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Convolution Inequalities in l_p Weighted Spaces

Dedicated to the 65th anniversary of Prof. Saburou Saitoh

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Abstract. Various weighted l_p-norm inequalities in convolutions are derived by a simple and general principle whose l_2 version was obtained by using the theory of reproducing kernels. Applications to the Riemann zeta function and a difference equation are also considered.

1 Introduction

Let $\xi = \{\xi_n\}_{n\in\mathbb{Z}} \in l_p$, $\eta = \{\eta_n\}_{n\in\mathbb{Z}} \in l_q$, and $p^{-1} + q^{-1} > 1$. Young's inequality (see [3, pp. 178–179]) says that the convolution

$$\xi * \eta := \{\xi_n * \eta_n\}_{n\in\mathbb{Z}}, \quad \xi_n * \eta_n := \sum_{r\in\mathbb{Z}} \xi_r \eta_{n-r} \tag{1.1}$$

belongs to l_r, where $r^{-1} = p^{-1} + q^{-1} - 1$, and moreover,

$$\|\xi * \eta\|_r \le \|\xi\|_p \, \|\eta\|_q. \tag{1.2}$$

Note that for the typical case of $\xi, \eta \in l_2$, the inequality (1.2) does not hold.

In [6] by using the theory of reproducing kernels (see [5]), S. Saitoh obtained the following inequality

$$\sum_{n=0}^{\infty} \frac{|\sum_{r+s=n} \xi_r^{(1)} \xi_s^{(2)}|^2}{\sum_{r+s=n} \rho_r^{(1)} \rho_s^{(2)}} \le \Big(\sum_{n=0}^{\infty} \frac{|\xi_n^{(1)}|^2}{\rho_n^{(1)}}\Big)\Big(\sum_{n=0}^{\infty} \frac{|\xi_n^{(2)}|^2}{\rho_n^{(2)}}\Big) \tag{1.3}$$

for some sequences of strictly positive numbers $\{\rho_n^{(j)}\}_{n=0}^{\infty}$ $(j = 1, 2)$ belonging to l_1 and for $\{\xi_n^{(j)}\}_{n=0}^{\infty}$ $(j = 1, 2)$ such that the right hand side of (1.3) is finite.

Some other remarkable results are in [1], in which D. Borwein and W. Kratz dealt with conditions for the validity of the weighted convolution inequality

$$\sum_{n\in\mathbb{Z}} \Big| b_n \sum_{r\in\mathbb{Z}} a_{n-r} x_r \Big|^p \le C^p \sum_{n\in\mathbb{Z}} |x_n|^p \tag{1.4}$$

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when $p \geq 1$.

In the following, our fundamental convolution norm inequalities will be given in the form

$$\|\xi * \eta\|_p \leq \|\xi\|_p \, \|\eta\|_p \tag{1.5}$$

by considering the l_p-norm in some naturally determined weighted spaces. Our new type l_p convolution norm inequalities are obtained easily by elementary means and furthermore, in general l_p ($p > 1$) versions of Saitoh's inequality (1.3). Various reverse convolution inequalities in l_p weighted spaces are also considered. Moreover, we shall state in this paper two typical examples, as applications.

2 Preliminaries

We suppose throughout that

$$1 < p \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We always assume that a weight sequence is a sequence of strictly positive numbers belonging to l_1. For a weight sequence $\rho = \{\rho_n\}_{n=0}^{\infty}$ and a complex sequence $\xi = \{\xi_n\}_{n=0}^{\infty}$, we define

$$\|\xi\|_{l_p(\rho)} := \Big(\sum_{n=0}^{\infty} \rho_n |\xi_n|^p\Big)^{\frac{1}{p}}, \quad \text{for } 1 < p < \infty \quad \text{and} \quad \|\xi\|_{l_\infty(\rho)} := \sup_n \rho_n |\xi_n|,$$

and we say that $\xi \in l_p(\rho)$ if $\|\xi\|_{l_p(\rho)} < \infty$.

For two complex sequences $\xi^{(j)} = \{\xi_n^{(j)}\}_{n=0}^{\infty}$ ($j = 1, 2$), we shall denote by $\xi^{(1)} * \xi^{(2)}$ the convolution of $\xi^{(1)}$ and $\xi^{(2)}$, which is defined by

$$\xi^{(1)} * \xi^{(2)} := \{\xi_n^{(1)} * \xi_n^{(2)}\}_{n=0}^{\infty}, \quad \xi_n^{(1)} * \xi_n^{(2)} := \sum_{r=0}^{n} \xi_r^{(1)} \xi_{n-r}^{(2)}. \tag{2.1}$$

Given a positive integer m, define the convolution product $\prod_{j=1}^{m} *\xi^{(j)}$ by

$$\prod_{j=1}^{m} *\xi^{(j)} := \Big[\prod_{j=1}^{m-1} *\xi^{(j)}\Big] * \xi^{(m)}. \tag{2.2}$$

The $\mathcal{Z}$ transform (see [2, Chapter 12]) of a sequence $\{f(n)\}$ as the function $F(z)$ of a complex variable z defined by

$$\mathcal{Z}\{f(n)\} := F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}. \tag{2.3}$$

It is assumed that there exists an R such that (2.3) converges for $|z| > R$.

The inverse $\mathcal{Z}$ transform is given by the complex integral

$$\mathcal{Z}^{-1}\{F(z)\} := f(n) = \frac{1}{2\pi i}\oint_C F(z)z^{n-1}\,dz, \tag{2.4}$$

where C is a simple closed contour enclosing the origin and lying outside the circle $|z| = R$.

The relationship between the convolution product and the $\mathcal{Z}$ transform is based on the following lemma.

Lemma 2.1 *If $\mathcal{Z}\{f_j(n)\} = F_j(z)$ $(j = 1, \dots, m)$ then the $\mathcal{Z}$ transform of the convolution product $\prod_{j=1}^{m} * f_j(n)$ is given by*

$$\mathcal{Z}\Big\{\prod_{j=1}^{m} * f_j(n)\Big\} = \prod_{j=1}^{m} \mathcal{Z}\{f_j(n)\}. \tag{2.5}$$

Or, equivalently,

$$\mathcal{Z}^{-1}\Big\{\prod_{j=1}^{m} F_j(z)\Big\} = \prod_{j=1}^{m} * f_j(n). \tag{2.6}$$

Our main results follow.

3 Convolution Inequalities in Weighted l_p Spaces

We obtain new inequalities by considering Hölder's inequality and by using interchange of order of summation.

Theorem 3.1 *Let $\rho^{(j)} = \{\rho_n^{(j)}\}_{n=0}^{\infty}$ be some weight sequences and $\xi^{(j)} = \{\xi_n^{(j)}\}_{n=0}^{\infty} \in l_p(\rho^{(j)})$ $(j = 1, \dots, m)$. Then, we have the weighted l_p inequality*

$$\Big\| \prod_{j=1}^{m} *(\xi^{(j)}\rho^{(j)}) \Big(\prod_{j=1}^{m} *\rho^{(j)}\Big)^{(1-p)/p} \Big\|_{l_p} \le \prod_{j=1}^{m} \|\xi^{(j)}\|_{l_p(\rho^{(j)})}. \tag{3.1}$$

Equality holds here if and only if $\xi^{(j)}$ are represented in the form

$$\xi^{(j)} = c_j\{a^n\}_{n=0}^{\infty}, \quad c_j\text{: constants}, \tag{3.2}$$

where $a \in \mathbb{C}$ is a constant such that $\xi^{(j)} \in l_p(\rho^{(j)})$ $(j = 1, \dots, m)$.

Unlike Young's inequality, inequality (3.1) holds also in case $p = 2$.

Remark 3.2 Saitoh's inequality (1.3) is a special case of the above result when $p = 2$ and $m = 2$.

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Remark 3.3 For the convolution as in (1.1), we also have the inequality

$$(3.3)\qquad \sum_{n\in\mathbb{Z}}\Big|\prod_{j=1}^{m}*(\xi_n^{(j)}\rho_n^{(j)})\Big|^{p}\Big(\prod_{j=1}^{m}*\rho_n^{(j)}\Big)^{1-p}\le\prod_{j=1}^{m}\sum_{n\in\mathbb{Z}}|\xi_n^{(j)}|^{p}\rho_n^{(j)}$$

for $\rho^{(j)}=\{\rho_n^{(j)}\}_{n\in\mathbb{Z}}$ be some sequences of strictly positive numbers belonging to l_1 and for some two-sided complex sequences $\xi^{(j)}\in l_p(\rho^{(j)})$ $(j=1,\dots,m)$.

In particular, for $\rho^{(m)}=\{1\}_{n\in\mathbb{Z}}$, we have

$$(3.4)\qquad \sum_{n\in\mathbb{Z}}\Big|\Big[\prod_{j=1}^{m-1}*(\xi_n^{(j)}\rho_n^{(j)})\Big]*\xi_n^{(m)}\Big|^{p}$$

$$\le\Big(\prod_{j=1}^{m-1}\sum_{n\in\mathbb{Z}}\rho_n^{(j)}\Big)^{p-1}\Big(\sum_{n\in\mathbb{Z}}|\xi_n^{(m)}|^{p}\Big)\prod_{j=1}^{m-1}\sum_{n\in\mathbb{Z}}|\xi_n^{(j)}|^{p}\rho_n^{(j)}.$$

Let $\rho^{(j)}=\{1\}_{n=0}^{\infty}$ $(j=1,\dots,m)$. Upon simple computation, we get

$$\mathcal{Z}\{\rho_n^{(j)}\}=\frac{z}{z-1},\qquad j=1,2,\dots,m,$$

and so, by Lemma 2.1,

$$\prod_{j=1}^{m}*\rho_n^{(j)}=\frac{1}{(m-1)!}\prod_{j=1}^{m-1}(n+j).$$

Then, in view of Theorem 3.1, we have the following corollary.

Corollary 3.4 *If $\xi^{(j)}=\{\xi_n^{(j)}\}_{n=0}^{\infty}\in l_p$ $(j=1,\dots,m)$, then*

$$(3.5)\qquad \sum_{n=0}^{\infty}\Big|\prod_{j=1}^{m}*\xi_n^{(j)}\Big|^{p}\Big[\prod_{j=1}^{m-1}(n+j)\Big]^{1-p}\le\Big[\frac{1}{(m-1)!}\Big]^{p-1}\prod_{j=1}^{m}\sum_{n=0}^{\infty}|\xi_n^{(j)}|^{p}.$$

The constant $[(m-1)!]^{1-p}$ is the best possible. Moreover the (nonzero) extremal complex sequences are of the form $\xi^{(j)}=c_j\{a^n\}_{n=0}^{\infty}$, where c_j are constants and $a\in\mathbb{C}$ such that $\xi^{(j)}\in l_p$ $(j=1,\dots,m)$.

In particular, for $m=2$ we have

$$(3.6)\qquad \sum_{n=0}^{\infty}\Big|\sum_{r=0}^{n}\xi_r^{(1)}\xi_{n-r}^{(2)}\Big|^{p}(n+1)^{1-p}\le\sum_{n=0}^{\infty}|\xi_n^{(1)}|^{p}\sum_{n=0}^{\infty}|\xi_n^{(2)}|^{p}.$$

Proof of Theorem 3.1 For $m\ge 1$, we first observe that

$$(3.7)\qquad \Big|\prod_{j=1}^{m}*(\xi_n^{(j)}\rho_n^{(j)})\Big|^{p}\le\Big(\prod_{j=1}^{m}*\rho_n^{(j)}\Big)^{p-1}\prod_{j=1}^{m}*(|\xi_n^{(j)}|^{p}\rho_n^{(j)}),\qquad n=0,1,\dots.$$

We use induction on m. When $m = 2$, the inequality (3.7) is reduced to Hölder's inequality. Now suppose (3.7) holds for some integer $m \geq 2$. We claim that it also holds for $m+1$. By induction hypothesis, we arrive at

$$\Big|\prod_{j=1}^{m+1} *(\xi_n^{(j)}\rho_n^{(j)})\Big|^p \leq \Big|\sum_{r=0}^{n}\Big[\prod_{j=1}^{m} *(|\xi_r^{(j)}|\rho_r^{(j)})\Big]\,|\xi_{n-r}^{(m+1)}|\rho_{n-r}^{(m+1)}\Big|^p$$
$$\leq \Big|\sum_{r=0}^{n}\Big(\prod_{j=1}^{m} *\rho_r^{(j)}\Big)^{(p-1)/p}\Big(\prod_{j=1}^{m} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big)^{1/p}|\xi_{n-r}^{(m+1)}|\rho_{n-r}^{(m+1)}\Big|^p,$$

which is, by Hölder's inequality,

$$\leq \Big(\prod_{j=1}^{m+1} *\rho_n^{(j)}\Big)^{p-1}\prod_{j=1}^{m+1} *(|\xi_n^{(j)}|^p\rho_n^{(j)}),$$

completing the proof of (3.7).

By using (3.7), we obtain

$$\sum_{n=0}^{\infty}\Big|\prod_{j=1}^{m} *(\xi_n^{(j)}\rho_n^{(j)})\Big|^p\Big(\prod_{j=1}^{m} *\rho_n^{(j)}\Big)^{1-p} \leq \sum_{n=0}^{\infty}\prod_{j=1}^{m} *(|\xi_n^{(j)}|^p\rho_n^{(j)})$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\Big[\prod_{j=1}^{m-1} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big]\,|\xi_{n-r}^{(m)}|^p\rho_{n-r}^{(m)}.$$

For the function

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0, \end{cases}$$

we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Big[\prod_{j=1}^{m-1} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big]\,|\xi_{n-r}^{(m)}|^p\rho_{n-r}^{(m)}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Big[\prod_{j=1}^{m-1} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big]\,|\xi_{n-r}^{(m)}|^p\rho_{n-r}^{(m)}\theta(n-r),$$

which is, by interchanging the order of summation and substituting $n - r = s$,

$$= \sum_{r=0}^{\infty}\Big[\prod_{j=1}^{m-1} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big]\sum_{s=-r}^{\infty}|\xi_s^{(m)}|^p\rho_s^{(m)}\theta(s)$$
$$= \sum_{r=0}^{\infty}\Big[\prod_{j=1}^{m-1} *(|\xi_r^{(j)}|^p\rho_r^{(j)})\Big]\sum_{s=0}^{\infty}|\xi_s^{(m)}|^p\rho_s^{(m)}.$$

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Therefore,

$$\sum_{n=0}^{\infty}\Big|\prod_{j=1}^{m} *(\xi_n^{(j)}\rho_n^{(j)})\Big|^p\Big(\prod_{j=1}^{m} *\rho_n^{(j)}\Big)^{1-p} \le \prod_{j=1}^{m}\sum_{n=0}^{\infty}|\xi_n^{(j)}|^p\rho_n^{(j)}. \tag{3.8}$$

Raising both sides of the inequality (3.8) to power $1/p$ yields the inequality (3.1).

Next we determine under what conditions equality can hold in (3.1). Equality in (3.1) implies that equality holds in (3.7) for each positive integer m. In the case $m = 2$, this happens only if equality holds in Hölder's inequality, *i.e.*, only if for $n \ge 0$ there exists a number $\xi_n \in \mathbb{C}$ such that

$$\xi_r^{(1)}\xi_{n-r}^{(2)} = \xi_n, \quad r = 0, 1, \dots, n. \tag{3.9}$$

Suppose that $\xi_0^{(1)} = 0$ and $\xi_r^{(1)} \ne 0$ for some $r > 0$. Then, by (3.9), we have $\xi_{n-r}^{(2)} = 0$ $n \ge r$, that is $\xi^{(2)} = \{0\}_{n=0}^{\infty}$. So, $\xi_0^{(1)} = 0$ implies that $\xi^{(1)} = \{0\}_{n=0}^{\infty}$ or $\xi^{(2)} = \{0\}_{n=0}^{\infty}$. Now we assume that $\xi_0^{(1)}$ and $\xi_0^{(2)}$ are nonzero. From (3.9), the complex number $\eta_r := \xi_r^{(1)}/\xi_0^{(1)} = \xi_r^{(2)}/\xi_0^{(2)}$ satisfies $\eta_r\eta_{n-r} = \xi_n/[\xi_0^{(1)}\xi_0^{(2)}] = \eta_n$ for all $0 \le r \le n$. Put $a = \eta_1$, we have $\eta_0 = 1$ and $\eta_n = a^n$ for $n \ge 1$. This implies that $\xi_n^{(1)} = c_1a^n$ and $\xi_n^{(2)} = c_2a^n$. Finally, by inducting on m, we have (3.2). ∎

Generally, let $\rho^{(j,r)}$ $(j = 1, \dots, m;\ r = 1, \dots, s)$ be some weight sequences and $\xi^{(j,r)} \in l_p(\rho^{(j,r)})$. For $r = 1, \dots, s$, by setting

$$a_r = \Big(\prod_{j=1}^{m} *(\xi^{(j,r)}\rho^{(j,r)})\Big)\Big(\prod_{j=1}^{m} *\rho^{(j,r)}\Big)^{-1/q}, \quad b_r = \Big(\prod_{j=1}^{m} *\rho^{(j,r)}\Big)^{1/q}$$

and by using the inequality

$$\Big|\sum_{r=1}^{s} a_rb_r\Big|^p \le \Big(\sum_{r=1}^{s}|a_r|^p\Big)\Big(\sum_{r=1}^{s}|b_r|^q\Big)^{p-1},$$

we observe that

$$\Big|\sum_{r=1}^{s}\Big(\prod_{j=1}^{m} *(\xi^{(j,r)}\rho^{(j,r)})\Big)\Big|^p \le \sum_{r=1}^{s}\Big|\prod_{j=1}^{m} *(\xi^{(j,r)}\rho^{(j,r)})\Big|^p\Big(\prod_{j=1}^{m} *\rho^{(j,r)}\Big)^{1-p}\Big\{\sum_{r=1}^{s}\prod_{j=1}^{m} *\rho^{(j,r)}\Big\}^{p-1}.$$

Therefore,

$$\Big|\sum_{r=1}^{s}\Big(\prod_{j=1}^{m} *(\xi^{(j,r)}\rho^{(j,r)})\Big)\Big|^p\Big\{\sum_{r=1}^{s}\prod_{j=1}^{m} *\rho^{(j,r)}\Big\}^{1-p} \le \sum_{r=1}^{s}\Big|\prod_{j=1}^{m} *(\xi^{(j,r)}\rho^{(j,r)})\Big|^p\Big(\prod_{j=1}^{m} *\rho^{(j,r)}\Big)^{1-p}. \tag{3.10}$$

Combining with Theorem 3.1 we get the following inequality

$$\text{(3.11)}\qquad \Big\| \Big(\sum_{r=1}^{s} \prod_{j=1}^{m} *(\xi^{(j,r)} \rho^{(j,r)}) \Big) \Big(\sum_{r=1}^{s} \prod_{j=1}^{m} *\rho^{(j,r)} \Big)^{1/p-1} \Big\|_{l_p}^{p} \leq \sum_{r=1}^{s} \prod_{j=1}^{m} \| \xi^{(j,r)} \|_{l_p(\rho^{(j,r)})}^{p}.$$

4 Reverse Convolution Inequalities in Weighted l_p Spaces

Various weighted l_p-norm inequalities in convolutions are derived by using Hölder's inequality. Therefore, by using reverse Hölder inequality one can obtain reverse weighted l_p-norm inequalities.

We first restate the reverse Hölder inequality ([8], see also [3, pp. 125–126]) here.

Proposition 4.1 *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be some positive numbers such that $0 < m \leq a_k / b_k \leq M$. Then we have*

$$\text{(4.1)}\qquad \Big(\sum_{k=1}^{n} a_k \Big)^{1/p} \Big(\sum_{k=1}^{n} b_k \Big)^{1/q} \leq A_{p,q} \Big(\frac{m}{M} \Big) \sum_{k=1}^{n} a_k^{1/p} b_k^{1/q},$$

if the right hand side of (4.1) *is finite. Here*

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{(1-t^{\frac{1}{p}})^{\frac{1}{p}}(1-t^{\frac{1}{q}})^{\frac{1}{q}}}.$$

In connection with Proposition 4.1 we note the following version, whose proof is surprisingly simple.

Theorem 4.1 *In Proposition 4.1, replacing a_k and b_k by a_k^p and b_k^q ($k = 1, 2, \dots, n$), respectively, we obtain the reverse Hölder type inequality*

$$\text{(4.2)}\qquad \Big(\sum_{k=1}^{n} a_k^p \Big)^{1/p} \Big(\sum_{k=1}^{n} b_k^q \Big)^{1/q} \leq \Big(\frac{m}{M} \Big)^{-\frac{1}{pq}} \sum_{k=1}^{n} a_k b_k.$$

Proof Since $a_k^p / b_k^q \leq M$, it follows that

$$a_k b_k \geq M^{-1/q} a_k^p$$

and so,

$$\text{(4.3)}\qquad \Big(\sum_{k=1}^{n} a_k^p \Big)^{1/p} \leq M^{\frac{1}{pq}} \Big(\sum_{k=1}^{n} a_k b_k \Big)^{1/p}.$$

On the other hand, from $m \le a_k^p / b_k^q$, we obtain

$$\sum_{k=1}^{n} a_k b_k \ge m^{1/p} \sum_{k=1}^{n} b_k^q,$$

and this establishes

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^{1/q} \ge m^{\frac{1}{pq}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{1/q}.$$

Combining with (4.3), we have the desired inequality (4.2). ■

Remark 4.2 Theorem 4.1 can be derived from Proposition 4.1 by using the inequality

$$A_{p,q}(t) < t^{-\frac{1}{pq}},$$

which was proved by Lars-Erik Persson (see [7] for more details).

Theorem 4.3 *Let $\xi^{(j)} = \{\xi_n^{(j)}\}_{n=0}^{\infty}$ $(j = 1, \dots, m)$ be some sequences of strictly positive numbers satisfying*

$$0 < m_j^{1/p} \le \xi_n^{(j)} \le M_j^{1/p} < \infty, \quad j = 1, 2, \dots, m, \quad n = 0, 1, \dots. \tag{4.4}$$

Then for any weight sequences $\rho^{(j)} = \{\rho_n^{(j)}\}_{n=0}^{\infty}$, we have the reverse l_p weighted convolution inequality

$$\begin{aligned}\Big\| \prod_{j=1}^{m} *(\xi^{(j)}\rho^{(j)}) \Big(\prod_{j=1}^{m} *\rho^{(j)}\Big)^{(1-p)/p} \Big\|_{l_p}^{p} &\ge \prod_{j=2}^{m} A_{p,q}^{-1}\Big(\prod_{k=1}^{j} \frac{m_k}{M_k}\Big) \prod_{j=1}^{m} \|\xi^{(j)}\|_{l_p(\rho^{(j)})}^{p} \\ &\ge \Big(\prod_{j=2}^{m}\prod_{i=1}^{j} \frac{m_i}{M_i}\Big)^{\frac{1}{pq}} \prod_{j=1}^{m} \|\xi^{(j)}\|_{l_p(\rho^{(j)})}^{p}.\end{aligned} \tag{4.5}$$

Proof For $m = 2$, from the inequality (4.1) we have

$$\begin{aligned}\prod_{j=1}^{2} *(\xi_n^{(j)}\rho_n^{(j)}) &= \sum_{r=0}^{n} \xi_r^{(1)}\rho_r^{(1)}\xi_{n-r}^{(2)}\rho_{n-r}^{(2)} \\ &\ge A_{p,q}^{-1}\Big(\frac{m_1 m_2}{M_1 M_2}\Big) \Big(\prod_{j=1}^{2} *[(\xi_n^{(j)})^p \rho_n^{(j)}]\Big)^{1/p} \Big(\prod_{j=1}^{2} *\rho_n^{(j)}\Big)^{1/q}.\end{aligned}$$

We suppose that

$$\prod_{j=1}^{m} *(\xi_n^{(j)}\rho_n^{(j)}) \ge \prod_{j=2}^{m} A_{p,q}^{-1}\Big(\prod_{i=1}^{j} \frac{m_i}{M_i}\Big) \Big(\prod_{j=1}^{m} *[(\xi_n^{(j)})^p \rho_n^{(j)}]\Big)^{1/p} \Big(\prod_{j=1}^{m} *\rho_n^{(j)}\Big)^{1/q}.$$

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Then, we have

$$\prod_{j=1}^{m+1} *(\xi_n^{(j)}\rho_n^{(j)})$$

$$= \sum_{r=0}^{n}\Big[\prod_{j=1}^{m} *(\xi_r^{(j)}\rho_r^{(j)})\Big]\,\xi_{n-r}^{(m+1)}\rho_{n-r}^{(m+1)}$$

$$\geq \prod_{j=2}^{m} A_{p,q}^{-1}\Big(\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)\sum_{r=0}^{n}\Big(\prod_{j=1}^{m} *[(\xi_r^{(j)})^p\rho_r^{(j)}]\Big)^{1/p}\Big(\prod_{j=1}^{m} *\rho_r^{(j)}\Big)^{1/q}\xi_{n-r}^{(m+1)}\rho_{n-r}^{(m+1)},$$

which is, by the inequality (4.1),

$$\geq \prod_{j=2}^{m} A_{p,q}^{-1}\Big(\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)A_{p,q}^{-1}\Big(\prod_{i=1}^{m+1}\frac{m_i}{M_i}\Big)\Big(\prod_{j=1}^{m+1} *[(\xi_n^{(j)})^p\rho_n^{(j)}]\Big)^{1/p}\Big(\prod_{j=1}^{m+1} *\rho_n^{(j)}\Big)^{1/q}$$

and so the assertion follows.

Therefore,

$$\Big(\prod_{j=1}^{m} *(\xi_n^{(j)}\rho_n^{(j)})\Big)^p\Big(\prod_{j=1}^{m} *\rho_n^{(j)}\Big)^{1-p} \geq \prod_{j=2}^{m} A_{p,q}^{-p}\Big(\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)\prod_{j=1}^{m} *[(\xi_n^{(j)})^p\rho_n^{(j)}]$$

and hence that

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\prod_{j=1}^{m} *(\xi_n^{(j)}\rho_n^{(j)})\Big)^p\Big(\prod_{j=1}^{m} *\rho_n^{(j)}\Big)^{1-p} &\geq \prod_{j=2}^{m} A_{p,q}^{-p}\Big(\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)\sum_{n=0}^{\infty}\prod_{j=1}^{m} *[(\xi_n^{(j)})^p\rho_n^{(j)}]\\ &= \prod_{j=2}^{m} A_{p,q}^{-p}\Big(\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)\prod_{j=1}^{m}\sum_{n=0}^{\infty}(\xi_n^{(j)})^p\rho_n^{(j)}\\ &\geq \Big(\prod_{j=2}^{m}\prod_{i=1}^{j}\frac{m_i}{M_i}\Big)^{\frac{1}{q}}\prod_{j=1}^{m}\sum_{n=0}^{\infty}(\xi_n^{(j)})^p\rho_n^{(j)}.\end{aligned}$$

Thus, the theorem is proved. ■

Inequality (4.5) can be generalized further as follows:

Theorem 4.4 *Let $\xi^{(j)} = \{\xi_n^{(j)}\}_{n=0}^{\infty}$ ($j = 1, \dots, m$) be some sequences of strictly positive numbers and let*

$$m_n^{(j)} = \Big(\min_{r_1+\cdots+r_j=n}\Big\{\prod_{i=1}^{j}\xi_{r_i}^{(i)}\Big\}\Big)^p, \quad j = 2, 3, \dots, m, \quad n = 0, 1, \dots$$

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and

$$M_n^{(j)} = \Big(\max_{r_1+\cdots+r_j=n}\Big\{\prod_{i=1}^{j}\xi_{r_i}^{(i)}\Big\}\Big)^p, \quad j=2,3,\dots,m, \quad n=0,1,\dots.$$

Then for any weight sequences $\rho^{(j)} = \{\rho_n^{(j)}\}_{n=0}^{\infty}$, *we have the reverse* l_p *weighted convolution inequality*

$$\sum_{n=0}^{\infty}\Big(\prod_{j=1}^{m}*(\xi_n^{(j)}\rho_n^{(j)})\Big)^p\Big(\prod_{j=1}^{m}*\rho_n^{(j)}\Big)^{1-p}\Big(\prod_{j=2}^{m}\prod_{i=1}^{j}\frac{m_n^{(i)}}{M_n^{(i)}}\Big)^{-\frac{1}{q}} \tag{4.6}$$
$$\geq \sum_{n=0}^{\infty}\Big(\prod_{j=1}^{m}*(\xi_n^{(j)}\rho_n^{(j)})\Big)^p\Big(\prod_{j=1}^{m}*\rho_n^{(j)}\Big)^{1-p}\prod_{j=2}^{m}A_{p,q}^p\Big(\prod_{i=1}^{j}\frac{m_n^{(i)}}{M_n^{(i)}}\Big)$$
$$\geq \prod_{j=1}^{m}\sum_{n=0}^{\infty}(\xi_n^{(j)})^p\rho_n^{(j)}.$$

We note that the reverse convolution inequality (4.5) does not hold for $m_j = 0$. In this case, we have the following theorem:

Theorem 4.5 *Let* $p \geq 1$, $N_j \in \mathbb{N}$ $(j=1,\dots,m)$, $0 \leq N < N_1$, *and* $\xi_k^{(j)}$ *satisfy*

$$0 \leq \xi_k^{(j)} \leq M < \infty, \quad 0 \leq k \leq \sum_{j=1}^{m}N_j, \quad j=1,\dots,m. \tag{4.7}$$

Then

$$\sum_{k=N}^{N_1}(\xi_k^{(1)})^p\prod_{j=2}^{m}\sum_{k=0}^{N_j}(\xi_k^{(j)})^p \tag{4.8}$$
$$\leq M^{m(p-1)}\sum_{k_m=N}^{N_1+\cdots+N_m}\sum_{k_{m-1}=N}^{k_m}\cdots\sum_{k_1=N}^{k_2}\xi_{k_1}^{(1)}\prod_{j=2}^{m}\xi_{k_j-k_{j-1}}^{(j)}.$$

In particular, for $N=0$, *we have*

$$\prod_{j=1}^{m}\sum_{k=0}^{N_j}(\xi_k^{(j)})^p \leq M^{m(p-1)}\sum_{n=0}^{N_1+\cdots+N_m}\prod_{j=1}^{m}*\xi_n^{(j)}. \tag{4.9}$$

Proof Since $0 \leq \xi_k^{(j)} \leq M$ for $0 \leq k \leq \sum_{j=1}^{m}N_j$, $1 \leq j \leq m$, it follows that

$$\sum_{k_m=N}^{N_1+\cdots+N_m}\sum_{k_{m-1}=N}^{k_m}\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p \tag{4.10}$$
$$\leq M^{m(p-1)}\sum_{k_m=N}^{N_1+\cdots+N_m}\sum_{k_{m-1}=N}^{k_m}\cdots\sum_{k_1=N}^{k_2}\xi_{k_1}^{(1)}\prod_{j=2}^{m}\xi_{k_j-k_{j-1}}^{(j)}.$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k_m=N}^{N_1+\cdots+N_m}\sum_{k_{m-1}=N}^{k_m}\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\\
&=\sum_{k_m=N}^{N_1+\cdots+N_m}\sum_{k_{m-1}=N}^{N_1+\cdots+N_m}\Big\{\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m-1}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\Big\}[\xi_{k_m-k_{m-1}}^{(m)}]^p\theta(k_m-k_{m-1})\\
&=\sum_{k_{m-1}=N}^{N_1+\cdots+N_m}\Big\{\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m-1}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\Big\}\sum_{k_m=N}^{N_1+\cdots+N_m}[\xi_{k_m-k_{m-1}}^{(m)}]^p\theta(k_m-k_{m-1})\\
&=\sum_{k_{m-1}=N}^{N_1+\cdots+N_m}\Big\{\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m-1}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\Big\}\sum_{r=0}^{N_1+\cdots+N_m-k_{m-1}}[\xi_r^{(m)}]^p\\
&\geq\sum_{k_{m-1}=N}^{N_1+\cdots+N_{m-1}}\Big\{\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m-1}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\Big\}\sum_{r=0}^{N_1+\cdots+N_m-k_{m-1}}[\xi_r^{(m)}]^p\\
&\geq\sum_{k_{m-1}=N}^{N_1+\cdots+N_{m-1}}\Big\{\cdots\sum_{k_1=N}^{k_2}\Big[\xi_{k_1}^{(1)}\prod_{j=2}^{m-1}\xi_{k_j-k_{j-1}}^{(j)}\Big]^p\Big\}\sum_{r=0}^{N_m}[\xi_r^{(m)}]^p,
\end{aligned}$$

which is, by inducting on m,

$$\geq\Big(\sum_{k=N}^{N_1}(\xi_k^{(1)})^p\Big)\prod_{j=2}^{m}\Big(\sum_{k=0}^{N_j}(\xi_k^{(j)})^p\Big).$$

Combining with (4.10), we have the desired inequality (4.8). ■

5 Applications

Following our fundamental convolution inequalities, we shall give two typical examples as applications.

5.1 Example 1

Let

$$\xi_n^{(1)}=\xi_n^{(2)}=\frac{1}{n+1},\quad n=0,1,2,\ldots,$$

we have

$$\xi_n^{(1)}*\xi_n^{(2)}=\sum_{m=0}^{n}\frac{1}{m+1}\frac{1}{n-m+1}=\frac{2}{n+2}\sum_{m=0}^{n}\frac{1}{m+1}$$

and

$$\frac{4}{(n+2)^2}\leq\frac{1}{m+1}\frac{1}{n-m+1}\leq\frac{1}{n+1}.$$

Then, from the inequality (3.6) and Theorem 4.4 we obtain

$$\sum_{n=1}^{\infty} \frac{(\sum_{m=1}^{n} \frac{1}{m})^p}{n^{p-1}(n+1)^p} < \frac{\zeta^2(p)}{2^p} < 4^{1-1/p} \sum_{n=1}^{\infty} \frac{(\sum_{m=1}^{n} \frac{1}{m})^p}{n^{p+1/p-2}(n+1)^{p+2/p-2}}, \tag{5.1}$$

where the Riemann zeta function $\zeta(s)$ defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{5.2}$$

In particular, for $p = 2$ we have

$$\sum_{n=1}^{\infty} \frac{(\sum_{m=1}^{n} \frac{1}{m})^2}{n(n+1)^2} < \frac{\pi^4}{144} < 2 \sum_{n=1}^{\infty} \frac{(\sum_{m=1}^{n} \frac{1}{m})^2}{\sqrt{n}(n+1)}. \tag{5.3}$$

5.2 Example 2

A discrete system that smooths the input signal x_n is described by the difference equation

$$y_n = a y_{n-1} + (1-a)x_n, \quad n = 0, 1, 2, \dots, \tag{5.4}$$

where a is a constant such that $|a| < 1$.

By repeated substitution and assuming zero initial condition $y_{-1} = 0$, the output of the system is given by

$$y_n = (1-a) \sum_{m=0}^{n} a^{n-m} x_m, \quad n = 0, 1, 2, \dots. \tag{5.5}$$

Then, from the inequality (3.6), we obtain the estimate

$$\sum_{n=0}^{\infty} \frac{|y_n|^p}{(n+1)^{p-1}} \le \frac{|1-a|^p}{1-|a|^p} \sum_{n=0}^{\infty} |x_n|^p \tag{5.6}$$

for $\{x_n\}_{n=0}^{\infty} \in l_p (p > 1)$.

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