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ALGEBRAIC VALUES OF TRANSCENDENTAL FUNCTIONS AT ALGEBRAIC POINTS

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Abstract

It is shown that any subset of $\overline{\mathbb{Q}}$ can be the exceptional set of some transcendental entire function. Furthermore, we give a much more general version of this theorem and present a unified proof.

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1. Introduction

In 1886, Weierstrass gave an example of a transcendental entire function which takes rational values at all rational points. He also suggested that there exist transcendental entire functions which take algebraic values at any algebraic point. Later, in [3], Stäckel proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(\Sigma) \subseteq T$. Another construction due to Stäckel produces an entire function f whose derivatives $f^{(s)}$, for $s = 0, 1, 2, \ldots$, all map $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}$; see [4]. A more thorough discussion on this subject can be found in [2, 6]. There are recent results due to Surroca on the number of algebraic points where a transcendental analytic function takes algebraic values, see [5]. We were able to generalize these two results of Stäckel to the following general theorem.

THEOREM 1. *Given a countable subset $A \subseteq \mathbb{C}$ and for each integer $s \geq 0$ with $\alpha \in A$, fix a dense subset $E_{\alpha,s} \subseteq \mathbb{C}$. Then there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$, for all $\alpha \in A$ and all $s \geq 0$.*

Let f be given, and denote by S_f the set of all algebraic points $\alpha \in \mathbb{C}$, for which $f(\alpha)$ is also algebraic. An interesting problem is to determine properties of S_f, which we call the exceptional set of f. In the conclusion we will show that for any $A \subseteq \overline{\mathbb{Q}}$ there is a transcendental entire function f such that A is the exceptional set of f.

Without referring to Theorem 1, we have the following special examples.

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EXAMPLE 2. Arbitrary finite subsets of algebraic numbers are easily seen to be exceptional. For instance, if $f_1(z) = e^{(z-\alpha_1)\cdots(z-\alpha_k)}$, then the Hermite–Lindemann theorem implies that $S_{f_1} = \{\alpha_1, \ldots, \alpha_k\}$. If $f_2(z) = e^z + e^{z+1}$and $f_3(z) = e^{z\pi+1}$, then the Lindemann–Weierstrass and Baker theorems imply that $S_{f_2} = S_{f_3} = \emptyset$.

EXAMPLE 3. Some well-known infinite sets are also exceptional; for instance, if $f_4(z) = 2^z$, $f_5(z) = e^{i\pi z}$, then $S_{f_4} = S_{f_5} = \mathbb{Q}$, by the Gelfond–Schneider theorem.

EXAMPLE 4. Assuming Schanuel's conjecture to be true, it is easy to prove that if

$$f_6(z) = \sin(\pi z)e^z, \quad f_7(z) = 2^{3^z} \quad \text{and} \quad f_8(z) = 2^{2^{2^{z-1}}},$$

then $S_{f_6} = S_{f_7} = \mathbb{Z}$ and $S_{f_8} = \mathbb{N}$.

These examples are just special cases of our Theorem 1; they can be proved uniformly here.

2. Preliminary results

In order to prove Theorem 1, we need several lemmas.

LEMMA 5. *Let $\{P_n(z)\}_{n\geq 0}$ be a sequence of complex polynomials, where* deg $P_n = n$. *Also let $\{C_n\}_{n\geq 0}$ be a sequence of positive constants such that $|P_n(z)| \leq C_n \max\{|z|, 1\}^n$. If a sequence of complex numbers $\{a_n\}_{n\geq 0}$ satisfies $|a_n| \leq 1/C_n n!$, then the series $\sum_{n=0}^{\infty} a_n P_n(z)$ converges absolutely and uniformly on any compact set; in particular, this gives an entire function.*

PROOF. When $|a_n| \leq 1/C_n n!$,

$$\sum_{n=0}^{\infty} |a_n||P_n(z)| \leq \sum_{n=0}^{\infty} \frac{1}{C_n n!} C_n \max\{|z|, 1\}^n \leq \exp(\max\{|z|, 1\}),$$

so $\sum_{n=0}^{\infty} a_n P_n(z)$ converges absolutely and uniformly on any compact set. Therefore this series will produce an entire function. □

Let us now enumerate the set A in Theorem 1 as $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$.

For $n \geq 1$, define m_n and j_n by $n = 1 + 2 + 3 + \cdots + m_n + j_n$, where $m_n \geq 0$ and $1 \leq j_n \leq m_n + 1$. Next, construct a sequence of polynomials by letting $P_0(z) = 1$ and defining recursively

$$P_n(z) = (z - \alpha_{j_n})P_{n-1}(z) \quad \text{for } n \geq 1.$$

Here we list the first few polynomials:

$$\begin{aligned}
P_0(z) &= 1 \\
P_1(z) &= (z - \alpha_1) \\
P_2(z) &= (z - \alpha_1)^2 \\
P_3(z) &= (z - \alpha_1)^2(z - \alpha_2) \\
P_4(z) &= (z - \alpha_1)^3(z - \alpha_2)
\end{aligned}$$

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$$\begin{aligned} P_5(z) &= (z-\alpha_1)^3(z-\alpha_2)^2 \\ P_6(z) &= (z-\alpha_1)^3(z-\alpha_2)^2(z-\alpha_3) \\ P_7(z) &= (z-\alpha_1)^4(z-\alpha_2)^2(z-\alpha_3) \\ &\vdots \end{aligned}$$

Let $i_n = m_n + 1 - j_n$. For any given $i \geq 0$ and $j \geq 1$ there exists a unique $n \geq 1$ such that $i_n = i$ and $j_n = j$, namely $n = \frac{1}{2}(i+j)(i+j-1)+j$.

LEMMA 6. *For $n \geq 1$, we have $P_{n-1}^{(i_n)}(\alpha_{j_n}) \neq 0$ and $P_l^{(i_n)}(\alpha_{j_n}) = 0$ when $l \geq n$.*

PROOF. From the definition of $P_n(z)$, we can write explicitly

$$P_l(z) = (z-\alpha_1)^{m_l}(z-\alpha_2)^{m_l-1}\cdots(z-\alpha_{m_l})(z-\alpha_1)\cdots(z-\alpha_{j_l}).$$

It follows that α_{j_n} is a zero of $P_{n-1}(z)$ with multiplicity i_n, which means that $P_{n-1}^{(i_n)}(\alpha_{j_n}) \neq 0$. On the other hand, if $l \geq n$, then α_{j_n} is a zero of $P_l(z)$ with multiplicity at least $i_n + 1$, which implies that $P_l^{(i_n)}(\alpha_{j_n}) = 0$. □

LEMMA 7. *If $\sum_{k=0}^{\infty} a_k P_k(z) = \sum_{k=0}^{\infty} b_k P_k(z)$ for all $z \in \mathbb{C}$, then $a_k = b_k$ for each $k \geq 0$.*

PROOF. It suffices to prove that if $g(z) := \sum_{k=0}^{\infty} a_k P_k(z) = 0$ for all $z \in \mathbb{C}$, then $\{a_k\}_{k\geq 0}$ is identically 0. Notice that $a_0 = g(\alpha_1) = 0$. Assuming that $a_0, a_1, \ldots, a_{n-1}$ are all 0, by Lemma 6,

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) \\ &= \sum_{k=0}^{n-1} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) + a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) + \sum_{k=n+1}^{\infty} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) \\ &= a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}). \end{aligned}$$

Since $P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \neq 0$, we have $a_n = 0$. The proof is completed by induction. □

We are now able to prove our theorem.

3. Proof of Theorem 1

We will construct the desired transcendental entire function by fixing the coefficients in the series $\sum_{k=0}^{\infty} a_k P_k(z)$ recursively, where the sequence $\{P_k\}_{k\geq 0}$ has been defined in Section 2.

First, by Lemma 5, the condition $|a_k| \leq 1/C_k k!$ will ensure that $\sum_{k=0}^{\infty} a_k P_k(z)$ is entire.

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Now we will fix the coefficients a_k recursively. For $n \geq 1$, we denote $E_n = E_{\alpha_{j_n}, i_n}$ and let the numbers $\beta_n = \sum_{k=0}^{\infty} a_k P_k^{(i_n)}(\alpha_{j_n})$. We will choose the value of a_k so that $\beta_n \in E_{\alpha_{j_n}, i_n} = E_n$ for all $n \geq 1$.

By Lemma 6, we know that $P_l^{(i_n)}(\alpha_{j_n}) = 0$ when $l \geq n$, so β_n is actually the finite sum $\sum_{k=0}^{n-1} a_k P_k^{(i_n)}(\alpha_{j_n})$. Notice that $\beta_1 = a_0 P_0^{(0)}(\alpha_1) = a_0$ and E_1 is dense; we can fix a value for a_0 such that $0 < |a_0| \leq 1/C_0$ and $\beta_1 \in E_1$. Now suppose that the values of $\{a_0, a_1, \ldots, a_{n-1}\}$ are well fixed such that $0 < |a_k| \leq 1/C_k k!$ and $\beta_k \in E_k$ for $0 \leq k \leq n-1$. By Lemma 6, we know $P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \neq 0$, so we can pick a proper value of a_n such that $0 < |a_n| \leq 1/C_n n!$ and

$$\beta_n = \sum_{k=0}^{n-1} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) + a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \in E_n.$$

So now by induction all the a_k are well chosen such that for all $k \geq 1$ we have $0 < |a_k| \leq 1/C_k k!$ and $\beta_k \in E_k$. Thus by Lemma 5, the function $f(z) = \sum_{k=0}^{\infty} a_k P_k(z)$ is entire and for any $i \geq 0$, $j \geq 1$ we have $f^{(i)}(\alpha_j) = \sum_{k=0}^{\infty} a_k P_k^{(i)}(\alpha_j) = \beta_n \in E_n = E_{\alpha_j, i}$ where n is the unique integer such that $i_n = i$, $j_n = j$. Taking into account that every polynomial can be expressed as a finite linear combination of the $\{P_k\}$, and all the $\{a_k\}$ here are not 0, so by Lemma 7 we conclude that $f(z)$ is not a polynomial. Hence $f(z)$ is the desired transcendental entire function, and the proof is complete.

From the construction of the proof, we can easily see that in fact there are uncountably many functions satisfying the properties required in Theorem 1.

4. Applications to exceptional sets

We recall the following definition.

DEFINITION 8. Let f be an entire function. We define *the exceptional set of* f to be

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\}.$$

We list some of the more interesting consequences of Theorem 1 with the choice of A, $E_{\alpha,s}$ noted in parentheses.

COROLLARY 9. *For each countable subset $\Sigma \subseteq \mathbb{C}$ and for each dense subset $T \subseteq \mathbb{C}$ there is a transcendental entire function f such that $f^{(s)}(\Sigma) \subseteq T$ for $s \geq 0$. ($A = \Sigma$, $E_{\alpha,s} = T$.)*

COROLLARY 10. *Let $A \subseteq \mathbb{C}$ be countable and dense in $\mathbb{C}$. Then there is a transcendental entire function f such that $f^{(s)}(A) \subseteq A$, for $s \geq 0$. ($E_{\alpha,s} = A$.)*

COROLLARY 11. *There exists a transcendental entire function such that $f^{(s)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for $s \geq 0$. ($A = \overline{\mathbb{Q}}$, $E_{\alpha,s} = \mathbb{Q}(i)$.)*

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A set A is said to be *closed* (with respect to $\overline{\mathbb{Q}}$) if it has the following property: if α is algebraic and α' is any algebraic conjugate of α, then $\alpha \in A$ implies that also $\alpha' \in A$. In 1965, Mahler [1] proved that every closed set is the exceptional set of some transcendental entire function. Our next result shows, in particular, that another interesting consequence of Theorem 1 is that every $A \subseteq \overline{\mathbb{Q}}$ is an exceptional set of a transcendental entire function.

THEOREM 12. *If $A \subseteq \overline{\mathbb{Q}}$, then there is a transcendental entire function such that $S_{f^{(s)}} = A$ for $s \geq 0$.*

PROOF. Suppose that A and $\overline{\mathbb{Q}} \setminus A$ are both infinite, thus we can enumerate $\overline{\mathbb{Q}} = \{\alpha_1, \alpha_2, \ldots\}$ where $A = \{\alpha_1, \alpha_3, \ldots, \alpha_{2n+1}, \ldots\}$. Set $E_{\alpha_{2n+2},s} = \mathbb{C} \setminus \overline{\mathbb{Q}}$ and $E_{\alpha_{2n+1},s} = \overline{\mathbb{Q}}$ for all $n, s \geq 0$. Now by Theorem 1, there exists a transcendental entire function f with $f^{(s)}(\alpha_{2n+1}) \in \overline{\mathbb{Q}}$ and $f^{(s)}(\alpha_{2n+2}) \in \mathbb{C} \setminus \overline{\mathbb{Q}}$, for each $n, s \geq 0$. Therefore it is plain that $S_{f^{(s)}} = A$.

For the case where A is finite, we can suppose that $A = \{\alpha_1, \ldots, \alpha_m\}$. Take $E_{\alpha_1,s} = \cdots = E_{\alpha_m,s} = \overline{\mathbb{Q}}$ for all $s \geq 0$, and set $E_{\alpha_k,s} = \mathbb{C} \setminus \overline{\mathbb{Q}}$ for all $k > m$, $s \geq 0$. If $\overline{\mathbb{Q}} \setminus A = \{\alpha_1, \ldots, \alpha_m\}$, we take $E_{\alpha_1,s} = \cdots E_{\alpha_m,s} = \mathbb{C} \setminus \overline{\mathbb{Q}}$ for all $s \geq 0$, and set $E_{\alpha_k,s} = \overline{\mathbb{Q}}$ for all $k > m$, $s \geq 0$. Then for these two cases we proceed as in the proof above. □

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