x,t)$ is a square in $\overline{\mathbb{Q}}(t)$, the Galois group over $\overline{\mathbb{Q}}(t)$ is A_4.

We can see easily that $f(x,t)$ has 4 complex roots. If θ_t is a root, it is a unit. Let $K_t = \mathbb{Q}[\theta_t]$.

Proposition 8.1 *The regulator R_{K_t} satisfies $R_{K_t} \ll \log t$.*

Proof By considering $\alpha = 6t$ in Lemma 5.3, we can see that

$$2t - \frac{1}{54t^3} < |\theta_t| < 6t.$$

Therefore, $R_{K_t} \leq 2\log|\theta_t| \ll \log t$. ∎

Note that $f(x,t) = (x+t)(x-3t)^3 + 27t^4 + 1$. Take t such that $27t^4+1$ is cube-free. Let $p \mid (27t^4+1)$, $p > 3$. Then $p \nmid t$ and $f(x,t) \equiv (x+t)(x-3t)^3 \mod p$. The vertices of the Newton polygon with respect to $x - 3t$ are (0,0), (1,0), $(4, i)$ with $i = 1, 2$. By Cohen [5, p. 315], $p\mathbb{Z}_{K_t} = \mathfrak{p}_1\mathfrak{p}_2^3$, with prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$. Hence $p \mid d_{K_t}$. Therefore, $d_{K_t} \geq \prod_{p \mid (27t^4+1)} p$.

But $27t^4 + 1 \leq \big(\prod_{p\mid(27t^4+1)} p\big)^2$. Hence $d_{K_t} \gg t^2$, and Assumption 4.1 is verified. We have proved the following Proposition.

Proposition 8.2 *If $27t^4+1$ is cubic free, $R_t \ll \log d_{K_t}$.*

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As described in Section 4, we consider a set $L(X)$ of cube-free integers

$$L(X) = \left\{ \frac{X}{2} < t < X : \ 27t^4 + 1 \text{ cube-free and } t \equiv t_M \pmod{M} \right\}.$$

We prove the following lemma, which is a direct consequence of [8, p. 69].

Lemma 8.3 *Let $f(x)$ be an irreducible polynomial of degree $d \geq 3$ in $\mathbb{Z}[x]$. Let M be a positive integer and $\gcd(a, M) = 1$. Suppose that if $p \mid M$, then $f(a) \not\equiv 0 \pmod{p}$. Let $N(X, f, M)$ be the number of integers $1 \leq n < X$ and $n \equiv a \pmod{M}$, with the property that $f(n)$ is $(d-1)$-free. Then*

$$N(X, f, M) = C(M)\frac{X}{M} + O\Big(\frac{X}{M}\Big(\log\frac{X}{M}\Big)^{\frac{2}{d+1}-1}\Big),$$

where $C(M) = \prod_{p \nmid M}(1 - \rho(p^{d-1})/p^{d-1})$, and $\rho(p^k)$ is the number of solutions for $f(x) \equiv 0 \pmod{p^k}$.

Proof Let $n = Mm + a$, and $g(m) = f(Mm + a)$. Then $1 \leq m < X/M$, and $g(x)$ is an irreducible polynomial of degree d. Hooley [8, p. 69] showed that the number of $1 \leq m < X/M$ with the property that $g(m)$ is $(d-1)$-free is

$$C(M)\frac{X}{M} + O\Big(\frac{X}{M}\Big(\log\frac{X}{M}\Big)^{\frac{2}{d+1}-1}\Big),$$

where

$$C(M) = \prod_{p \nmid M}\Big(1 - \frac{\rho'(p^{d-1})}{p^{d-1}}\Big),$$

and $\rho'(p^{d-1})$ is the number of solutions for $g(m) \equiv 0 \pmod{p^{d-1}}$. If $p \mid M$, $g(m) \equiv f(a) \not\equiv 0 \pmod{p}$. Hence $\rho'(p^{d-1}) = 0$. If $p \nmid M$, then since $Mm + a \equiv 0 \pmod{p^{d-1}}$ has a unique solution mod p^{d-1}, $\rho'(p^{d-1}) = \rho(p^{d-1})$. Our result follows. ∎

Note that by the definition of M, t_M, if $p \mid M$, then $c_f \leq p \leq y$, $t_M \equiv t_p \pmod{p}$, and p splits completely in $\widehat{K}_{t_p}$. Then $27(Mm + t_M)^4 + 1 \equiv 27t_p^4 + 1 \not\equiv 0 \pmod{p}$. Hence, $\rho(p) = 0$. This implies that $\rho(p^3) = 0$. If $p \nmid M$, by Nagell [15, p. 87], $\rho(p^3) = \rho(p) \leq 4$. So

$$\prod_{p \nmid M}\Big(1 - \frac{\rho(p^3)}{p^3}\Big) \gg \prod_{p \nmid M}(1 - 4p^{-3}) \geq \prod_{p \nmid M}(1 - p^{-3})^5 \geq \zeta(3)^{-5}.$$

Hence, by the above lemma,

$$|L(X)| = C(M)\frac{X}{2M} + O\Big(\frac{X}{M(\log\frac{X}{M})^{\frac{3}{5}}}\Big) \quad \text{and} \quad |L(X)| \gg X^{1-\epsilon}.$$

In the remark below, we use the recent result of Heath-Brown [9] to obtain a better error term in $|L(X)|$.

Here different $t_1, t_2 \in L(X)$ may give rise to the same L-function. We need to know the locations of the roots more precisely to distinguish the L-functions. By applying Lemma 5.3 with $\alpha = 4t + 1.4ti$, for sufficiently large t, we find a complex root inside the circle of radius $0.03\sqrt{2}t$ centered at $4.015t+1.385ti$. Again by applying Lemma 5.3 with $\alpha = 0.23i/t$,for sufficiently large t, we find a complex root inside the circle of radius $0.0115/t$ centered at $0.2415i/t$.

We order the roots of $f(x,t)$ in the following way. Let θ_t^1 be the root near the origin whose imaginary part is positive and $\theta_t^2 = \overline{\theta_t^1}$. Let θ_t^3 be the other root whose imaginary part is positive and $\theta_t^4 = \overline{\theta_t^3}$. Let τ be the complex embedding of K_t that maps θ_t^1 to θ_t^3.

If $t_1, t_2 \in L(X)$ give rise to the same L-function, $\mathbb{Q}(\theta_{t_1}^1)$ and $\mathbb{Q}(\theta_{t_2}^1)$ are isomorphic, since they are quartic fields. Hence $\mathbb{Q}(\theta_{t_1}^1) = \mathbb{Q}(\theta_{t_2}^j)$ for some $1 \le j \le 4$. Assume that 33 different $t_1, t_2, \dots, t_{33}$ give rise to the same L-function. Then we can see that there are at least nine $t_{i_1}, t_{i_2}, \dots, t_{i_9}$ with $\mathbb{Q}(\theta_{t_{i_1}}^k) = \mathbb{Q}(\theta_{t_{i_2}}^k) = \cdots = \mathbb{Q}(\theta_{t_{i_9}}^k)$ for some $1 \le k \le 4$. Without loss of generality, we assume that $k = 1$. Then there are at least two t_{i_l}, t_{i_m} such that $\tau\colon \theta_{t_{i_l}}^1 \to \theta_{t_{i_l}}^3; \tau\colon \theta_{t_{i_m}}^1 \to \theta_{t_{i_m}}^3$. Now we further assume that $0.55X < t < X$. Then

$$N(\theta_{t_{i_l}}^1 - \theta_{t_{i_m}}^1) < \left(\frac{0.253}{0.55X}\right)^2 \times (4.277X - 4.199 \times 0.55X)^2 \approx 0.8232840 < 1.$$

Since $\theta_{t_{i_l}}^1 - \theta_{t_{i_m}}^1 \neq 0$, it induces a contraction. So there are at most 32 t's giving rise to the same L-function. Hence Assumption 4.2 holds. Let $\widetilde{L}(X)$ be the set of distinct L-functions coming from $L(X)$. By the above argument, we have

$$X^{1-\epsilon} \ll |\widetilde{L}(X)| \ll X.$$

Let $L(s,\rho,t) = \zeta_{K_t}(s)/\zeta(s)$. Then ρ is equivalent to a twist of $\mathrm{Sym}^2(\sigma)$ by a character, where σ is the 2-dimensional representation of $\widetilde{A}_4 \simeq SL_2(\mathbb{F}_3)$. Hence ρ is modular and $L(s,\rho,t)$ is the cuspidal automorphic L-function of $GL_3/\mathbb{Q}$. See [2] for the details. We have the following theorem.

Theorem 8.4 *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, A_4, 0, 2)$ with arbitrary large discriminant d_K for which*

$$h_K > c\, d_K^{\frac{1}{2}} \frac{(\log\log d_K)^3}{\log d_K}.$$

Remark 8.5 Let $M \ll X^{\delta'}$ with $0 < \delta' < \delta$, where δ is the constant in [9]. Then we can show that

$$|L(X)| = \prod_{p \nmid M} \left(1 - \frac{\rho(p^3)}{p^3}\right) \frac{X}{M} + O(X^{1-\delta}),$$

where $\rho(p^3)$ is the number of solutions to $27t^4 + 1 \equiv 0 \pmod{p^3}$.

Remark 8.6 Let $f(x,t)$ be a family of quintic polynomials and let θ_t be a root, and $K_t = \mathbb{Q}[\theta_t]$. Let $\widehat{K_t}$ be the Galois closure of K_t. Assume that $G = \mathrm{Gal}(\widehat{K_t}/\mathbb{Q})$ is isomorphic to A_5 and that K_t has signature $(1,2)$. We also assume that $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$. For example,

$$f(x,t) = x^5 - 5(5t^2-1)x^4 - (4(5t^2-1))^4 \quad \text{or} \quad f(x,t) = x^5 + 5(5t^2-1)x + 4(5t^2-1)$$

satisfies those properties. Then G has a subgroup H isomorphic to A_4 such that $\widehat{K_t}^H = K_t$. Let $\mathrm{Ind}_H^G\, 1_H = 1_G \oplus \rho$, where ρ is the 4-dimensional representation of A_5. Then

$$L(s,\rho,t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}.$$

Now by [11, p. 498], ρ is equivalent to a twist of $\sigma \otimes \sigma^\tau$ by a character, where σ, σ^τ are the icosahedral 2-dimensional representations of $\widetilde{A}_5 \simeq SL_2(\mathbb{F}_5)$. Since K_t is not totally real, σ and σ^τ are odd. Hence by [10, Corollary 10.2], σ, σ^τ are modular, *i.e.,* they are attached to cuspidal representations π, π^τ of $GL_2/\mathbb{Q}$. By [18], the functorial product $\pi \boxtimes \pi^\tau$ is a cuspidal representation of $GL_4/\mathbb{Q}$. Hence $L(s,\rho,t)$ is a cuspidal automorphic L-function of $GL_4/\mathbb{Q}$. In particular, we can prove that there exists an arbitrarily large t such that $L(1,\rho,t) \gg (\log\log d_{K_t})^4$. Unfortunately, for the above polynomials, numerical calculation shows that the regulator of K_t seems large. We have not been able to find a family of A_5 quintic polynomials with small regulators, of size $(\log t)^2$.

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