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On the Regularity of Powers of Ideal Sheaves

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Abstract. We use the geometry of the secant variety to an embedded smooth curve to prove some vanishing and regularity theorems for powers of ideal sheaves.

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1. Introduction

The original motivation for this work was an attempt to understand an unpublished manuscript of J. Rathmann in which he proves the following nontrivial result via a fairly lengthy calculation on the triple product $C \times C \times C$:

THEOREM 1.1 ([15]). *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle of degree at least $2g + 3$. Then $H^i(\mathbb{P}^n, \mathcal{I}_C^2(k)) = 0$ for $k \geqslant 3$, $i > 0$.* □

This result was used by A. Bertram to obtain the following:

THEOREM 1.2 ([4, 4.2]). *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle of degree at least $(8g + 2)/3$. Then $H^i(\mathbb{P}^n, \mathcal{I}_C^a(k)) = 0$ for $k \geqslant 2a - 1$, $i > 0$.* □

Bertram proceeds quite differently than Rathmann: using the GIT flip construction of Thaddeus [18], as well as results from [2, 3], he constructs useful log canonical divisors on the blow up of $\mathbb{P}^n$ along C, and then obtains vanishing results from a Kodaira-type vanishing theorem.

We work in the same general context as Bertram, though we mostly avoid the explicit use of flips and of generalized Kodaira-type vanishing, to give a new proof (Corollary 3.10) of Rathmann's result and then to prove an extension of Theorem 1.2 suggested in [4]:

THEOREM 1.3. *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle of degree at least $2g + 3$. Then $H^i(\mathbb{P}^n, \mathcal{I}_C^a(k)) = 0$ for $k \geqslant 2a - 1$, $i > 0$.* □

See Theorem 4.1 for a slightly more general statement.

As the title suggests, we use these to make statements regarding the regularity of powers of ideal sheaves (Corollary 3.10). We also include some statements for canonical curves (Proposition 2.5) which should not be considered at all optimal.

Finally, we mention a closely related conjecture of Wahl, toward which we hope to adapt these techniques:

CONJECTURE 1.4 ([23]). *Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve with* **CliffC $\geqslant$ 3**. *Then $H^1(\mathscr{I}_C^2(3)) = 0$.* □

Note that many of the results in Section 2 (Corollary 2.2 though Proposition 2.5) can be derived from more general results on point sets [9, 10]. Further, these results, along with Rathmann's Theorem 1.1, can be used to derive Theorem 4.1 (and, hence, Theorem 1.3). However, we have chosen to retain these results and their proofs as they give context to the main results and illustrate that fact that only the $k = 3$ statement in Theorem 1.1 is not elementary. Specifically, the proofs in Section 2 are quite short and are in much the same spirit as the proof of the main result, Theorem 3.3.

2. Elementary Vanishing

In this section, we collect a few fairly elementary vanishing statements that are somewhat broader than those mentioned in the introduction. The first is due to Lazarsfeld (cf. [22, 2.3]):

LEMMA 2.1. *Let $C \subset \mathbb{P}^n$ be a smooth curve, scheme theoretically defined by forms of degree r. If $H^1(C, \mathcal{O}_C(t)) = 0$, then $H^1(C, N^{*\otimes a}(k)) = 0$ for $k \geqslant ra + t$.* □

We do not present a proof, as we will next describe a direct extension of the technique. Let $C \subset \mathbb{P}^n$ be a smooth curve scheme theoretically cut out by hypersurfaces of degree r. Tensor the resolution of the ideal sheaf:

$$\to \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \Gamma(\mathscr{I}_C(r)) \otimes \mathcal{O}_{\mathbb{P}^n}(-r) \to \mathscr{I}_C \to 0$$

by $\mathcal{O}_C(k)$, and break the sequence into two short exact sequences:

$$0 \to \mathscr{K}_2 \to \bigoplus_i \mathcal{O}_C(k - a_i) \to \mathscr{K}_1 \to 0,$$
$$0 \to \mathscr{K}_1 \to \Gamma(\mathscr{I}_C(r)) \otimes \mathcal{O}_C(k - r) \to N_C^*(k) \to 0.$$

Suppose $a_i \geqslant a_{i+1}$ for all i. If $H^1(\mathcal{O}_C(k - a_1)) = 0$, then $\Gamma(\mathscr{I}_C(r)) \otimes \Gamma(\mathcal{O}_C(k - r)) \to \Gamma(N_C^*(k))$ is surjective (note that $H^1(\mathcal{O}_C(k - a_i)) = 0$ because C is a curve and by maximality of a_1). Now if $H^1(\mathscr{I}_C(k-r))=0$, we have the diagram

$$\begin{array}{ccccc} \Gamma(\mathscr{I}_C(r))\otimes\Gamma(\mathcal{O}_{\mathbb{P}}(k-r)) & \longrightarrow & \Gamma(\mathscr{I}_C(k)) & & \\ \downarrow & & \downarrow & & \\ \Gamma(\mathscr{I}_C(r))\otimes\Gamma(\mathcal{O}_C(k-r)) & \longrightarrow & \Gamma(N_C^*(k)) & \longrightarrow & 0 \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

where the first vertical map is surjective by the normality hypothesis just mentioned, and the second horizontal map is surjective by the above discussion. Therefore, the second vertical map is surjective and we have:

COROLLARY 2.2. *Let $C\subset\mathbb{P}^n$ be a smooth curve scheme theoretically cut out by hypersurfaces of degree r, with syzygies generated by forms of degree at most s. If $H^1(\mathscr{I}_C(k-r))=H^1(\mathcal{O}_C(k-r-s))=0$, then $H^1(\mathscr{I}_C^2(k))=0$.*

Proof. This follows by the above discussion and the sequence

$$0\to\mathscr{I}_C^2(k)\to\mathscr{I}_C(k)\to N_C^*(k)\to 0$$

(note that the vanishing of $H^1(\mathscr{I}_C(k))$ is implied by our assumptions). □

TERMINOLOGY 2.3. For the remainder of the paper, we will be interested in curves that are at least projectively normal and whose homogeneous ideals are generated by quadrics. This is usually referred to as Green's condition (N_1). If, further, the syzygies among the defining quadrics are generated by linear relations, we have condition (N_2). Recall that a smooth curve embedded by a line bundle of degree at least $2g+1+p$ satisfies condition (N_p) ([11]). □

Proceeding inductively, we just as easily deduce vanishing statements for higher powers of the ideal sheaf. We will assume, however, that $H^1(\mathcal{O}_C(1))=0$; the more general case may be similarly worked out. In particular, tensoring the resolution of the ideal by $S^aN_C^*(2a+1)$ and applying Lemma 2.1, we see that the map

$$\Gamma(\mathscr{I}_C(2))\otimes\Gamma(S^aN_C^*(2a+1))\to\Gamma(S^{a+1}N_C^*(2a+3))$$

is surjective. There is an analogous diagram

$$\begin{array}{ccccc} \Gamma(\mathscr{I}_C(2))\otimes\Gamma(\mathscr{I}_C^a(2a+1)) & \longrightarrow & \Gamma(\mathscr{I}_C^{a+1}(2a+3)) & & \\ \downarrow & & \downarrow & & \\ \Gamma(\mathscr{I}_C(2))\otimes\Gamma(S^aN_C^*(2a+1)) & \longrightarrow & \Gamma(S^{a+1}N_C^*(2a+3)) & \longrightarrow & 0 \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

where the first vertical map is surjective by the previous stage.

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PROPOSITION 2.4. *Let $C \subset \mathbb{P}^n$ be a smooth curve with $H^1(\mathcal{O}_C(1)) = 0$.*

(1) *If C satisfies (N_1), then:*
   (a) $H^1(\mathcal{I}_C^a(k)) = 0$ *for* $k \geqslant 2a+1$
   (b) $H^2(\mathcal{I}_C^a(k)) = 0$ *for* $k \geqslant 2a-1$ *hence* $\mathcal{I}_C^a$ *is* $(2a+2)$*-regular*

(2) *If C satisfies (N_2), then:*
   (a) $H^1(\mathcal{I}_C^a(k)) = 0$ *for* $k \geqslant 2a$,
   (b) $\mathcal{I}_C^a$ *is* $(2a+1)$*-regular.*

Proof. The first part follows directly from the discussion above and Lemma 2.1. The second part follows from the fact that the $\mathcal{O}(-4)$ term in the second stage of the resolution of the ideal may be removed. □

For the sake of completeness, we include a result not covered by the above statements, but which may be of interest:

PROPOSITION 2.5 (cf. [23]). *Let $C \subset \mathbb{P}^{g-1}$ be the canonical embedding of a smooth curve with* **CliffC ⩾ 3***. Then $H^1(\mathcal{I}_C^a(k)) = 0$ for $k \geqslant 2a+1$ and $H^2(\mathcal{I}_C^a(k)) = 0$ for $k \geqslant 2a-1$.*

Proof. This follows exactly as above taking into account:

(1) $H^1(\mathcal{O}_C(2)) = 0$,
(2) $C \subset \mathbb{P}^{g-1}$ satisfies condition (N_2) ([16, 21]),
(3) $H^1(S^a N_C^*(k)) = 0$ for $k \geqslant 2a+1$ ([6, Thm 2; 17]). □

We conclude this section by recalling a pair of basic lemmas; the first describes some situations where the cohomology of powers of ideal sheaves vanishes 'automatically', the second gives the relationship between powers of ideal sheaves and divisors on the blow-up along the subvariety.

LEMMA 2.6. *Let $X \subset \mathbb{P}^n$ be a nondegenerate smooth variety of dimension r. Then*

(1) $H^i(\mathbb{P}^n, \mathcal{I}_X^a(k)) = 0$ *for* $i \geqslant r+2$ *and* $a, k \geqslant 1$
(2) *If* $H^{r+1}(\mathbb{P}^n, \mathcal{I}_X^a(k)) = 0$ *then* $H^{r+1}(\mathbb{P}^n, \mathcal{I}_X^a(k+\sigma)) = 0$ *for* $\sigma \geqslant 0$, $a \geqslant 1$
(3) $H^0(\mathbb{P}^n, \mathcal{I}_X^a(k)) = 0$ *for* $k \leqslant a$

Proof. The first two statements follow immediately from the basic sequence

$$0 \to \mathcal{I}_X^{a+1} \to \mathcal{I}_X^a \to S^a N_X^* \to 0.$$

The third is just the statement that a form of degree k cannot vanish k times on a nondegenerate variety. □

LEMMA 2.7. ([5, 1.2,1.4]). *Let $Y \subset X$ be a smooth subvariety of codimension e of a smooth projective variety, L an invertible sheaf on X, $\pi: B = \mathrm{Bl}_Y(X) \to X$ the blow up along Y with exceptional divisor E. Then*

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(1) *If* $0 \leqslant t \leqslant e-1$, *then* $H^i(B, \pi^* L(tE)) = H^i(X, L)$, $\forall i$
(2) $\pi_* \pi^* L(-kE) = \mathscr{I}_Y^k \otimes L$ *and* $R^i \pi_*(\pi^* L(-kE)) = 0$ *for* $k \geqslant 0$, $i > 0$; *hence*
(3) $H^*(X, \mathscr{I}_Y^k \otimes L) = H^*(B, \pi^* L(-kE))$, $k \geqslant 0$ □

3. The Square of the Ideal Sheaf

This section is devoted to the proof of Theorem 1.1 stated in the Introduction. We denote the ith secant variety to an embedded curve C by $\Sigma_i C$, or just by Σ_i when no confusion will result. The following construction and 'Terracini Recursiveness' result of A. Bertram provides the means for our vanishing results. Recall that a line bundle L on a curve C is said to be. *k-very ample* if $h^0(C, L(-Z)) = h^0(C, L) - k$ for all $Z \in S^k C$.

Let $C \subset X_0 = \mathbb{P}(H^0(C, L))$ be a smooth curve embedded by a $2k$-very ample line bundle L. Construct a birational morphism $f: \widetilde{X} \to X_0$ which is a composition of the following blow-ups:

$f^1: X_1 \to X_0$ is the blow up of X_0 along $C = \Sigma_0$,
$f^2: X_2 \to X_1$ is the blow up along the proper transform of Σ_1,
$\vdots$
$f^k: \widetilde{X} = X_k \to X_{k-1}$ is the blow up along the proper transform of Σ_{k-1}.

We then have:

THEOREM 3.1 ([2, Theorem 1], [3, 3.6]). *Hypotheses as above:*

(1) *For* $i \leqslant k-1$, *the proper transform of each* Σ_i *in* X_i *is smooth and irreducible of dimension* $2i+1$, *transverse to all exceptional divisors, and so in particular* $\widetilde{X}$ *is smooth. Let* E_i *be the proper transform in* $\widetilde{X}$ *of each* f^i*-exceptional divisor. Then* $E_1 + \cdots + E_k$ *is a normal crossings divisor on* $\widetilde{X}$ *with* k *smooth components.*
(2) *Suppose* $i \leqslant k-1$ *and* $x \in \Sigma_i \setminus \Sigma_{i-1}$. *Then the fiber* $(f^i)^{-1}(x) \subset X_i$ *is naturally isomorphic to* $\mathbb{P}(H^0(C, L(-2Z)))$, *where* Z *is the unique divisor of degree* $i+1$ *whose span contains* x. *Moreover, the fiber* $f^{-1}(x) \subset E_{i+1} \subset \widetilde{X}$ *is isomorphic to* $\widetilde{X}_Z$, *the variety obtained by applying the above construction to the line bundle* $L(-2Z)$. □

Write $\mathbf{Pic}\widetilde{\mathbf{X}} = \mathbb{Z}\mathbf{H} + \mathbb{Z}\mathbf{E_1} + \cdots + \mathbb{Z}\mathbf{E_k}$. We collect a few technical implications of the construction:

LEMMA 3.2. *Hypotheses and notation as above:*

(1) $\Sigma_i \subset X_0$ *is normal for* $i \leqslant k-1$,
(2) $f: E_1 \to C$ *is a smooth morphism,*
(3) $f_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{X_0}$ *and* $R^j f_* \mathcal{O}_{\widetilde{X}} = 0$ *for* $j \geqslant 1$,
(4) $f_* \mathcal{O}_{E_i} = \mathcal{O}_{\Sigma_{i-1}}$.

Proof. The first statement follows from the fact that E_i is smooth and $f: E_i \to \Sigma_{i-1}$ has reduced, connected fibers. The second follows from the smoothness of C and the description of the fibers $f^{-1}(x) \cong \widetilde{X}_Z \subset E_1$ above.

For the third, $f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{X_0}$ is Zariski's Main Theorem. $R^j f_*\mathcal{O}_{\widetilde{X}} = 0$ follows from the fact that $H^j(\widetilde{X}, f^*\mathcal{O}_X(mH)) = 0$ for $m, j > 0$ (cf. [13, 2.69]).

To show $f_*\mathcal{O}_{E_i} = \mathcal{O}_{\Sigma_{i-1}}$, note that $f: E_i \to \Sigma_{i-1}$ is the composition of birational morphisms to smooth varieties, followed by a projective bundle, followed by a birational morphism to $Sec^{i-1}C$ which is normal by the first statement. □

We recover Rathmann's result (Corollary 3.10) from the main result of this section:

THEOREM 3.3. *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a nonspecial line bundle L. Suppose there exists a point $p \in C$ such that $L(2p)$ is* 6-*very ample and such that $C \subset \mathbb{P}H^0(C, L(2p-2q))$ satisfies condition (N_2) for all $q \in C$. Then $H^i(\mathbb{P}\Gamma(C, L(2p-2q)), \mathcal{I}_C^2(k)) = 0$, $k \geqslant 3$, $i > 0$.*

The idea of the proof is to take weak (i.e. asymptotic) vanishing statements on the spaces X_i from Bertram's construction and to descend them to effective vanishing results along the fibers of $f: \widetilde{X} \to X_0$. For this we need the following special case of [12, III.12.11b]:

PROPOSITION 3.4 ([14, p. 52, Cor $1\frac{1}{2}$]). Let $\rho: X \to Y$ be a flat morphism of projective varieties, $\mathcal{F}$ a locally free sheaf on X. If $R^i\rho_*\mathcal{F} = 0$ for all $i \geqslant i_0$, then $H^i(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$ and all $i \geqslant i_0$. □

Aside from Lemma 3.6, the proof of Theorem 3.3 is fairly straightforward. We hope to clarify the idea by giving the proof now, referencing the necessary lemmas below:

Proof of Theorem 3.3. The case $i > 2$ is automatic by Lemma 2.6. The case $i = 2$ is contained in Proposition 2.4. For $i = 1$, by Proposition 2.4 we need only prove the result for $k = 3$.

First note that as $L(2p-2q)$ is 4-very ample, we can apply Theorem 3.1 to $L(2p-2q)$ to obtain $f: X_2 \to X_0$. Furthermore, as the restriction of $\mathcal{O}_{X_1}(3H - 2E_1)$ to a fiber of the $\mathbb{P}^1$ bundle $\widetilde{\Sigma}_1 \to S^2C$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$, we see immediately

$$\begin{aligned} H^i(X_2, \mathcal{O}(3H - 2E_1 - E_2)) &= H^i(X_1, \mathcal{O}(3H - 2E_1)) \\ &= H^i(\mathbb{P}\Gamma(C, L(2p-2q)), \mathcal{I}_C^2(3)) \end{aligned} \tag{1}$$

Now, beginning anew with $L(2p)$, apply Theorem 3.1 to $L(2p)$. This yields

$$f: \widetilde{X} = X_3 \to X_2 \to X_1 \to X_0 = \mathbb{P}H^0(C, L(2p))$$

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where $X_{i+1} = \mathrm{Bl}_{\tilde{\Sigma}_i}(X_i)$. We deduce the desired vanishing from the sequence

$$0 \to \mathcal{O}_{\tilde{X}}(kH - 4E_1 - 2E_2 - E_3) \to \mathcal{O}_{\tilde{X}}(kH - 3E_1 - 2E_2 - E_3)$$
$$\to \mathcal{O}_{E_1}(kH - 3E_1 - 2E_2 - E_3) \to 0,$$

where $k \in \mathbb{Z}$ is arbitrary. The fact that $H^2(\mathbb{P}H^0(C, L(2p - 2q)), \mathcal{I}_C^2(3)) = 0$ (this is true from Equation (1) and Proposition 2.4) implies that

$$R^2 f_* \mathcal{O}_{E_1}(kH - 3E_1 - 2E_2 - E_3) = 0.$$

By Proposition 3.4, if $R^1 f_* \mathcal{O}_{E_1}(kH - 3E_1 - 2E_2 - E_3) = 0$, then the cohomology along the fibers vanishes, implying the groups in (1) vanish (note the higher direct images vanish by Lemma 2.6).

$R^1 f_* \mathcal{O}_{\tilde{X}}(kH - 3E_1 - 2E_2 - E_3) = 0$ is shown in Lemma 3.5.

$R^2 f_* \mathcal{O}_{\tilde{X}}(kH - 4E_1 - 2E_2 - E_3) = 0$ is more difficult and is shown in Lemma 3.9. □

LEMMA 3.5. *Under the hypotheses of Theorem 3.3, apply Theorem 3.1 to obtain* $f: X_3 \to X_0$. *Then* $R^1 f_* \mathcal{O}_{X_3}(kH - 3E_1 - 2E_2 - E_3) = 0$.

Proof. From Lemma 3.2 parts 3 and 4, we have $R^1 f_* \mathcal{O}_{X_3}(kH - E_3) = 0$. Using part 3 of Lemma 2.6 to check the vanishing of $R^0 f_*$ of the rightmost term of sequences of the form

$$0 \to \mathcal{O}_{X_3}(kH - E_2 - E_3) \to \mathcal{O}_{X_3}(kH - E_3) \to \mathcal{O}_{E_2}(kH - E_3) \to 0$$

and

$$0 \to \mathcal{O}_{X_3}(kH - E_1 - 2E_2 - E_3) \to \mathcal{O}_{X_3}(kH - 2E_2 - E_3)$$
$$\to \mathcal{O}_{E_1}(kH - 2E_2 - E_3) \to 0$$

gives $R^1 f_* \mathcal{O}_{X_3}(kH - 3E_1 - 2E_2 - E_3) = 0$. □

LEMMA 3.6. *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a nonspecial line bundle L satisfying* (N_2). *Apply Theorem 3.1 to obtain* $f: X_2 \to X_0$. *Then* $H^2(X_2, \mathcal{O}(kH - 2E_1 - E_2)) = 0$ *for* $k \geqslant 3$.

Remark 3.7. It may appear as if we have already shown this in the proof of Theorem 3.3. Indeed, we know the result holds for $k = 3$. However, the special circumstances involved in the demonstration that $H^i(X_2, \mathcal{O}(3H - 2E_1 - E_2)) = H^i(\mathcal{I}_C^2(3))$ do not apply when $k \geqslant 4$. Specifically, we need not have $H^0(\mathcal{I}_C^2(k)) \subseteq H^0(\mathcal{I}_{\Sigma_1}(k))$ for $k \geqslant 4$.

Proof of Lemma 3.6. As before, Equation (1) and Proposition 2.4 imply the result for $k = 3$. Hence, we show $H^2(X_2, \mathcal{O}(kH - E_1 - E_2)) = 0$ for $k \geqslant 4$. By restricting to E_1 and computing direct images (recall $E_1 \to C$ is flat), this immediately implies $H^2(X_2, \mathcal{O}(kH - 2E_1 - E_2)) = 0$ for $k \geqslant 4$.

Because $H^2(X_2, \mathcal{O}(kH - E_1)) = 0$, it suffices to show

$$H^1(\tilde{\Sigma}_1, \mathcal{O}(kH - E_1)) = 0.$$

We prove $H^i(\tilde{\Sigma}_1, \mathcal{O}((4-i)H - E_1)) = 0$ and the result follows by a regularity argument (note that $\tilde{\Sigma}_1$ is smooth and $\mathcal{O}(H)$ globally generated). As before, we have $H^3(\tilde{\Sigma}_1, \mathcal{O}(H - E_1)) = 0$ because the restriction of $\mathcal{O}(H - E_1)$ to a fiber of the $\mathbb{P}^1$-bundle is $\mathcal{O}_{\mathbb{P}^1}(-1)$. The fact that $H^1(\tilde{\Sigma}_1, \mathcal{O}(3H - E_1)) = 0$ follows immediately from projective normality and the first paragraph.

The final step is to note $\tilde{\Sigma}_1 \cap E_1 = C \times C$ (this follows from [2]), hence we have the exact sequence

$$0 \to \mathcal{O}_{\tilde{\Sigma}_1}(2H - 2E_1) \to \mathcal{O}_{\tilde{\Sigma}_1}(2H - E_1) \to \mathcal{O}_{C\times C}(2H - E_1) \to 0$$

which we push down $f\colon \tilde{\Sigma}_1 \to S^2C$. As $\mathcal{O}_{\tilde{\Sigma}_1}(2H - E_1)$ is trivial along the fibers, it is the pull back of a line bundle $\mathscr{L}$ on S^2C. As the restriction of f to $C \times C$ is flat of degree two, $f_*\mathcal{O}_{C\times C}(2H - E_1) \cong \mathscr{L} \otimes (\mathcal{O}_{S^2C} \oplus \mathscr{E})$ for some line bundle $\mathscr{E}$. Therefore, $H^2(C \times C, \mathcal{O}(2H - E_1)) = 0$ implies $H^2(\tilde{\Sigma}_1, \mathcal{O}(2H - E_1)) = 0$. It is, however, not difficult to verify that $\mathcal{O}_{C\times C}(2H - E_1) \cong \mathcal{O}_{C\times C}(L \boxtimes L \otimes \mathscr{I}_\Delta^2)$, and the vanishing follows from the fact that L is nonspecial and very ample. □

As we will make three Formal Function calculations of essentially the same type, we state an elementary result:

PROPOSITION 3.8. *Let $\rho\colon X \to Y$ be a morphism of projective varieties; X smooth, Y normal. Let $\mathcal{F}$ be a locally free sheaf on X and assume $y \in Y$ is a point such that:*

(1) *X_y is smooth*
(2) *$H^i(X_y, \mathcal{F}_y) = 0$*
(3) *$H^i(X_y, N^{*\otimes a}_{X_y/X} \otimes \mathcal{F}_y) = 0$ for all $a \geqslant 1$*

Then $R^i\rho_*\mathcal{F}$ is not supported at y.

Proof. This follows by induction after tensoring the sequence:

$$0 \to S^aN^* \to \mathcal{O}_X/\mathscr{I}^{a+1}_{X_y} \to \mathcal{O}_X/\mathscr{I}^{a}_{X_y} \to 0$$

by $\mathcal{F}$. As we work over $\mathbb{C}$, hypothesis 3 implies H^i of the left term vanishes. By 2, the Theorem on Formal Functions [12, III.11.1] implies the completion $(R^i\rho_*\mathcal{F})^\wedge_y = 0$. As $R^i\rho_*\mathcal{F}$ is coherent, the result follows (e.g. by [1, Ex.10.3]). □

LEMMA 3.9. *With notation and hypotheses as in Lemma 3.5, we have $R^2f_*\mathcal{O}_{X_3}(kH - rE_1 - 2E_2 - E_3) = 0$ for $r \geqslant 3$.*

Proof. We proceed via Proposition 3.8. As

$$R^2f_*\mathcal{F} := R^2f_*\mathcal{O}_{X_3}(kH - rE_1 - 2E_2 - E_3)$$

is supported on $\Sigma_2 \subset X_0$, we need to check three classes of fibers.

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First, let $x \in \Sigma_2 \setminus \Sigma_1$. Then $B_x = f^{-1}(x) \cong \mathbb{P}H^0(C, L(2p-2Z)) = \mathbb{P}^{n-4}$ where $Z \in S^3C$ determines the unique 3-secant $\mathbb{P}^2$ containing x. The restriction of $\mathcal{F}$ to such a fiber is simply $\mathcal{O}_{\mathbb{P}^{n-4}}(1)$, hence $H^2(\mathcal{O}_{B_x}(\mathcal{F})) = 0$. The conormal sequence for $B_x \subset E_3 \subset X_3$ is

$$0 \to \mathcal{O}_{B_x}(-E_3) \to N^*_{B_x/X_3} \to \oplus \mathcal{O}_{B_x} \to 0$$

The required vanishings follow after twisting by $(N^*_{B_x})^{\otimes a}(\mathcal{F})$.

Let $x \in \Sigma_1 \setminus C$. Then $B_x = f^{-1}(x) \cong \mathrm{Bl}_C(\mathbb{P}^{n-2})$ with the embedding $C \hookrightarrow \mathbb{P}H^0(C, L(2p-2Z)) = \mathbb{P}^{n-2}$ where $Z \in S^2C$ determines the unique secant line containing x. The restriction of $\mathcal{F}$ to such a fiber is $\mathcal{O}_{B_x}(2H-E)$ where $\mathbf{Pic(B_x)} = \mathbb{Z}\mathbf{H} + \mathbb{Z}\mathbf{E}$. Therefore, $H^2(\mathcal{O}_{B_x}(\mathcal{F})) = 0$ by projective normality of the above embedding. The conormal sequence for $B_x \subset E_2 \subset X_3$ is

$$0 \to \mathcal{O}_{B_x}(-E_2) \to N^*_{B_x/X_3} \to \oplus \mathcal{O}_{B_x} \to 0$$

and as above the required vanishing follows after twisting by $(N^*_{B_x})^{\otimes a}(\mathcal{F})$.

If $x \in C$, then

$$B_x = f^{-1}(x) \cong \mathrm{Bl}_{\tilde{\Sigma}_1}(\mathbb{P}^n C),$$

where

$$\mathbb{P}^n = \mathbb{P}H^0(C, L(2p-2x)).$$

The restriction of $\mathcal{F}$ to such a fiber is $\mathcal{O}_{B_x}(rH - 2E_1 - E_2)$ where $\mathbf{Pic(B_x)} = \mathbb{Z}\mathbf{H} + \mathbb{Z}\mathbf{E_1} + \mathbb{Z}\mathbf{E_2}$. By Lemma 3.6, $H^2(\mathcal{O}_{B_x}(\mathcal{F})) = 0$ for $r \geqslant 3$ and the vanishing of tensor powers of the conormal bundle follows exactly as above. □

We immediately recover Rathmann's result:

COROLLARY 3.10. *Assume* $\deg(L) \geqslant 2g+3$. *Then*

(1) $H^1(\mathbb{P}^n, \mathcal{I}^2_C(k)) = 0$ for $k \geqslant 3$ and $\mathcal{I}^2_C$ is 5-regular
(2) $\mathcal{I}^2_C$ is 4-regular if and only if the Gauss–Wahl map $\Phi_L\colon \wedge^2\Gamma(L) \to \Gamma(K \otimes L^2)$ is surjective (e.g. when $\deg(L) \geqslant 3g+2$, [6])

Proof. For the first, we need only note that a line bundle of degree at least $2g+3$ satisfies condition (N_2) [11].

For the second statement, we need only add that under our hypotheses, surjectivity of Φ_L is equivalent to the vanishing $H^2(\mathbb{P}^n, \mathcal{I}^2_C(2)) = 0$ ([23, 1.7.3]). □

4. Extended Vanishing

In this section we extend Corollary 3.10 to a result suggested by Bertram [4, 4.3]:

THEOREM 4.1. *Let* $C \subset \mathbb{P}^n$ *be a smooth curve satisfying* (N_1) *and assume* $H^1(\mathbb{P}^n, \mathcal{I}^2_C(k)) = 0$ *for* $k \geqslant 3$. *Then* $H^i(\mathbb{P}^n, \mathcal{I}^a_C(k)) = 0$ *for* $k \geqslant 2a-1$, $i \geqslant 1$.

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Proof. For $i > 2$ the result is again automatic by Lemma 2.6, and $i = 2$ is in Proposition 2.4.

Recall $\widetilde{\mathbb{P}^n} = \mathrm{Bl}_C(\mathbb{P}^n)$. To settle the case $i = 1$, let $V = \Gamma(\widetilde{\mathbb{P}^n}, \mathcal{O}(2H - E))$ and let $\varphi: \widetilde{\mathbb{P}^n} \to \mathbb{P}^s$ be the morphism induced by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H - E)$. We have the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \varphi^*(\Omega^1_{\mathbb{P}^s}(1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(H - E) & \to & V \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(H - E) & \to & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(3H - 2E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \varphi^*(\Omega^1_{\mathbb{P}^s}(1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(H) & \to & V \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(H) & \to & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(3H - E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \varphi^*(\Omega^1_{\mathbb{P}^s}(1)) \otimes \mathcal{O}_E(H) & \to & V \otimes \mathcal{O}_E(H) & \to & \mathcal{O}_E(3H - E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Twisting the entire diagram by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H - E)$, we see from the top row that showing $H^1(\mathcal{I}^3_C(5)) = 0$ is equivalent to showing

$$H^1(E, \varphi^*(\Omega^1_{\mathbb{P}^s}(1)) \otimes \mathcal{O}_E(3H - E)) = 0.$$

However, from the pictured diagram, it is easy to see that $H^1(\mathcal{I}^2_C(3)) = 0$ is equivalent to the vanishing $H^1(E, \varphi^*(\Omega^1_{\mathbb{P}^s}(1)) \otimes \mathcal{O}_E(H)) = 0$. Therefore, twisting the last row by $\varphi^*(\Omega^1_{\mathbb{P}^s}(1))$ we need $H^2(E, \varphi^*(\Omega^1_{\mathbb{P}^s}(1))^{\otimes 2} \otimes \mathcal{O}_E(H)) = 0$.

Clearly, it suffices to prove that the higher direct images of the blow down to the curve vanish. As $E \to C$ is flat, we only need vanishing along the fibers, which are isomorphic to $\mathbb{P}^{n-2}$. However, φ maps a fiber of $E \to C$ isomorphically to a *linearly embedded* subspace $\mathbb{P}^{n-2} \subset \mathbb{P}^s$, and the vanishing follows easily.

Repeating this argument after tensoring by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(m(2H - E))$ yields the stated result. □

Analogous to Corollary 3.10, we have:

COROLLARY 4.2. *Assume* $\deg(L) \geqslant 2g + 3$. *Then*

(1) $H^1(\mathbb{P}^n, \mathcal{I}^a_C(k)) = 0$ *for* $k \geqslant 2a - 1$ *and* $\mathcal{I}^a_C$ *is* $(2a + 1)$*-regular*
(2) $\mathcal{I}^a_C$ *is* $2a$*-regular if and only if* Φ_L *is surjective.*

Proof. Part 1 is Theorem 3.3 applied to Theorem 4.1. Part 2 is the earlier statement that Φ_L is surjective exactly when $H^2(\mathbb{P}^n, \mathcal{I}^2_C(2)) = 0$ applied to Theorem 4.1. □

We further have the immediate result on points:

COROLLARY 4.3. *Let $\Gamma = C \cap H$ be a hyperplane section of a linearly normal smooth curve of degree at least $2g+3$. Then $H^1(\mathscr{I}_\Gamma^a(k)) = 0$ for $k \geqslant 2a$, and the vanishing holds for $k = 2a-1$ if and only if Φ_L is surjective.* □

Note that the vanishing for $k \geqslant 2a$ is, more generally, true for any set of $2n-1$ points in $\mathbb{P}^{n-1}$ in linearly general position by [7, 8].

Combining [6, Thm 1] with [8, 3.7] yields the amusing:

PROPOSITION 4.4. *Let C be a smooth curve embedded by a line bundle of degree $3g+1$. Then C is hyperelliptic if and only if the general hyperplane section lies on a rational normal curve (in the hyperplane).* □

The procedure detailed in Section 3 should be extendible via Theorem 3.1 to give further vanishing statements for higher degree embeddings. In the very interesting cases of canonical embeddings and higher dimensional varieties it seems that some sort of converse ('ascending degree') procedure must be worked out. The main difficulty in the canonical case is that canonical curves cannot arise in the fibers of the blow up. For varieties of higher dimension, similar problems occur in that the fibers in the blow up are copies of the original variety blown up at a point (though the technique should at least reveal information in these cases). A somewhat greater obstacle is the lack of a structure theorem as strong as Theorem 3.1, though parts of this have been worked out in [19] and [20].

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